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Efficient Numerical Approximation Methods for Solving High-Order Integro-Differential Equations

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In this work, we developed an approximate solution of high-order integro-differential equations (IDEs.) via the third kind of Chebyshev and Hermite Polynomials as basis functions using standard collocation method for Volterra and Fredholm integro-differential equations (IDEs). An assumed approximate solution is substituted into the given problem considered. After simplifications, the like terms of the unknown constants to be determined were collected and collocate at point $z = z_j$, where z_j are the zeros of the Chebyshev and Hermite polynomials. The resulting equations are then put into matrix form which is then solved via Maple 18 software to obtain the unknown constants $c_i(i \geq 0)$**.** These are substituted back to obtain our approximate solution. Comparison is made with the two basis functions aforementioned in terms of errors obtained. Given numerical examples shows that the methods are efficient, reliable and less computational for the numerical solution of the integro-differential equation.

Keywords: Approximate Solution; Hermite Polynomial; Integro-Differential Equations; Standard Collocation Method; Third kind of Chebyshev Polynomial.

1. Introduction

The term "Integro-differential equations" plays an important role in Mathematics right from the onset. It has been of great theoretical and practical importance. It is noticed recently that a score of problems in various fields such as theoretical physics, engineering and other disciplines do lead to the linear and nonlinear integral equation or integro-differential equation. To get these problems solved, some numerical approaches have been proposed by the researchers.
Among these researchers are ([1], [6], & [11]), just to mention but few. Also, numerous polynomials like another new algorithm for calculating Adomian polynomials was established by [12] using parametrization approach, Chebyshev and Legendre by [10], [9] employed trial solution constructed as Chebyshev form of fourier cosine series , Hermite polynomials [5] presents an approximate solution of non-homogeneous multi-pantograph based on Hermite polynomials, [4] also proposes a techniques ton approximate the solutions of non-linear initial value problem with Hermite interpolation polynomial,Variational Iteration Decomposition Method (VIDM) [8] and Homotopy Perturbation Method (HPM) [2] and others have been used to derive solutions of some classes of integro-differential equations.

The great work did by the researchers aforementioned motivated us and eventually led to the proposal of a numerical approximation method that is efficient, accurate and less computational to obtain an approximate solution of high order linear Volterra and Fredholm integro-differential equations of the form

$$P_{01}\,\zeta^{(m)}(z) + P_{11}\,\zeta^{(m-1)}(z) + P_{21}\,\zeta^{(m-2)}(z) + \cdots + P_{m1}\,\zeta(z)\lambda\int_{\vartheta(z)}^{\sigma(z)} K(z,t)\zeta(t)dt = f(z) \qquad (1)$$

Subject to the conditions

$$\zeta(0) = C_{0,}\ \zeta'(0) = C_{1,}\ \ \zeta''(0) = C_{2,} + \cdots + \zeta^{(m-1)}(0)(z) = C_{m-1} \qquad (2)$$

where $\zeta^{(m)}(z)$ is the kth derivative of $\zeta(z)$. The kernel $K(z,s)$ and $f(z)$ are given real-valued function, λ is a complex-valued parameter. P'^{s} are functions of the independent variable.

2. Basic Definitions

2.1 Integro-Differential Equation

An integro-differential Equations (IDEs.) is an equation in which the unknown function $\zeta(z)$ appears under the integral sign and contains an ordinary derivative $\zeta^{(k)}$ as well. A standard integro-differential equation is of the form:

$$\zeta^{(k)}(z) = f(z) + \lambda\int_{\vartheta(z)}^{\sigma(z)} K(z,s)\zeta(s)ds \qquad (3)$$

where $i(z)$ and $h(z)$ are limits of integration which may be constants, variables or combined. λ is a constant parameter, $f(z)$ is a given function and $K(z,s)$ is a known function of two variables $z\ and\ s,$ called the kernel.

We have Fredholm integro-differential equation if the limits of integration are constants and it is called Volterra integro-differential equation if the limit $\zeta(z)$ is replaced with a variable of integration z.

2.2 Collocation Method

Collocation method is a method involving the determination of an approximate solution in a suitable set of functions sometimes called trial solution and also is a method of evaluating a given differential equation at some points in order to nullify the values of an ordinary differential equation at those points.

2.3 Exact Solution

A solution is called exact solution if it can be expressed in a closed form, such as a polynomial, exponential function, trigonometric function or the combination of two or more of these elementary functions.

$$U_m(z) = \cos\frac{\left(m+\frac{1}{2}\right)\vartheta}{\cos\left(\frac{\vartheta}{2}\right)}, \tag{5}$$

where $z = \cos\vartheta$ Source [7]

This class of Chebyshev Polynomials satisfied the following recurrence relation is given as

$$U_0(z) = 1,\ \ U_1(z) = 2z - 1, U_m(z) = 2zU_{m-1}(z) - U_{m-2}(z),\ m = 2,3,\cdots \tag{6}$$

The third kind of Chebyshev Polynomial in $[\alpha,\beta]$ of degree, m is denoted by $V_m^*(z)$ and is defined by

$$U_m^*(z) = \cos\frac{\left(m+\frac{1}{2}\right)\vartheta}{\cos\left(\frac{\vartheta}{2}\right)}, \cos\vartheta = \frac{2z-(\alpha+\beta)}{\beta-\alpha},$$

$$\vartheta\ \epsilon\ [0,\pi] \tag{7}$$

All the results of Chebyshev polynomials of the third kind can be easily transformed to give the corresponding results for their shifted ones. The orthogonality relations of $U_m^*(z)$ on $[\alpha,\beta]$ with respect to the weight functions $\sqrt{\frac{z-\alpha}{\beta-z}}$ is given by

$$\int_\alpha^\beta \sqrt{\frac{z-\alpha}{\beta-z}} = \begin{cases}(\beta-\alpha)\frac{\pi}{2}, m = n \\ 0,\ \ m \neq n\end{cases} \tag{8}$$

Source [3]

2.4 Approximate Solution

An approximate solution is an inexact representation of the exact solution that is still close enough to be used instead of exact and it is denoted by $\zeta_M(z)$, where M is the degree of the approximant used in the calculation. Methods of the approximate solution are usually adopted because complete information needed to arrive at the exact solution may not be given. In this work, the approximate solution used is given as

$$\zeta_m(z) = \sum_{i=0}^{M} c_i\varphi_i(z) \tag{4}$$

where $c_{i,}\ i = 0,1,2,\ldots,M$ are unknown constants to be determined, $\varphi_i(z)(i \geq 0)$ is the basis functions which is either third kind of Chebyshev or Hermite Polynomials and M is the degree of approximating Polynomials.

2.5 Third kind Chebyshev polynomials

The third kind of Chebyshev Polynomial in [-1, 1] of degree m is denoted by $U_m(z)$ and defined by

2.6 Hermite polynomials

The Hermite polynomials are defined as

$$H_m(z) = \sum_{j=0}^{M}(-1)^j\frac{m!}{j!\,(m-2)!}\ z^{(m-2j)},$$

$$-1 \leq z \leq 1. \tag{9}$$

where $M = \frac{m}{2}$, for m is even and $M = \frac{(m-2)}{2}$ for m is odd.

and the Hermite Polynomials are

$H_0(z) = 1,\ H_1(z) = 2z$

The recurrence relation is

$$H_m(z) = (-1)^m e^{z^2}\frac{d^m}{dz^m}\left(e^{-z^2}\right) \tag{10}$$

is known as Rodrigue's formula, and offer another method of expressing the Hermite Polynomials.

Since

$$H_{m+1}(z) = 2zH_m(z) - 2mH_{m-1}(z),\ for\ \ m \geq 1 \tag{11}$$

we have

$H_2(z) = 4z^2 - 2,\ H_3(z) = 8z^3 - 2z,$

$H_4(z) = 16z^4 - 48z^2 + 12,$

$H_5(z) = 32z^5 - 160z^2 + 120z$, and so on.

Source [5]

3. Problem Considered and Methodology

Here, we applied standard collocation method to solve equation (1) using the following basis functions:

i. Third kind of Chebyshev Polynomials, and
ii. Hermite Polynomials.

3.1 Standard Collocation Method by Third kind Chebyshev Polynomials

To solve the general problem given in equation (1) subject to the conditions given in equation (2) using the standard Collocation Method, we assumed an approximate solution of the form:

$$\zeta_m(z) = \sum_{i=0}^{M} c_i U_i^*(z) \qquad (12)$$

where $c_{i,}\ i = 0,1,2,\ldots,M$ are unknown constants and $U_i^*(z)(i \geq 0)$ are Chebyshev polynomials of the third kind defined in equation (5) to (7). M is the degree of approximating Polynomials, where in most cases the better approximate solution (i.e. closer to the exact solution) is produced by larger M, and c_i is the specialized coordinate called Degree of freedom.

Thus, differentiating equation (12) with respect to z mth-times, we obtain

$$\left.\begin{array}{l} \zeta'_m(z) = \sum_{i=0}^{M} c_i U_i^{*'}(z) \\ \zeta''_m(z) = \sum_{i=0}^{M} c_i U_i^{*''}(z) \\ \quad\vdots \\ \zeta^{(m)}(z) = \sum_{i=0}^{M} c_i c_i U_i^{*(m)} \end{array}\right\} \qquad (13)$$

Hence, substituting equation (12) & (13) into equation (1), we obtain

$$P_{01}\sum_{i=0}^{M} c_i U_i^{*(m)}(z) + P_{11}\sum_{i=0}^{M} U_i^{*(m-1)}(z)$$
$$+P_{21}\sum_{i=0}^{M} c_i U_i^{*(m-2)} + \cdots + P_{m1}\sum_{i=0}^{M} U_i^{*(m)}(z)$$
$$+\lambda\int_{\vartheta(z)}^{\sigma(z)} K(z,t)\left(\sum_{i=0}^{M} c_i U_i^{*(m)}(t)\right)dt = f(z) \quad (14)$$

Evaluating the integral part of the equation (14) to obtain

$$P_{01}\sum_{i=0}^{M} c_i U_i^{*(m)}(z) + P_{11}\sum_{i=0}^{M} c_i U_i^{*(m-1)}(z)$$
$$+P_{21}\sum_{i=0}^{M} c_i U_i^{*(m-2)}(z) + \cdots + P_{m1}\sum_{i=0}^{M} c_i U_i^{*}(z)$$
$$+\lambda G(z) = f(z) \qquad (15)$$

where $G(z) = \int_{\vartheta(z)}^{\sigma(z)} K(z,t)(\sum_{i=0}^{M} c_i U_i^*(z))dz$

Thus, collocating equation (15) at the point

$z = z_j$, we obtain

$$P_{01}\sum_{i=0}^{M} c_i U_i^{*(m)}(z_j) + P_{11}\sum_{i=0}^{M} c_i U_i^{*(m-1)}(z_j)$$
$$+P_{21}\sum_{i=0}^{M} c_i U_i^{*(m-2)}(z_j) + \cdots + P_{m1}\sum_{i=0}^{M} c_i U_i^{*}(z_j)$$
$$+\lambda G(z_j) = f(z_j) \qquad (16)$$

and

$$z_j = \alpha + \frac{(\beta-\alpha)j}{M+1};$$
$$j = 1,2,\ldots,M \qquad (17)$$

Thus, equation (16) is then put into matrix form as

$$P\underline{z} = \underline{q} \qquad (18)$$

where

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & & p_{1,m} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2,m} \\ p_{31} & p_{32} & p_{33} & & p_{3,m} \\ & \vdots & & \ddots & \vdots \\ p_{m,1} & p_{m,2} & p_{m,3} & \cdots & p_{m,m} \end{pmatrix} \qquad (19)$$

$$\underline{z} = (z_1, z_2, z_3, \cdots, z_m)^T \qquad (20)$$

$$\underline{q} = (f(q_1), f(q_2), f(q_3)\ldots, f(q_m))^T \qquad (21)$$

Thus, equation (16) gives rise to $(M+1)$ system of linear algebraic equations in $(M+1)$ unknown constants and n extra equations are obtained using the conditions given in equation (2). Altogether, we now have $(M+n+1)$ system of linear algebraic equations. These equations are then solved via Maple 18 software to obtain $(M+1)$ unknown constants $c_i\ (i \geq 0)$ which are then substituted back into the approximate solution given by equation (6).

3.2 Standard Collocation Method by Hermite Polynomial Basis

To solve the general problem given in equation (1) subject to the conditions given in equation (2) using the standard Collocation Method, we assumed an approximate solution of the form

$$\zeta_m(z) = \sum_{i=0}^{M} c_i H_i(z) \qquad (22)$$

where $c_{i,}\ i = 0, 1, 2, \dots, M$ are unknown constants and $H_i(z)(i \geq 0)$ are Hermite polynomials defined in equations (8) and (9). M is the degree of approximating Polynomials, where in most cases the better approximate solution (i.e. closer to the exact solution) is produced by larger M.

Thus, differentiating equation (22) with respect to z mth-times, we obtain

$$\left.\begin{array}{l} \zeta'_m(z) = \sum_{i=0}^{M} c_i H'_i(z) \\ \zeta''_m(z) = \sum_{i=0}^{M} c_i H''_i(z) \\ \quad\vdots \\ \zeta^{(m)}(z) = \sum_{i=0}^{M} c_i H_i^{(m)} \end{array}\right\} \qquad (23)$$

Hence, substituting equations (22) and (23) into equation (1), we obtain

$$P_{01}\sum_{i=0}^{M} c_i H_i^{(m)}(z) + P_{11}\sum_{i=0}^{M} c_i H_i^{(m-1)}(z)$$
$$+P_{21}\sum_{i=0}^{M} c_i H_i^{(m-2)}(z) + \cdots + P_{m1}\sum_{i=0}^{M} c_i H_i(z)$$
$$+\lambda\int_{h(z)}^{\sigma(z)} K(z,t)\left(\sum_{i=0}^{M} c_i H_i(t)\right)dt = f(z) \qquad (24)$$

Evaluating the integral part of the equation (24)

$$P_{01}\sum_{i=0}^{M} c_i H_i^{(m)}(z) + P_{11}\sum_{i=0}^{M} c_i H_i^{(m-1)}(z)$$
$$+P_{21}\sum_{i=0}^{M} c_i H_i^{(m-2)}(z) + \cdots + P_{m1}\sum_{i=0}^{M} c_i H_i(z)$$
$$+\lambda G(z) = f(z) \qquad (25)$$

where $G(z) = \int_{\vartheta(z)}^{\sigma(z)} K(z,t)(\sum_{i=0}^{M} c_i H_i(t))dt$

Thus, collocating equation (25) at the point

$z = z_j$, we obtain

$$P_{01}\sum_{i=0}^{M} c_i H_i^{(m)}(z_j) + P_{11}\sum_{i=0}^{M} c_i H_i^{(m-1)}(z_j)$$
$$+P_{21}\sum_{i=0}^{M} c_i H_i^{(m-2)}(z_j) + \cdots + P_{m1}\sum_{i=0}^{M} c_i H_i(z_j)$$
$$+\lambda G(z_j) = f(z_j) \qquad (26)$$

and

$$z_j = \alpha + \frac{(\beta-\alpha)j}{M+1};$$
$$j = 1, 2, \dots, M \qquad (27)$$

Thus, equation (27) is then put into matrix form as

$$Q\underline{z} = \underline{d} \qquad (28)$$

where

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} & & q_{1,m} \\ q_{21} & q_{22} & q_{23} & \cdots & q_{2,m} \\ q_{31} & q_{32} & q_{33} & & q_{3,m} \\ & \vdots & & \ddots & \vdots \\ q_{m,1} & q_{m,2} & q_{m,3} & \cdots & q_{m,m} \end{pmatrix} \qquad (29)$$

$$\underline{z} = (z_1, z_2, z_3, \cdots, z_m)^T \qquad (30)$$

$$\underline{d} = (f(d_1), f(d_2), f(d_3) \dots, f(d_m))^T \qquad (31)$$

Thus, equation (26) gives rise to $(M+1)$ system of linear algebraic equations in $(M+1)$ unknown constants and n extra equations are obtained using the conditions given in equation (2). Altogether, we now have $(M+n+1)$ system of linear algebraic equations. These equations are then solved via Maple 18 software to obtain $(M+1)$ unknown constants $c_i\ (i \geq 0)$ which are then substituted back into the approximate solution given by equation (22).

4. Numerical Experiments

In this section, we have demonstrated the standard collocation approximation method on high-order integro-differential equations using Chebyshev of the third kind and Hermite Polynomial as the basis functions, and our result are compared with each other on three problems to test for the effectiveness and efficiency of our methods via the Maple 18 software.

4.1 Numerical Example 1

Consider the second-order linear Fredholm integro-differential equation

$$\zeta''(z) = e^z - z + z\int_0^1 t\zeta(t)dt \qquad (32)$$

with initial conditions

$\zeta(0) = 1, \qquad \zeta'^{(0)} = 1$ (33)

The exact solution is given as

$\zeta(z) = e^z$ (34)

Source [2]

4.2 Numerical Example 2

Consider the third-order linear Volterra integro-differential equation

$$\zeta'''(z) = -1 + z - \int_0^z (z-t)\zeta(t)dt \qquad (35)$$

with initial conditions

$\zeta(0) = 1, \quad \zeta'(0) = -1, \quad \zeta''(0) = 1$ (36)

The exact solution is given as

$\zeta(z) = e^{-z}$ (37)

Source [2]

4.3 Numerical Example 3

Consider the fouth-order linear Fredholm integro-differential equation

$$\zeta^{(iv)}(z) = 3e^z + e^{2z} - \int_0^1 e^{2(z-t)}\zeta(t)dt \qquad (38)$$

with initial conditions

$\zeta(0) = 0, \quad \zeta'(0) = 1, \quad \zeta''(0) = 2,$
$\zeta'''(0) = 3$ (39)

The exact solution is given as

$\zeta(z) = ze^z$ (40)

Source [2]

Remark: We defined absolute error as:

$Error = |\zeta(z) - \zeta_M(z)|;$

where, $\zeta(z)$ is the exact solution and $\zeta_M(z)$ is our approximate solution obtained for the various value of *M*.

T**able 1.** Comparison of Results and Errors between Chebyshev and Hermite for example 1

Z	**Exact Result**	**Chebyshev Computed result for case M = 5**	**Hermite Computed result for case M = 5**	**Error In Chebyshev.**	**Error in Hermite.**
0.0	1.0000000000000	1.0000000000000000	0.999999999899999	0.00000e+00	1.00000e-10
0.2	1.2214027581602	1.2213929138390528	1.221392913813536	9.84432e-06	9.84500e-06
0.4	1.4918246976413	1.4918027202345216	1.491802720054752	2.19774e-05	2.19790e-05
0.6	1.8221188003905	1.8220841490924544	1.822084148146848	3.46513e-05	3.46520e-05
0.8	2.2255409284925	2.2254924488172032	2.225492446228064	4.84797e-05	4.84820e-05
1.0	2.7182818284590	2.7181704811200000	2.718170476020000	1.11347e-04	1.11352e-04

Table 2. Comparison of Results and Errors between Chebyshev and Hermite for example 2

Z	**Exact Result**	**Chebyshev Computed result for case M = 5**	**Hermite Computed result for case M = 5**	**Error In Chebyshev.**	**Error in Hermite.**
0.0	1.00000000000000	0.99999999980000	1.00000000000000	2.0000e-10	0.00000e+00
0.2	0.81873075307798	0.81873352529753	0.81873352526950	2.7722e-06	2.7722e-06
0.4	0.67032004603564	0.67033384005778	0.67033383978527	1.3794e-05	1.3798e-05
0.6	0.54881163609403	0.54884481246803	0.54884481193343	3.3176e-05	3.3176e-05
0.8	0.44932896411722	0.44938546598356	0.44938546516737	5.6502e-05	5.6501e-05
1.0	0.36787944117144	0.36792015475300	0.36792015363300	4.0714e-05	4.0712e-05

Table 3. Comparison of Results and Errors between Chebyshev and Hermite for example 3

Z	**Exact Result**	**Chebyshev Computed result for case M = 8**	**Hermite Computed result for case M = 8**	**Error In Chebyshev.**	**Error in Hermite.**
0.0	0.00000000000000	0.00000003034775	-0.00000151539200	3.0300e-08	1.5154e-06
0.2	0.24428055163203	0.24428058337374	0.244280249234661	3.1800e-08	3.0240e-07
0.4	0.59672987905651	0.59672996415055	0.596730754193489	8.5100e-08	8.7500e-07
0.6	1.09327128023431	1.09327152428490	1.093273088792861	2.4400e-07	1.8090e-06
0.8	1.78043274279397	1.78043330528754	1.780435083327691	5.6300e-07	2.3420e-06
1.0	2.71828182845905	2.71828239893951	2.718283701152300	5.7100e-07	1.8730e-06

Table 1, 2, and, 3 show the numerical solution obtained in terms of approximate solution and the errors for the linear integro-differential equations solved through third kind Chebyshev and Hermite Polynomials basis function. We also observed from the examples solved that both methods converge close to the exact solution in a view iterations and lower error.

5. Conclusion

In this work, we have demonstrated Collocation Approximation Method for Solving high-order integro-differential equations of various order using the third kind of Chebyshev Polynomial and Hermite Polynomial as basis functions and compared it with each other. The results obtained by the third kind of Chebyshev Polynomial basis performed better over the Hermite Polynomial in some examples. However, we also observed that as M (degree of approximant) increases, the results obtained yield a good approximation to the exact solution only in a few iterations in all the problems considered (as it can be seen from tables of results). We, thus, conclude that both methods were feasible and effective for the class of problems considered. This work is limited to linear integro-differential equation, it is therefore recommended for the immediate solution of other types of equations, for example, Fractional differential equations, Integro-differential difference equations, and Partial differential equations.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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