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CANONICAL BASIS INTERPOLATION METHOD OF SOLVING INITIAL-VALUE PROBLEMS

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Abstract

This paper is focused on the numerical solution of initial-value problems of ordinary differential equations, using Newton's interpolation method and Lagrange method. New canonical polynomials are constructed and used as basis functions. The Lanczos method is adopted for the construction of the polynomials to certain degrees, whereby a recursive relation is developed to generate a set of canonical polynomials. Hence, these Canonical polynomials in combination with the Newton interpolation and Lagrange method are utilized for the approximation of the unknown functions and differential functions in the given differential equations. The effectiveness and efficiency of the method is evidently ascertained as it is applied on some test problems.

Key Words: Lanczos Methods, Canonical Polynomials, Lagrange and Newton Interpolation Method.

Introduction.

For decades, differential equations (DEs) have been consistently solved using series of numerical methods, as not all DEs can be solved by the usual known analytical methods. Therefore, overcoming the challenges of the analytical methods using numerical methods in order to provide the required approximate solutions, cannot be overemphasized. However, some numerical methods are based on polynomial approximations. These methods are very relevant and useful in the development of softwares for digital computers, due to their stabilities in the approximation of functions. For instance, in (Owolanke et al, 2017) and (Yakusak and Owolanke, 2018), polynomials of the power series form were used to solve Initial Value Problems (IVPs) of ordinary differential equations. The approximate solutions were obtained via collocation techniques.

In (Lanczos, 1956), a polynomial known as canonical polynomial was introduced, and the idea was purposefully to approximate first order differential equations. The method became popular when it was later used in (Ortiz, 1969). The paper reveals that the solution y(x) can easily be expressed as a linear combination of polynomials, whereby the approximation does not have to depend on integration range or initial conditions. Canonical polynomials have been revealed to be very effective in the approximation of fractional differential equations (FDEs), as well as fractional-integro differential equations (FIDEs), which was evidently proved in (Owolanke et al, 2020). Canonical polynomials were used in the paper to solve exponentially fitted multi-order fractional integro-differential equations. The polynomial also came into play in (Alexandros, 2018), where it was used to decompose higher order differential operators into direct sum of two factor operators.

Interpolation method has remained a viable and reliable method in the solution of ordinary differential equations. It is used in (Faith, 2018) in combination with the Lagrange method to solve first order ordinary differential equations. The Combination of two existing methods have been very effective and efficient in finding approximate solutions to DEs. For instance, this system was adopted by (Salisu, 2023) and (IDE, 2020), whereby the authors solved first order differential equations using the Newton's interpolation method combined with Lagrange interpolation method and Newton's interpolation method combined with Aitken's method respectively. The errors in the numerical results were significantly small when compared with the exact solutions of the given differential equations. Thus, in view of the recent works of (Salisu, 2023) and (IDE, 2020), this paper is purposefully focused on the solution of first order differential equations adopting the Newton's interpolation method with Lagrange method in combination with canonical polynomials derived from linear differential operators.

Methodology

The kth-order ordinary differential equations is considered as follows

$$Ay(x)+By'(x)+Cy''(x)+...+Ny^{k}(x)=r(x) \qquad (1)$$

where, A, B, C, ..., N are constant coefficients and r(x) is the source term. The equation (1) is equivalently written as

$$(A+B\frac{d}{dx}+C\frac{d^2}{dx^2}+...+N\frac{d^k}{dx^k})y(x)=r(x) \qquad (2)$$

Then from equation (2), a differential operator D is defined as follow:

$$D\equiv A+B\frac{d}{dx}C\frac{d^2}{dx^2}+...+N\frac{d^k}{dx^k}$$

Therefore,

$$Dx^k = Ax^k + B(k)x^{(k-1)} + C(k)(k-1)x^{(k-2)} + ...$$
$$+N(k)(k-1)(k-2)...(k-q+1)x^{(k-q)} \qquad (3)$$

In addition, by Lanczos (1956):

$$DP_K(x) = x^k \qquad (4)$$

Hence, equation (1) becomes

$$Dx^k = ADP_k(x) + B(k)DP_{(k-1)}(x) + C(k)(k-1)DP_{(k-2)}(x)$$
$$+...+N(k)(k-1)(k-2)...(k-q+1)DP_{(k-q)}(x) \qquad (5)$$

Assuming the inverse of the operator D, denoted D^{-1} exists, equation (5) is then transformed to

$$x^k = AP_k(x) + B(k)P_{(k-1)}(x) + C(k)(k-1)P_{(k-2)}(x) +$$
$$N(k)(k-1)(k-2)...(k-q+1)P_{(k-q)}(x) \qquad (6)$$

Equation (6) equivalently rearranged to

$$AP_k(x) = x^k - [B(k)P_{(k-1)}(x) + C(k)(k-1)P(x) +$$
$$N(k)(k-1)(k-2)...(k-q+1)P_{(k-q)}(x)] \qquad (7)$$

Implying

$$P_k(x) = 1/A\{x^k - [B(k)P_{(k-1)}(x) + C(k)(k-1)P_{(k-2)}(x)$$
$$+...+N(k)(k-1)(k-2)...(k-q+1)P_{(k-q)}(x)] \qquad (8)$$

Further simplifying equation (8) is further yields the following set of Polynomials for first order initial-value problems

$$P_k(x) = \frac{1}{A}\{x^k - [B(k)P_{(k-1)}(x)]\} \qquad (9)$$

This generates the following polynomials

$$P_0(x) = \frac{1}{A}$$

$$P_1(x) = \frac{1}{A}[x - \frac{B}{A}]$$

$$P_2(x) = \frac{1}{A}[x^2 - \frac{2B}{A}x - \frac{2B^2}{A^2}]$$

Similarly, the polynomial for the second order ordinary differential equation is constructed from equation (8) to yield

$$P_k(x) = \frac{1}{A}\{x^k - [B(k)P_{(k-1)}(x) + C(k)(k-1)P_{(k-2)}(x)]\} \quad (10)$$

It can as well generate another set of polynomials that can be used in the approximation of functions. B and C represent the coefficients of the first and second order ordinary differential equation respectively.

Furthermore, the canonical polynomial basis functions can be written in linear combination with the Newton interpolation method as follows

$$y_k(x) = a_0(x)Q_0(x) + a_1(x)Q_1(x) + a_2(x)Q_2(x) + \dots + a_k(x)Q_k(x) \quad (11)$$

Newton's Divided Difference Method

Newton divided difference formula is a method, basically utilized when the values of arguments in their intervals are not equally spaced. The function values corresponding to the arguments are taking into consideration. Hence, from equation (11), it can be related that

$$a_0k = y_0 \quad (12)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \quad (13)$$

$$a_2 = \frac{\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)} \quad (14)$$

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$$a_n = f(x_k, x_{k-1}, \dots, x_1, x_0) = \frac{[x_k, x_{k-1}, \dots, x_2, x_1] - [x_{k-1}, x_{k-2}, \dots, x_1, x_0]}{(x_2 - x_0)} \quad (15)$$

Equation (15) is the kth Newton's divided differences.

Lagrange Interpolation Method

The Lagrange interpolation method is another versatile numerical interpolation method. It handles values of arguments that are equally and unequally spaced within any given intervals. That is the reason it is preferred to other methods of interpolation specifically, the forward and the backward interpolation methods as they are only useful in cases of equal intervals.

Corollary: Let $x_0, x_1, \dots x_n$ be (n+1) positive integers.

And function f(x) is continuous, closed and bounded. Also, let a unique polynomial $P_n(x)$ that is of degree n exists such that

$$f(x_j) = p(x_j), j = 0, 1, \dots, n \quad (16)$$

$$p(x) = \sum_{j=0}^{n} f(x_j) L_{n,j}(x), \quad j = 0, 1, \dots, n \quad (17)$$

$$p(x) = f(x_0)L_{n,0}(x) + f(x_1)L_{n,1}(x) + \dots + f(x_n)L_{n,n}(x) \quad (18)$$

$$L_{n,j} = \frac{(x - x_0)(x - x_1)\dots(x - x_{j-1})(x - x_{j+1})\dots(x - x_n)}{(x_k - x_0)(x_j - x_1)\dots(x_j - x_{j-1})(x_j - x_{j+1})\dots(x_j - x_n)} \quad (19)$$

for each j = 0, 1, ..., n.

At n= 2, j = 0,1,2. Then equation (19) becomes:

$$p(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \quad (20)$$

where, $f(x_i) = y_i$, i = 0,1,2

Implementation of the Algorithm

In this section, the effectiveness of the method is investigated with respect to equation (11), where the $Q_i(x)$'s are the canonical polynomials recursively generated using equation (10); $a_i(x)$'s represent the Newton's differences of equations (12-14); and $y_k(x)$ is the $f(_xk)$ in the Lagrange method of equation (18). Therefore, combining these schemes as stated above, form the linear combination in equation (11).

Example 1: y'=y-x, y(0) = 0.5, h=0.01

Exact solution: $1 + x - 0.5e^x$

$$y_n(x) = a_0 + a_1P_1 + a_2P_2 + \dots + a_nP_n$$

$P_k = \frac{1}{A}[x^k - BkP_{k-1}]$, where for each k, $P_{-1} = 0$

It implies that

$$P_0 = \frac{1}{1}(x^0 - 1(0)(P_{j-1}) = 1 - 0 = 1$$

Hence, with respect to equation (12)

$a_0 = y_0 = 0.5$

$y_1 = a_0 + a_1 p_1(x) = y + a_1 p_1(x)$ (21)

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = [\frac{dy}{dx}]_{0,0.5} = [y - x]_{0,0.5} = 0.5 - 0 = 0.5$$

$$P_1 = \frac{1}{1}(x^1 - 1(1)(P_0) = x - P_0 = x - 1$$

$$y_1 = 0.5 + 0.5(0.01 - 0) = 0.5 + 0.005 = 0.505$$

$y_2 = a_0 + a_1 p_1(x) + a_2 p_2(x)$ (22)

Applying equation (21) on equation (22), it becomes

$y_2 = y_1 + a_2 p_2(x)$

$$a_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$a_2 = \frac{[\frac{dy}{dx}]_{0.01,0.505} - [\frac{dy}{dx}]_{0,0.5}}{0.02 - 0} \quad (23)$$

This on simplification, equation (23) is equivalently

$$a_2 = \frac{[0.505 - 0.01] - [0.5 - 0]}{0.02}$$

$$a_2 = \frac{0.495 - 0.5}{0.02} = \frac{-0.005}{0.02} = -0.25$$

$$p_2 = \frac{1}{1}((x^2 - 1(2)(p_1))$$

$$p_2 = x^2 - 2p_1$$

$$p_2 = x^2 - 2(x - 1) = x^2 - 2x + 2$$

$$\Rightarrow [x_2^2 - 2x_1 + 2x_0$$

$$y_2 = 0.505 + (-0.25)[0.02^2 - 2(0.01) + 2(0)]$$

$$= 0.505 + 0.0049 = 0.5099$$

With the aid of Lagrange method, a quadratic is formed using the values x_0, x_1, x_2 and y_0, y_1, y_2 inserted in equation (22) as follows:

$$y(x) = \frac{(x-0.01)(x-0.02)}{(0-0.01)(0-0.02)}(0.5) + \frac{(x-0)(x-0.02)}{(0.01-0)(0.01-0.02)}(0.505) + \frac{(x-0)(x-0.01)}{(0.02-0)(0.02-0.01)}(0.5099)$$

$$y(x) = -0.5x^2 + 0.505x + 0.5$$

Example 2: $y' = x^2 - y, y(0) = 1, h = 0.01$

Exact solution: $x^2 - 2x + 2 - e^{-x}$

$$y_n(x) = a_0 + a_1 P_1 + a_2 P_1 + \cdots + a_n P_n$$

$$P_j = \frac{1}{A}(x^j - BjP_{j-1}), \qquad A \neq 0, B = 1, P_{-1} = 0$$

$$p_0 = \frac{1}{1}(x^0 - 1(0)(p - 1) = 1 - 0 = 1$$

$$a_0 = y_0 = 1 \quad (24)$$

$y_1 = a_0 + a_1 p_1$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (25)$$

Equation (25) is equivalently

$$[\frac{dy}{dx}]_{0,1} = [x^2 - y]_{0,1} \quad (26)$$

This implies that

$a_1 = -1$ (27)

Similarly,

$$P_1 = \frac{1}{1}[x^1 - 1](P_0)$$

$$P_1 = x - P_0$$

$$P_1 = x - 1 \qquad \Longrightarrow [x_1 - 1(x_0)]$$

$$y_1 = 1 + (-1)(0.01 - 0)$$

Then, $y_1 = 0.99$ (28)

Similarly, using the approximate solution of equation (11)

$y^2 = a_0 + a_1 p_1 + a_2 p_2 = y_1 + a_2 p_2$

Such that

$$a_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} \quad (29)$$

Equation (29) is equivalently written as

$$\frac{[\frac{dy}{dx}]_{0.01,0.99} - [\frac{dy}{dx}]_{0,1}}{0.02 - 0} = \frac{[0.01^2 - 0.99] - [0^2 - 1]}{0.02}$$

which implies that

$a_2 = 0.505$ (30)

$$P_2 = \frac{1}{1}(x^2 - 1(2)(P_1)$$
$$= x^2 - 2P_1$$
$$= x^2 - 2x + 2 \quad \Rightarrow [x_2^2 - 2x_1 + 2x_0]$$
$$y_2 = 0.505 + (-0.25)[0.02^2 - 2(0.01) + 2(0)]$$

$y_2 = 0.5099$ (31)

Thus, forming a quadratic using equations (24), (28), (31) and the values of x_0, x_1, x_2 equations (18-20) become

$$y(x) = \frac{(x-0.01)(x-0.02)}{(0-0.01)(0-0.02)}(0.5) + \frac{(x-0)(x-0.02)}{(0.01-0)(0.01-0.02)}(0.505) + \frac{(x-0)(x-0.01)}{(0.02-0)(0.02-0.01)}(0.5099)$$

$$y(x) = -0.5x^2 + 0.505x + 0.5$$

Discussion of Results

In this section, the errors in the method, in comparison with the exact solution and the error in (IDE, 2020) are shown.

Errors and Tables of Results

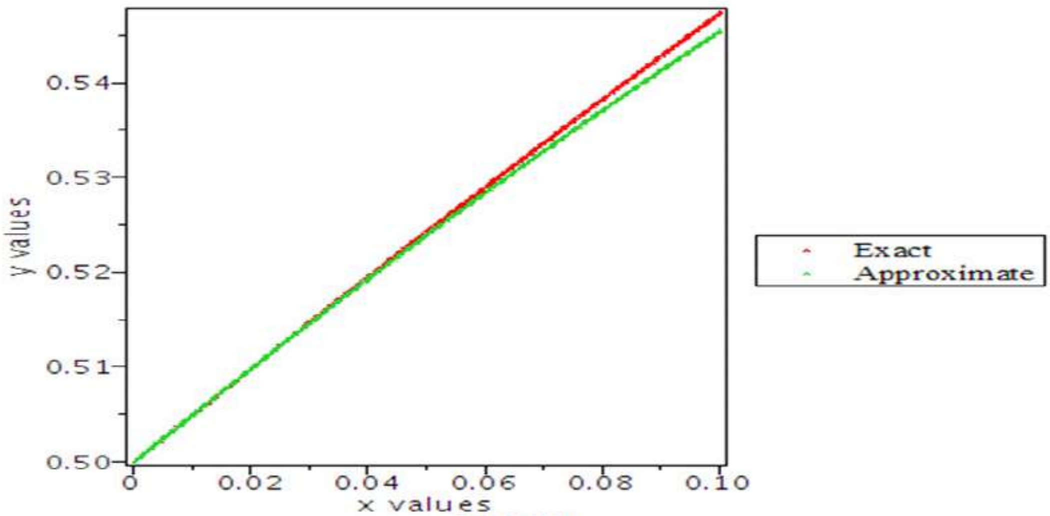
TABLE 1: FOR EXAMPLE 1; h=0.01

x	Exact Value	New Method	IDE (2020)	Errors in new method
0.00	0.500000000	0.5000	0.50000	0.0000000000
0.01	0.504974916	0.5050	0.50500	2.508350E-05
0.02	0.50989933	0.5099	0.50995	6.700130E-07
0.03	0.514772733	0.5147	0.51485	7.273300E-05
0.04	0.519594613	0.5194	0.51970	1.946130 E-04
0.05	0.524364452	0.5240	0.52450	3.644520 E-04
0.06	0.529081727	0.5285	0.52925	5.817270 E-04
0.07	0.533745909	0.5329	0.53395	8.459090 E-04
0.08	0.538356466	0.5372	0.53860	1.156466 E-03
0.09	0.542912858	0.5414	0.54320	1.512858 E-03
0.10	0.547414541	0.5455	0.54775	1.914541 E-03

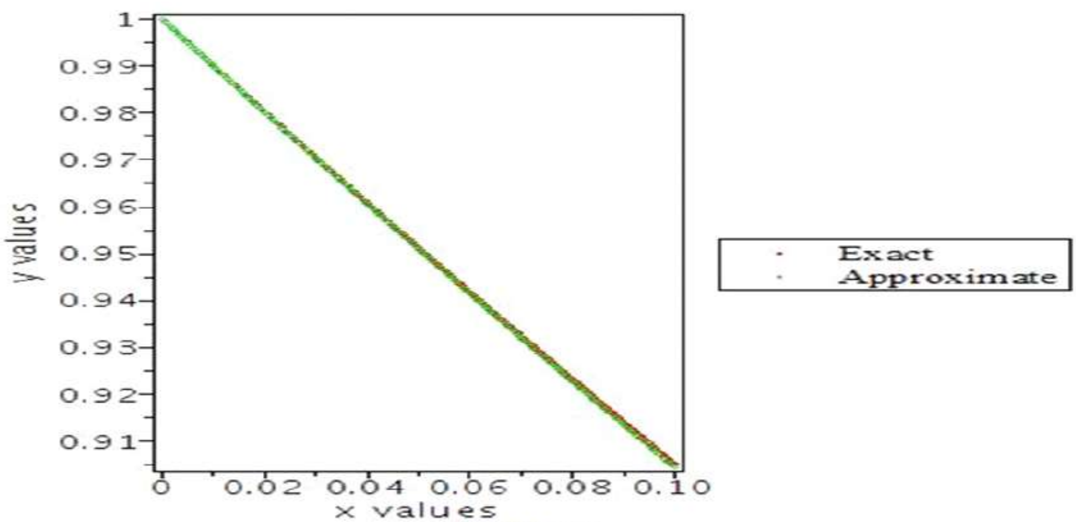
Table 2: FOR EXAMPLE 2; h=0.01

X	Exact Value	New Method	IDE (2020)	Errors in new method	Errors in IDE (2020)
0.00	1.000000000	1.000000	1.000000	0.00000000	0.000000000
0.01	0.990050166	0.990000	0.990000	5.0166**E-05**	5.01663**E-05**
0.02	0.980201327	0.980102	0.980101	9.9327**E-05**	1.00327**E-04**
0.03	0.970454466	0.970306	0.970303	1.4847**E-04**	1.51466**E-04**
0.04	0.960810561	0.960612	0.960606	1.9856**E-04**	2.04561**E-04**
0.05	0.951270575	0.951020	0.951010	2.5058**E-04**	2.60575**E-04**
0.06	0.941835466	0.941530	0.941515	3.0547**E-04**	3.20466**E-04**
0.07	0.932506180	0.932142	0.932121	3.6418**E-04**	3.85180**E-04**
0.08	0.923283654	0.922856	0.922828	4.2765**E-04**	4.55654**E-04**
0.09	0.914168815	0.913672	0.913636	4.9681**E-04**	5.32815**E-04**
0.10	0.905162582	0.904590	0.904545	5.7258**E-04**	6.17582**E-04**

Comparison Graphs



GRAPH 1



GRAPH 2

From the given examples and in view of their tables and graphs respectively, it is inferred that the newly constructed canonical polynomials utilized as the basis function in combination with the Newton-Lagrange interpolation method is a powerful tool. It is found that the approximate method is very efficient and reliable in the solution of ordinary differential equations of initial-value problems.

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