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Implicit Three-Step Hybrid Block Method for Solving First Order Initial Value Problems in Ordinary Differential Equations

Williams Barde,[1] Timothy Suleman,[2] Emmanuel E. Dan[3] and Sabo Zando[1]

[1]Department of Mathematics and Statistics, Federal University, Wukari-Nigeria
[2]Department of Mathematics, Federal College of Education, Yola.-Nigeria
[3]Department of Mathematics, Federal College of Education (T), Gombe.-Nigeria

***Corresponding author's email:**
w.barde@fuwukari.edu.ng

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Recently, the development of numerical method for approximating solutions of initial value problems (IVPs) in ordinary differential equations (ODEs) has attracted considerable attention and many researchers have shown interest in constructing efficient methods with good stability properties for the numerical integration of ODEs. This research focuses on the derivation of new implicit three step block hybrid method for the solution of first order IVPs in ODEs. The new method is derived based on multistep collocation using Chebyshev polynomials as bases functions at some selected points to get a continuous linear multistep method. The continuous methods are evaluated at some off-grid points to generate the discrete schemes for step number *k=3* which conveniently constitutes the block method. Basic properties of the developed method is examined and the method is found to be zero stable, consistent, convergent and of uniform order 8. The efficiency of the method is tested on some numerical examples in the literature. On comparison, the method developed performed favorably when compared with the existing methods. As such the method is recommended for the solution of general first order initial value problems in ordinary differential equations.

Keywords: Differential Equation**,** Initial Value Problems, Linear multistep method, Chebyshev Polynomial, Block Method.

1. Introduction

Although, a very wide variety of numerical methods have been proposed, the number of methods with high order and good stability properties remains relatively small. Solutions to ordinary differential equations were derived using analytic or even exact methods. Most of their solutions are very useful such that it provides excellent insight into the behavior of some system. These include those that can be approximated with linear model and those that have simple geometry and low dimensionality. Conversely, many differential equations cannot be solved analytically, because most real life problems are non-linear and involve complex shapes and processes.

Many researchers have developed different methods for solving first order ordinary differential equations among which are [1], [2], [5], multi-step collocation methods of [4], [10] just to mention but a few. In this research work, we considered the numerical computational methods for first ODEs of the form

$$y' = f(x, y),\ y(a) = y_0, x \in (a, b) \qquad (1)$$

where x_0 is the initial point y_0 is the solution at the initial point and f is assumed to be continuous and satisfy Lipschitz's condition for the existence and uniqueness of solution.

Equation (1.1). occurs in several areas of engineering, sciences and social sciences. Many physical problems are modeled into first order problems. Some of these problems have proved to be either difficult to solve or cannot be solved analytically, hence the need for numerical methods for solving such problems. [4] and [5] posited that there are many methods for solving first order ordinary differential equations. One of the popular methods for solving (1.1) is by Linear Multistep Methods (LMM). This method of solution had been developed in various forms such as discrete and continuous linear multistep methods. Continuous linear multistep methods have greater advantages over the discrete methods as they give better error estimation, provide a simplified form of coefficients for further evaluation at different points, and provides solution at all interior points within the interval of integration than the discrete one [13].

Recently we proposed, An Implicit Two-Step Hybrid Block Method based on Chebyshev Polynomial for Solving First Order Initial Value

Problems in Ordinary Differential Equations see [14], but in this paper effort is being made to extend the scope for k=3, combining the qualities of hybrid methods, block methods and approximate solution using Chebyshev polynomial as a trial function to derive a new method.

2. Derivation of the Method

The approach adopted in this section entails substituting into (1.1) an approximate solution of the form

$$y(x)=\sum_{j=0}^{p+q-1}\alpha_j T_j(x) \qquad (2)$$

where α_j is unknown coefficients and $T_j(x)$ are polynomial basis functions of degree in a manner that $1\le p\le k$ and $q>0$. The integer $k\ge 1$ denote the step number of the method.

$T_j(x)$ is the Chebyshev polynomial generated by the formula:

$$T_j(x)=\frac{(-2)^j}{(2j)!}\,j!\sqrt{1-x^2}\,\frac{d^j}{dx^j}(1-x^2)^{j-\frac{1}{2}} \qquad (3)$$

For sake of reporting, we present some few terms of Chebyshev polynomial as

$$T_0(x)=1,$$

$$T_1(x)=x,$$

$$T_2(x)=2x^2-1,$$

$$T_3(x)=4x^3-3x,$$

$$T_4(x)=8x^4-8x^2+1,$$

$$T_5(x)=16x^5-20x^3+5x,$$

$$T_6(x)=32x^6-48x^4+8x^2-1,$$

$$T_7(x)=64x^7-112x^5+56x^3-7x,$$

$$T_8(x)=128x^8-256x^6+160x^4-32x^2+1,$$

From (2)

$$y'(x)=\sum_{j=0}^{p+q-1}\alpha_j T'_j(x) \qquad (4)$$

Putting (4) into (1) we obtained

$$f(x,y)=\sum_{j=0}^{p+q-1}\alpha_j T'_j(x) \qquad (5)$$

To derive the method, some off step points are carefully introduced to guarantee zero stability.

3. Specification of the Method

We interpolating (2) at $x_{n+p},\ p=0,\ p=\frac{1}{2}$ and collocating (5) at $x_{n+q},\ q=0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3$ leads to the system of equations $AX=U$. (6)

Use Maple 18 software to solve (6) gives the unknown coefficient $\alpha_i\left(i=0,\frac{1}{2}\right),\beta_i\left(i=0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3\right)$ which are then substituted into equation (2) and simplified to give the continuous hybrid method of the form;

$$y(x)=\alpha_0(x)y_n+\alpha_{\frac{1}{2}}y_{n+\frac{1}{2}}+h\left[\sum_{j=0}^{3}\beta_j(x)f_{n+j}+\beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}}+\beta_{\frac{3}{2}}(x)f_{n+\frac{3}{2}}+\beta_{\frac{5}{2}}(x)f_{n+\frac{5}{2}}\right]$$

where

$$\alpha_0(x)=1-\frac{6912}{275}\frac{(-x_n+x)^2}{h^2}+\frac{112896}{1375}\frac{(-x_n+x)^3}{h^3}-\frac{155904}{1375}\frac{(-x_n+x)^4}{h^4}+\frac{112896}{1375}\frac{(-x_n+x)^5}{h^5}-\frac{1792}{55}\frac{(-x_n+x)^6}{h^6}+\frac{9216}{1375}\frac{(-x_n+x)^7}{h^7}-\frac{768}{1375}\frac{(-x_n+x)^8}{h^8}$$

$$\alpha_{\frac{1}{2}}(x)=\frac{6912}{275}\frac{(-x_n+x)^2}{h^2}-\frac{112896}{1375}\frac{(-x_n+x)^3}{h^3}+\frac{155904}{1375}\frac{(-x_n+x)^4}{h^4}-\frac{112896}{1375}\frac{(-x_n+x)^5}{h^5}+\frac{1792}{55}\frac{(-x_n+x)^6}{h^6}-\frac{9216}{1375}\frac{(-x_n+x)^7}{h^7}+\frac{768}{1375}\frac{(-x_n+x)^8}{h^8}$$

$$\beta_0(x)=-x_n+x-\frac{247021}{38500}\frac{(-x_n+x)^2}{h}+\frac{2963212}{185625}\frac{(-x_n+x)^3}{h^2}-\frac{9866993}{495000}\frac{(-x_n+x)^4}{h^3}+\frac{849779}{61875}\frac{(-x_n+x)^5}{h^4}-\frac{39329}{7425}\frac{(-x_n+x)^6}{h^5}+\frac{463588}{433125}\frac{(-x_n+x)^7}{h^6}-\frac{38174}{433125}\frac{(-x_n+x)^8}{h^7}$$

$$\beta_{\frac{1}{2}}(x)=-\frac{72474}{9625}\frac{\left(-x_n+x\right)^2}{h}+\frac{224106}{6875}\frac{\left(-x_n+x\right)^3}{h^2}-\frac{1059457}{20625}\frac{\left(-x_n+x\right)^4}{h^3}+\frac{826318}{20625}\frac{\left(-x_n+x\right)^5}{h^4}-\frac{13736}{825}\frac{\left(-x_n+x\right)^6}{h^5}+\frac{509896}{144375}\frac{\left(-x_n+x\right)^7}{h^6}-\frac{43408}{144375}\frac{\left(-x_n+x\right)^8}{h^7};$$

$$\beta_1(x)=\frac{41469}{19250}\frac{\left(-x_n+x\right)^2}{h}-\frac{165511}{13750}\frac{\left(-x_n+x\right)^3}{h^2}+\frac{4016593}{165000}\frac{\left(-x_n+x\right)^4}{h^3}-\frac{462079}{20625}\frac{\left(-x_n+x\right)^5}{h^4}+\frac{2861}{275}\frac{\left(-x_n+x\right)^6}{h^5}-\frac{344188}{144375}\frac{\left(-x_n+x\right)^7}{h^6}+\frac{30974}{144375}\frac{\left(-x_n+x\right)^8}{h^7}$$

$$\beta_{\frac{3}{2}}(x)=-\frac{32524}{28875}\frac{\left(-x_n+x\right)^2}{h}+\frac{1233004}{185625}\frac{\left(-x_n+x\right)^3}{h^2}-\frac{896482}{61875}\frac{\left(-x_n+x\right)^4}{h^3}+\frac{909668}{61875}\frac{\left(-x_n+x\right)^5}{h^4}-\frac{55208}{7425}\frac{\left(-x_n+x\right)^6}{h^5}+\frac{790096}{433125}\frac{\left(-x_n+x\right)^7}{h^6}-\frac{75008}{433125}\frac{\left(-x_n+x\right)^8}{h^7}$$

$$\beta_2(x)=\frac{17313}{38500}\frac{\left(-x_n+x\right)^2}{h}-\frac{18693}{6875}\frac{\left(-x_n+x\right)^3}{h^2}+\frac{1015343}{165000}\frac{\left(-x_n+x\right)^4}{h^3}-\frac{135829}{20625}\frac{\left(-x_n+x\right)^5}{h^4}+\frac{2933}{825}\frac{\left(-x_n+x\right)^6}{h^5}-\frac{134188}{144375}\frac{\left(-x_n+x\right)^7}{h^6}+\frac{13474}{144375}\frac{\left(-x_n+x\right)^8}{h^7}$$

$$\beta_{\frac{5}{2}}(x)=-\frac{1074}{9625}\frac{\left(-x_n+x\right)^2}{h}+\frac{4706}{6875}\frac{\left(-x_n+x\right)^3}{h^2}-\frac{32657}{20625}\frac{\left(-x_n+x\right)^4}{h^3}+\frac{36118}{20625}\frac{\left(-x_n+x\right)^5}{h^4}-\frac{272}{275}\frac{\left(-x_n+x\right)^6}{h^5}+\frac{39496}{144375}\frac{\left(-x_n+x\right)^7}{h^6}-\frac{4208}{144375}\frac{\left(-x_n+x\right)^8}{h^7}$$

$$\beta_3(x)=\frac{731}{57750}\frac{\left(-x_n+x\right)^2}{h}-\frac{29101}{371250}\frac{\left(-x_n+x\right)^3}{h^2}+\frac{91057}{495000}\frac{\left(-x_n+x\right)^4}{h^3}-\frac{12871}{61875}\frac{\left(-x_n+x\right)^5}{h^4}+\frac{901}{7425}\frac{\left(-x_n+x\right)^6}{h^5}-\frac{15212}{433125}\frac{\left(-x_n+x\right)^7}{h^6}+\frac{1726}{433125}\frac{\left(-x_n+x\right)^8}{h^7}$$

Now, evaluating (9) at $x_{n+v}\left(v=1,\frac{3}{2},2,\frac{5}{2},\frac{11}{4},3\right)$ and its first derivative at $x_{n+u}\left(u=\frac{11}{4}\right)$, yields seven discrete schemes which constitute the block form

$$y_{n+\frac{1}{2}}=y_n+\frac{4553}{30240}hf_n-\frac{200}{567}hf_{n+\frac{11}{4}}+\frac{107293}{181440}hf_{n+\frac{1}{2}}-\frac{3727}{6720}hf_{n+1}+\frac{19001}{30240}hf_{n+\frac{3}{2}}-\frac{4271}{7560}hf_{n+2}+\frac{3559}{6720}hf_{n+\frac{5}{2}}+\frac{4381}{60480}hf_{n+3}$$

$$y_{n+1}=y_n+\frac{4027}{27720}hf_n-\frac{8192}{31185}hf_{n+\frac{11}{4}}+\frac{2227}{2835}hf_{n+\frac{1}{2}}-\frac{103}{840}hf_{n+1}+\frac{26}{63}hf_{n+\frac{3}{2}}-\frac{3047}{7560}hf_{n+2}+\frac{41}{105}hf_{n+\frac{5}{2}}+\frac{137}{2520}hf_{n+3}$$

$$y_{n+\frac{3}{2}}=y_n+\frac{361}{2464}hf_n-\frac{24}{77}hf_{n+\frac{11}{4}}+\frac{345}{448}hf_{n+\frac{1}{2}}+\frac{243}{2240}hf_{n+1}+\frac{859}{1120}hf_{n+\frac{3}{2}}-\frac{18}{35}hf_{n+2}+\frac{1053}{2240}hf_{n+\frac{5}{2}}+\frac{143}{2240}hf_{n+3}$$

$$y_{n+2}=y_n+\frac{1517}{10395}hf_n-\frac{8192}{31185}hf_{n+\frac{11}{4}}+\frac{440}{567}hf_{n+\frac{1}{2}}+\frac{8}{105}hf_{n+1}+\frac{976}{945}hf_{n+\frac{3}{2}}-\frac{193}{945}hf_{n+2}+\frac{8}{21}hf_{n+\frac{5}{2}}+\frac{52}{945}hf_{n+3}$$

$$y_{n+\frac{5}{2}}=y_n+\frac{295}{2016}hf_n-\frac{200}{567}hf_{n+\frac{11}{4}}+\frac{28025}{36288}hf_{n+\frac{1}{2}}+\frac{125}{1344}hf_{n+1}+\frac{1975}{2016}hf_{n+\frac{3}{2}}+\frac{125}{1512}hf_{n+2}+\frac{955}{1344}hf_{n+\frac{5}{2}}+\frac{275}{4032}hf_{n+3}$$

$$y_{n+\frac{11}{4}}=y_n+\frac{905773}{6193152}hf_n-\frac{4213}{18144}hf_{n+\frac{11}{4}}+\frac{7180745}{9289728}hf_{n+\frac{1}{2}}+\frac{310123}{3440640}hf_{n+1}+\frac{7643933}{7741440}hf_{n+\frac{3}{2}}+\frac{2019127}{30965760}hf_{n+2}+\frac{1476079}{1720320}hf_{n+\frac{5}{2}}+\frac{1925957}{30965760}hf_{n+3}$$

$$y_{n+3}=y_n+\frac{41}{280}hf_n+\frac{27}{35}hf_{n+\frac{1}{2}}+\frac{27}{280}hf_{n+1}+\frac{34}{35}hf_{n+\frac{3}{2}}+\frac{27}{280}hf_{n+2}+\frac{27}{35}hf_{n+\frac{5}{2}}+\frac{41}{280}hf_{n+3}$$

(9)

Rewriting equation (9) in block form yields

$$\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+\frac{3}{2}}\\ y_{n+2}\\ y_{n+\frac{5}{2}}\\ y_{n+\frac{11}{4}}\\ y_{n+3} \end{bmatrix}=\begin{bmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} y_{n-\frac{5}{2}}\\ y_{n-2}\\ y_{n-\frac{3}{2}}\\ y_{n-1}\\ y_{n-\frac{1}{2}}\\ y_{n-\frac{1}{4}}\\ y_{n} \end{bmatrix}+h\begin{bmatrix} 0&0&0&0&0&0&\frac{4553}{30240}\\ 0&0&0&0&0&0&\frac{4027}{27720}\\ 0&0&0&0&0&0&\frac{361}{2464}\\ 0&0&0&0&0&0&\frac{1517}{10395}\\ 0&0&0&0&0&0&\frac{295}{2016}\\ 0&0&0&0&0&0&\frac{905773}{6193152}\\ 0&0&0&0&0&0&\frac{41}{280} \end{bmatrix}\begin{bmatrix} f_{n-\frac{5}{2}}\\ f_{n-2}\\ f_{n-\frac{3}{2}}\\ f_{n-1}\\ f_{n-\frac{1}{2}}\\ f_{n-\frac{1}{4}}\\ f_{n} \end{bmatrix}$$

$$+h\begin{bmatrix} \frac{107293}{181440} & -\frac{3727}{6720} & \frac{19001}{30240} & -\frac{4271}{7560} & \frac{3559}{6720} & -\frac{200}{567} & \frac{4381}{60480}\\ \frac{2227}{2835} & -\frac{103}{840} & \frac{26}{63} & -\frac{3047}{7560} & \frac{41}{105} & -\frac{8192}{31185} & \frac{137}{2520}\\ \frac{345}{448} & \frac{243}{2240} & \frac{859}{1120} & -\frac{18}{35} & \frac{1053}{2240} & -\frac{24}{77} & \frac{143}{2240}\\ \frac{440}{567} & \frac{8}{105} & \frac{976}{945} & -\frac{193}{945} & \frac{8}{21} & -\frac{8192}{31185} & \frac{52}{945}\\ \frac{28025}{36288} & \frac{125}{1344} & \frac{1975}{2016} & \frac{125}{1512} & \frac{955}{1344} & -\frac{200}{567} & \frac{275}{4032}\\ \frac{7180745}{9289728} & \frac{310123}{3440640} & \frac{7643933}{7741440} & \frac{2019127}{30965760} & \frac{1476079}{170320} & -\frac{4213}{18144} & \frac{1925957}{3096760}\\ \frac{27}{35} & \frac{27}{280} & \frac{34}{35} & \frac{27}{280} & \frac{27}{35} & 0 & \frac{41}{280} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+\frac{3}{2}}\\ f_{n+2}\\ f_{n+\frac{5}{2}}\\ f_{n+\frac{11}{4}}\\ f_{n+3} \end{bmatrix}$$

4. Analysis of the Method

4.1 Order and error constant of the Method

We associate the Linear operator L with the scheme and defined by:

$$L[y(x):h]=\sum_{j=0}^{k}\left[\alpha_j y(x_n+jh)-h\beta_j y'(x_n+jh)\right] \quad (11)$$

where α_0 and β_0 are both non-zero and $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,b]$. Expanding $y(x_n+jh)$ and $y'(x_n+jh)$ in Taylor series about x_n and collecting like terms in h and y gives:

$$L[y(x):h]=c_o y(x)+c_1 hy'(x)+c_2 h^2 y''(x)+\ldots+c_p h^p y^{(p)}(x)+\ldots \quad (12)$$

Accordingly we say that the method has order p if,

$$c_0=c_1=\ldots=c_p=0,$$

$$c_{p+1}\neq 0$$

Then, c_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{(p+1)}(x_n)$ is the principal local truncation error at the point x_n

In this paper, since $c_0=c_1=c_2=\ldots=c_8=0$ and $c_9=c_{p+1}\neq 0$ which implies that the schemes are of uniform order 8 and the error constant are

$$C_9=\left[-\frac{209749}{14863564800},-\frac{653}{58060800},-\frac{2277}{183500800},-\frac{169}{14515200},-\frac{7325}{594542592},-\frac{46301497}{3805072588800},-\frac{9}{716800}\right]^T$$

4.2 Consistency

The hybrid block method is said to be consistent if it has an order more than or equal to one. Therefore, our method is consistent, since it is of eight (8). [9].

4.3 Zero Stability

The hybrid method, with four off grid collocation points expressed in the form

$$\rho(z)=\det\left[\sum_{j=0}^{k}A^{(i)}z^{k-i}\right]=0$$ gives

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{bmatrix}\right|=$$

$$\rho(z)=z^5(z-1)^2=0,\quad z=0,0,0,0,\ 0,1,1$$

Hence, the method is zero-stable.

4.4 Convergence

Theorem (Henrici, 1962).

Zero stability and consistency are sufficient conditions for a linear multistep method to be convergent. Since the method is consistent and zero-stable, by [8] the hybrid method is convergent.

4.5 Region of Absolute Stability

To plot the region of absolute stability of the method, the methods were formulated ([6]) and stability polynomial for method is computed as

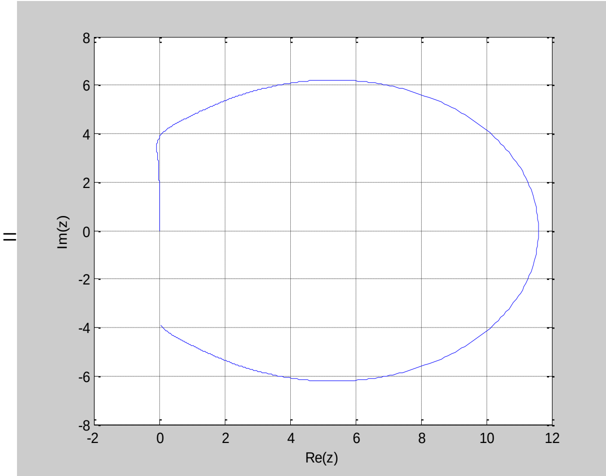
$$\left(-\frac{11}{14336}w^7-\frac{1}{14336}w^6\right)h^7+\left(\frac{579}{71680}w^7-\frac{89}{71680}w^6\right)h^6$$

$$+\left(-\frac{191}{3840}w^7-\frac{23}{1920}w^6\right)h^5+\left(\frac{1619}{7680}w^7-\frac{569}{7680}w^6\right)h^4$$

$$+\left(-\frac{485}{768}w^7-\frac{235}{768}w^6\right)h^3+\left(\frac{83}{64}w^7-\frac{53}{64}w^6\right)h^2$$

$$+\left(-\frac{53}{32}w^7-\frac{43}{32}w^6\right)h+w^7-w^6$$

Thus, the stability region of the method is plotted and shown below



4.5.2 Numerical Examples

In order to study the efficiency of the developed method, we present some numerical experiments with the following four problems.

Problem 4.1

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derived from the fact that they involve coupled equations relating the number of susceptible people $S(t)$, number of people infected $I(t)$ and the number of people who have recovered $R(t)$. This is a good and simple model for many infectious diseases including measles, mumps and rubella. The SIR model is described by the three coupled equations

$$\frac{dS}{dt}=\mu(1-S)-\beta IS$$

$$\frac{dI}{dt}=\mu I-\gamma I+\beta IS$$

$$\frac{dR}{dt}=-\mu R+\gamma I$$

where μ,γ *and* β are positive parameters. Defined y to be

$$y=S+I+R$$

Adding these equations gives

$$y'=\mu(1-y)$$

Taking $\mu=0.5$ and attaching an initial condition $y(0)=0.5$ (for a particular closed population), we obtain

$$y'(t)=0.5(1-y),\quad y(0)=0.5$$

Whose analytic solution is

$$y(t)=1-0.5e^{-0.5t}$$

Source: [11]

Problem 4.2

Highly stiff problem

$$y'+4y=20,\ \ y(0)=-4,\ \ y(0)=2\ ,\ \ 0\le x\le 1\ ,\ \ h=\frac{1}{100}$$

With the exact solution

$$y(x)=5-3e^{-4x}$$

Source: [3].

Problem 4.3

Highly stiff problem of ordinary differential equation which was solved by [11]

$y'=-\alpha(y_0-F(x))+F'(x)$ with initial $y(x_0)=y_0$, $x\in[0,0.1]$, $h=0.01$

With the exact solution

$y(x)=y_0+\exp(-\alpha x)-1$. where $\alpha=10$, $F(x)=0$, $x_0=0$ *and* $y_0=1$

Source: [11].

Table 4.1 Comparing the absolute errors in the new methods with error from [11] for problem 4.1

X	Error in new method	Error in [11]
0.1	9.6400e-18	1.714000e-14
0.2	9.5000e-18	3.260000e-14
0.3	9.8400e-18	4.653000e-14
0.4	1.7660e-17	5.902000e-14
0.5	1.7070e-17	7.018000e-14
0.6	1.6950e-17	8.011000e-14
0.7	2.3270e-17	8.891000e-14
0.8	2.23800e-17	9.665000e-14
0.9	2.18700e-17	1.034200e-13
1.0	2.69700e-17	1.093100e-13

Table 4.2 Comparing the absolute errors in the new methods with error from [3] for problem 4.2

X	Error in our new method	Error in [3]
0.1	7.7000(-18)	2.0000(-14)
0.2	7.900(-18)	3.0000(-14)
0.3	8.300(-18)	8.0000(-14)
0.4	1.500(-17)	1.2000(-12)
0.5	1.4700(-17)	1.0000(-12)
0.6	1.4800(-17)	2.6970(-11)
0.7	2.0300(-17)	5.5800(-12)
0.8	1.9900(-17)	6.2140(-10)
0.9	1.9700(-17)	2.7048(-10)
1.0	2.4400(-17)	1.4571(-8)

Table 4.3 Comparing the absolute errors in the new methods with error from [12] for problem 4.3

X	Error in our new method	Error in [11]
0.1	8.6877e-15	1.079154e-12
0.2	8.1407e-15	1.952918e-12
0.3	8.0997e-15	2.650610e-12
0.4	1.37648e-14	3.197828e-12
0.5	1.26622e-14	3.616893e-12
0.6	1.20007e-14	3.927240e-12
0.7	1.56266e-14	4.145766e-12
0.8	1.42931e-14	4.287136e-12
0.9	1.33356e-14	4.364056e-12
1.0	1.55987e-14	4.387513e-12

5. Conclusion

The efficiency of the new method has been demonstrated on some standard numerical examples. Details of the numerical results are displayed in Table 4.1, 4.2 and 4.3. The inclusion of off grid points allowed the adoption of linear multistep procedure which circumvents the ‘zero-stability barrier, up graded the order of accuracy of the methods and to obtain very low error constants. The order, error constant, consistency, zero stability and region of absolute stability of the methods including its hybrids form was determined.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

References

[1] Abdullahi, A., Chollom, J. P., &Yahaya, Y. A. (2014). A family of two-step block generalized Adams methods for the solution of non-stiff IVP in ODEs. *Journal of Mathematical Association of Nigeria (ABACUS),* ***41****(2A),* 67-77.

[2] Adeniyi, R. B. & Alabi, M. O. (2007). Continuous formulation of a class accurate implicit linear multi-step methods with Chebyshev basis function in a collocation technique. *Journal of Mathematical Association of Nigeria (ABACUS),* ***34****(2A),* 58-77*.*

[3] Badmus, A. M., Yahaya, Y. A & Subair, A. O. (2014). Implicit Stoner Cowbell k-step hybrid block methods for solution of first ODEs.*International Journal of Applied Mathematical Resaerch* **3**(4), 464-472.

[4] Lambert, J.D. 1991. *Numerical methods for ordinary differential systems.* New York: John Wiley.

[5] Hairer E, Wanner G. Solving ordinary differential equations II. New York: Springer; 1996.

[6] Kayode S. J, Awoyemi D. O. A Multi derivative collocation method for fifth order ordinary differential equation. J Math Stat 2010;6(1):60–3.

[7] Badmus, A.M. &Mshelia, D.W. (2012). Uniform Order zero stable k-step Block methods for initial problems of ODEs. *Journal of Nigerian Association of Mathematical Physics* ***20****,* 65-74.

[8] Chu, M. T. and Hamilton, H. (1987). Parallel Solution of Ordinary Differential Equations by Multiblock Methods*. SIAM Journal of Scientific and Statistical Computation,* ***8****,* 342-553.

[9] Ehigie, J. O., Okunuga, S. A., & Sololuwa, A. B. (2011). A class of 2-step continuous hybrid implicit linear multi-step method for IVPs. *Journal of Nigerian Mathematical Society,* ***30****, 145-161.*

[10] Lambert J. D. (1973a). *Computational Methods in Ordinary Differential Equations.* New York: John Wiley. 973pp*.*

[11] Mshelia, D. W., Badmus, A.M., Yakubu, D.G. & Manjak N. H. (2016). A fifth stage Runge-kutta method for the solution of ODEs. *The Pacific Journal of Science and Technology,* ***17****(2), 87-100.*

[12] Rufai, M. A., Duromola, M. K., & Ganiyu, M. K., (2016). Derivation of One-Sixth Hybrid Block Method for Solving General First Order Ordinary Differential Equations. *Journal of Mathematics (IOSR-JM)*. *12 (5):20-2*

[13] Ukpebor, L.A & Adoghe, L.O. (2019) Continuous fourth derivative block method for solving stiff systems first order ordinary differential equations. *Abacus (Mathematics Science Series) Vol. 44, No 1*.

[14] Barde, W., Timothy S. & Solomon A. U. (2021). An implicit two-step hybrid block method based on chebyshev polynomial for solving first order initial value problems in ordinary differential equations. *International Journal of science for Global Sustainability. IJSGS FUGUSAU VOL. 7(1)* 80-89.