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ON COUPLED TIME-FRACTIONAL HEAT-EQUATIONS

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ABSTRACT
In this paper, we consider a coupled time-fractional heat equations, one homogeneous and the other a non-homogeneous. Analytic solutions of the involved variables, the main and auxiliary, are presented with a special attention on the fractional order derivative α on the range of (0, 1]. The main variable may represent some density function while the auxiliary function plays the role of a source function. We consider the well-posedness of solution of the equation in the sense of Lipschitz and energy method. Scaling becomes important tools for the study of the solution near the singularity and larger numerical resolutions. We present the analytic solutions of the system using the method of Fourier and Laplace transforms, where one finds the solution in terms of a Green's function, which is a scaled Mittag-Leffler's function. The behaviour of the main and auxiliary variables is depicted using numerical approximation.
Keywords: Caputo's Fractional derivative, Laplace transform, Fourier transform, heat equations, well-posedness.

INTRODUCTION

Heat equation is famously known for the description of heat diffusion in a a certain region. The equation was firstly studied by the Josheph Fourier (1822), for instance see Cannon (1984) and Elsaid et al. (2016). As an evolution equation, the main function $u(t, x)$ depends on the temporal-variable t. It appears in several areas of mathematics and application such as financial mathematics to model options, image analysis (Perona & Malik 1990) and in machine learning. Several analytic and numerical techniques are used in solving the heat equation including Fourier-Laplace methods, Crank-Nicolson (1947), ... These approaches are mostly applicable to the both homogeneous and non-homogeneous heat equations, where the latter is meant to represent heat problem with a source term.

In this work, we intend to study the solution behaviour of the fractional time version of the non-homogeneous heat equation where the source term satisfies a homogeneous heat problem. This problem can be thought as a heat conduction problem where heat diffuses through two connected media with different diffusion coefficients. References on the integer and fractional heat diffusion problems are respectively Thambynayagam (2011) and Cosiglio (2019).

Fractional time heat equation has received a significant attention in recent years in the area of fractional calculus. In Mclean (2011), regular type of solution to the time-fractional heat equation is studied. The evolution behaviour of the fractional diffusive wave equation considered in Cosiglio & Mainardi (2019). Several other additional results on the issue of time-fraction heat equations are available in Kochubei (2008), Li et al. (2015), Mainardi et. al. (2007), Mainardi et. al. (2008), Wilmott et. al. (1995), Zecova (2014), and the references therein.

Here we are concerned in the analytic solutions to the system of fractional problem, for $0 < \alpha < 1$,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \nu_1 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu T(x,t), \quad \Omega = (I, \mathbb{R}^+) \subset \mathbb{R} \times \mathbb{R}^+ \tag{1}$$

$$u(0,x) = f(x), \quad u(t,x) = 0, \quad \text{on } \partial I \text{ for } t > 0, \tag{2}$$

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = \nu_2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < \alpha < 1, x \in I \subset \mathbb{R}, t > 0 \tag{3}$$

$$u(0,x) = f(x), \quad u(t,x) = 0, \quad \text{on } \partial I \text{ for } t > 0 \tag{4}$$

where ν_1, ν_2 are the thermal diffusivity coefficients of two different media. By taking the domain $I = \mathbb{R}$, it simply means that the influence of the actual boundaries is negligible. Moreover, the fractional-time derivative of a function $f(t)$ is $D_t^\alpha f := \frac{\partial^\alpha f}{\partial t^\alpha}$ in Caputo's sense, reads

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)}{(t-\tau)^{1+\alpha-m}} \mathrm{d}\tau, & m-1 < \alpha < m \\ f^{(m)}(t), & m = \alpha \end{cases} \qquad (5)$$

This system of heat equation problem can be described as one which deals with heat transfer between two media of different thermal diffusivity coefficients. The behaviour of the heat flow in one media, the non-homogeneous heat equation, depends on the behaviour of the other, the homogeneous one. Or equivalently, the source of heat from one material depends on the heat transferred from the other material.

WELL-POSEDNESS

In this section, we consider the question of well-posedness of the Cauchy problem. This is the case of existence, uniqueness and continuous dependence of solution on the initial data.

Existence & Uniqueness. Let us first define fractional integral operator $D_t^{-\alpha}$. Since

$$D_t^{-1} u(t,x) = \int_0^t u(\tau, x) \mathrm{d}\tau \quad \text{then } D_t^{-\alpha} u(t,x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau,x) \mathrm{d}\tau .$$

Expressing the heat equation, using the fractional-time derivative D_t^α, we've

$$D_t^\alpha u = \nu_1 \partial_x^2 u + F(t,x,T), \qquad (6)$$

where $F(t,x,T) = \mu_1 T(t,x)$. Applying direct fractional-time derivative $D_t^{-\alpha}$ on both-sides to have

$$\begin{aligned} u(t,x) - u(0,x) &= \nu_1 D_t^{-\alpha}(\partial_x^2 u) + D_t^{-\alpha}(F(t,x,T)) \\ &= \frac{\nu_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \partial_x^2 u(\tau,x) \mathrm{d}\tau + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(\tau,x,T) \mathrm{d}\tau \end{aligned}$$

By the Duhamel's principle the solution reads

$$u(t,x) = D_t^{-\alpha} \int_{\mathbb{R}} \widetilde{K}(x-y) F(t,y,T) \mathrm{d}y = \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}} (t-\tau)^{\alpha-1} K(x-y) F(t,y,T) \mathrm{d}y \mathrm{d}\tau \qquad (7)$$

where $\widetilde{K}(t,x) = t^{\alpha-1} K(x)$ and $K(x)$ are the heat kernel functions for the integrals with respect to t and x respectively. Then, with the initial condition $u(0,x) = u_0(x) = g(x)$, equation (7) reads

$$u(t,x) = \frac{1}{\Gamma(\alpha)} \left[\int_{\mathbb{R}} \widetilde{K}(x-y,t) g(y) \mathrm{d}y + \int_0^t \int_{\mathbb{R}} \widetilde{K}(x-y, t-\tau) F(\tau,y,T) \mathrm{d}y \mathrm{d}\tau \right] \qquad (8)$$

The result of existence of such solution is provided in Luchko (2009) where *u* is expressed in terms of Fourier series. In (Li et al. 2015), existence and uniqueness results for coupled diffusion systems were discussed using eigenfunction expansion. For more results on the existence and uniqueness of solution see Luchko (2009) and the references therein.

Theorem 2.1. *Suppose the IVP above has a continuous solution in* $\overline{\Omega} \times [0, t^*]$ *for* $t^* > 0$ *and finite real number, then the solution is unique.*

To check for uniqueness, we assume there are two different solutions u_1 and u_2. We let $v = u_1 - u_2$. Consequently, the system of the equations (1)-(4), as all the operators are linear, reduces to a single one-dimensional problem in v only:

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} - \nu_1 \frac{\partial^2 v(x,t)}{\partial x^2} = 0, \quad \Omega \times \mathbb{R}^+$$

$$v(0,x) = 0, \quad v(t,x) = 0, \qquad \text{on } \partial\Omega \text{ for } t > 0.$$

The equation (3) in the system (1)-(4) is not considered, as the equation for *v* is independent of the temperature variable $T(t,x)$. By using the energy method, the mass of the solution is given as

$$J(t) = \int_\Omega v^2 \, \mathrm{d}\Omega.$$

The fractional-time derivative of $J(t)$, using $m = 1$ in the general derivative form (5), yields

$$\frac{\partial^\alpha J(t)}{\partial t^\alpha} = 2\int_\Omega v\cdot\left(\frac{\partial^\alpha v}{\partial t^\alpha}\right)\mathrm{d}\Omega = 2\nu_1\int_\Omega v\,\frac{\partial^2 v}{\partial x^2}\mathrm{d}\Omega = 2\nu_1\int_{\partial\Omega} v\frac{\partial v}{\partial\eta}\mathrm{d}(\partial\Omega) - 2\nu_1\int_\Omega |\nabla v|^2\mathrm{d}\Omega.$$

The last equality is obtained by applying Divergence theorem. Again, by the virtue of the relation that $\partial_t^\alpha v - \nu_1\partial_x^2 v = 0$ one gets

$$\frac{\partial^\alpha J(t)}{\partial t^\alpha} = -2\nu_1\int_\Omega |\nabla v|^2\mathrm{d}\Omega.$$

Obviously, $\frac{\partial^\alpha J(t)}{\partial t^\alpha} \le 0$ if $\nu_1 > 0$ which has been the case considered in this paper. As a result, the $J(t)$ is a nonincreasing function, with the fact that $J(0) = 0$ and $J(t) \ge 0$ implies that $J(t) \equiv 0$. However, the integral $\int v^2$ being nonnegative and continuous w.r.t. all its arguments, it then follows that v is identically zero for $t \ge 0$. Thus $u_1 - u_2 = v = 0$ or $u_1 = u_2$, thereby proving that the solution is unique.

Continuous Dependence.

Theorem 2.2. *Suppose (6) is defined on and within $\mathbb{R}^+\times\Omega$. If f is continuous and $u \in S(\Omega)$, then, the Lipschitz condition is satisfied and the solution depends continuously on the initial data $u(0, x)$.*

Proof. Suppose $u = u(t,x)$ is a solution of the main fractional equation (1). Suppose further that $\partial_x^2 u$ is continuous on $\mathbb{R}^+\times\Omega$. Then, from the relation (in Fourier space)

$$\frac{d^\alpha \hat{u}(t,k)}{dt^\alpha} = -\nu_1 k^2\,\hat{u}(t,k) + \hat{F} \tag{9}$$

obtained by the Fourier transform $\mathcal{F}(u) := \hat{u} = \int_{\mathbb{R}} u(t,x)e^{-ikx}\mathrm{d}x$ of both sides of (6). Let us take the right-hand-side of (9) to be $\hat{G} \coloneqq -\nu_1 k^2\,\hat{u} + \hat{F}$, so we have

$$\left|\hat{G}_1(t,u_1) - \hat{G}_2(t,u_2)\right| = \left|-\nu_1 k^2\,\hat{u}(t_1,k) + \hat{F} + \nu_1 k^2\,\hat{u}(t_2,k) - \hat{F}\right| \le \nu_1 k^2|\hat{u}_1 - \hat{u}_2|.$$

As we can find Lipschitz's constant $L_c \coloneqq \nu_1 k^2 > 0$, the Lipschitz condition is satisfied. To check for continuous dependence, suppose $\tilde{\hat{u}} = \hat{u}(\tilde{t},k)$ be a perturbation to the solution $\hat{u}$ so that the perturbation of the corresponding initial data satisfies $\left|\tilde{\hat{u}}_0 - \hat{u}_0\right| \le \delta$. Then, we want to show that $\left|\tilde{\hat{u}} - \hat{u}\right| \le \delta$, for δ small positive. By direct integration of (9),

$$\left|\tilde{\hat{u}} - \hat{u}\right| \le \left|\tilde{\hat{u}}_0 - \hat{u}_0\right| + L_c\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left|\tilde{\hat{u}} - \hat{u}\right|\mathrm{d}\tau \tag{10}$$

Letting $W(t) := \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left|\tilde{\hat{u}} - \hat{u}\right|\mathrm{d}\tau$ we have

$$\frac{d^\alpha W}{dt^\alpha} - L_c W \le \delta.$$

Finding an integrating factor $I(t,k)$, we write

$$\frac{d^\alpha}{dt^\alpha}[W(t)\cdot I(t,k)] \le \delta\cdot I(t,k)$$

where $I(t,k) = E_{\alpha,1}\left(-D_t^{-\alpha}(L_c)\right) = E_{\alpha,1}\left(-\frac{L_c t^\alpha}{\Gamma(\alpha+1)}\right)$ is a Mittag-Leffler's function. Upon integrating with respect to t^α, and noting that $D_t^{-\alpha}\left[E_{\alpha,\beta}(\lambda\tau^\alpha)\right] = t^{\beta-1}\left[E_{\alpha,\beta}(\lambda t^\alpha) - 1\right]/\lambda$, one gets

$$W\cdot E_{\alpha,1}\left(-\frac{L_c t^\alpha}{\Gamma(\alpha+1)}\right) \le \frac{\delta}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,1}\left(-\frac{L_c\tau^\alpha}{\Gamma(\alpha+1)}\right)\mathrm{d}\tau = \delta D_t^{-\alpha}E_{\alpha,1}\left(-\frac{L_c t^\alpha}{\Gamma(\alpha+1)}\right)$$
$$\le \frac{\delta}{-\frac{L_c}{\Gamma(\alpha+1)}}\left[E_{\alpha,1}\left(-\frac{L_c}{\Gamma(\alpha+1)}t^\alpha\right) - 1\right] = \delta\frac{\Gamma(\alpha+1)}{L_c}\left[1 - E_{\alpha,1}\left(-\frac{L_c}{\Gamma(\alpha+1)}t^\alpha\right)\right]$$

Then,

$$W(t) \le \delta\frac{\Gamma(\alpha+1)}{L_c}\left[1 - E_{\alpha,1}\left(-\frac{L_c}{\Gamma(\alpha+1)}t^\alpha\right)\right].$$

And the inequality (10) becomes

$$\left|\tilde{\hat{u}}(t,k) - \hat{u}(t,k)\right| \le \delta + L_c\cdot\delta\frac{\Gamma(\alpha+1)}{L_c}\left[1 - E_{\alpha,1}\left(-\frac{L_c}{\Gamma(\alpha+1)}t^\alpha\right)\right] \le \frac{\delta\Gamma(\alpha+1)}{E_{\alpha,1}\left(-\frac{L_c}{\Gamma(\alpha+1)}t_{\max}^\alpha\right)} =: \varepsilon,$$

where $\varepsilon = \delta\Gamma(\alpha+1)/E_{\alpha,1}\left(-\frac{L_c}{\Gamma(\alpha+1)}t_{\max}^\alpha\right)$ for finite $|t| \le t_{\max}$. This establishes the proof.

By virtue of the Fourier inversion theorem for Schwartz functions, the proof above shows that the function $u(t,x)$ in the physical space depends continuously on the initial data $u_0(x) = f(x)$.

Furthermore, one concludes that the Cauchy problem (1)-(4) is well-posed having established the proofs of existence, uniqueness and continuous dependence.

SELF-SIMILAR SOLUTION

When a solution is known to exist, we may construct another solution through scaling. It is extremely important especially for solutions developing singularity, i.e., if *u* has the tendency to grow with no bound in either of space or time scale. A self-similar solution is constructed via the transformations

$$u \to \tilde{u} = \lambda^a u(\lambda^b t, \lambda^c x), \quad t \to \tilde{t} = \lambda^a t, \quad x \to \tilde{x} = \lambda^c x \quad \text{for } a, b, c \in \mathbb{R} \tag{11}$$

where λ is refer to as *scaling* parameter. The time derivative term, for $\alpha \neq m$, using (11) transforms to

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{u^{(m)}(x,\tau)}{(t-\tau)^{1+\alpha-m}} \mathrm{d}\tau = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\lambda^{-a}\tilde{u}^{(m)}(x,\tau)}{(\lambda^{-b}\tilde{t} - \lambda^{-b}\tilde{\tau})^{1+\alpha-m}} \lambda^{(1-m)b} \mathrm{d}\tilde{\tau}$$

This simplifies to

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \lambda^{-a+(\alpha-2m+2)b} \frac{\partial^\alpha \tilde{u}(t,x)}{\partial \tilde{t}^\alpha} \tag{12}$$

Similarly, for the second order spatial derivative we have

$$\frac{\partial\, u(x,t)}{\partial x} = \lambda^{c-a} \frac{\partial\, \tilde{u}(x,t)}{\partial \tilde{x}}, \qquad \frac{\partial^2\, u(x,t)}{\partial x^2} = \lambda^{c-2a} \frac{\partial^2\, \tilde{u}(x,t)}{\partial \tilde{x}^2} \tag{13}$$

Combining the two results (12) and (13), the main equation (1) transforms to

$$\lambda^{-a+(\alpha-2m+2)b} \frac{\partial^\alpha \tilde{u}(t,x)}{\partial \tilde{t}^\alpha} = \nu_1 \lambda^{c-2a} \frac{\partial^2\, \tilde{u}(t,x)}{\partial \tilde{x}^2} + \mu\, \tilde{T}(t,x) \tag{14}$$

The equation (14) stays invariant if $a = (\alpha - 2m + 2)b$ and $c = 2a$. Consequently, if $a = 1$ then $c = 2$ and $b = 1/(\alpha - 2m + 2)$ for $m \in \mathbb{N}$. The self-similar solution now becomes:

$$\tilde{u}(x,t) = \lambda\, u(\tilde{t}, \tilde{x}) = \lambda\, u\left(\lambda^{\frac{1}{\alpha-2m+2}} t, \lambda^2 x\right) \tag{15}$$

where $\tilde{t} = \lambda^{\frac{1}{\alpha-2m+2}} t$ and $\tilde{x} = \lambda^2 x$. The original function u from the scaled solution (15) reads

$$u(x,t) = \frac{1}{\lambda}\, \tilde{u}\left(\frac{\tilde{t}}{\lambda^{(\alpha-2m+2)^{-1}}}, \frac{\tilde{x}}{\lambda^2}\right).$$

The simulation of the solutions, especially, on high resolution are best improved through scaling. The amplitude $u(x, t)$ and the relevant variables x, t are equally scaled appropriately. The advantage here is that one reduces computational cost while simulating the solution as accurately as possible. This is done by taking the value(s) of λ as sufficiently small or large as appropriate, see Elsaid et. al. (2016) for more.

ANALYTIC APPROACH

We apply the Fourier-Laplace transform to the problem (1), however, we must solve the second equation first as the first equation involves the variable T. We take the Fourier transform of fractional derivative of function $u(t,x)$ as $\mathcal{F}(D_t^\alpha u)(t,k) = \frac{\partial^\alpha \boldsymbol{u}(t,k)}{\partial t^\alpha}$. The problem is defined on the whole $\mathbb{R} = (-\infty, \infty)$ so we can apply Fourier transform in space and Laplace transform in time. We impose the conditions that at the boundary $T(x, t) = 0$ as $x \to \pm\infty$ (i.e. temperature is negligible in the past or future in the region of study. In this regard, we are considering free temperature flow, negligible outside the region but accumulated inside the region of study.). Therefore, we apply the Fourier-Laplace transform technique to solve the problem (1).

We first write the Fourier transform of the fractional time derivatives for T of the first equation by applying integration by parts to the right-hand-sides:

$$\begin{aligned}
\frac{\partial^\alpha \boldsymbol{T}(t,k)}{\partial t^\alpha} &= \int_{-\infty}^{\infty} \frac{\partial^\alpha T(t,x)}{\partial t^\alpha} e^{-ikx} \mathrm{d}x = \nu_2 \int_{-\infty}^{\infty} \frac{\partial^2 T(t,x)}{\partial x^2} e^{-ikx} \mathrm{d}x \\
&= \nu_2 \left[\left[e^{-ikx} \frac{\partial T(t,x)}{\partial x}\right]_{x=-\infty}^{\infty} - (-ik) \int_{-\infty}^{\infty} \frac{\partial T(t,x)}{\partial x} e^{-ikx} \mathrm{d}x\right] = -(-ik)\nu_2 \int_{-\infty}^{\infty} \frac{\partial T(t,x)}{\partial x} e^{-ikx} \mathrm{d}x \\
&= -(-ik)\nu_2 \left[[e^{-ikx} T(t,x)]_{-\infty}^{\infty} - (-ik) \int_{-\infty}^{\infty} T(t,x)\, e^{-ikx} \mathrm{d}x\right] = (-ik)^2 \nu_2 \int_{-\infty}^{\infty} T(t,x)\, e^{-ikx} \mathrm{d}x \\
&= -k^2 \nu_2 \mathbf{T}(t,k).
\end{aligned} \tag{16}$$

where k is a wave-number and $\mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$ represents the Fourier-transform on (*-∞, ∞*). According to the Caputo's fractional derivative we write the *Laplace transform* w.r.t. time t as

$$\mathcal{L}_t(D_t^\alpha \mathbf{T}(t,k)) = s^\alpha\,\hat{\mathbf{T}}(s,k) - \sum_{\ell=0}^{m-1} s^{\alpha-\ell-1}\,\frac{\partial^\ell \mathbf{T}(t,k)}{\partial t^\ell}\,|_{t=0} \tag{17}$$

To determine the index m, we note that $0 < \alpha \leq 1$ and $m - 1 < \alpha < m$, thus m must equal 1. We also then use the initial condition $T(0, x) = g(x)$ with $\mathcal{F}(T(0,x)) = \mathbf{T}(0,k)$, so that the equation (17) becomes

$$\mathcal{L}_t(D_t^\alpha \mathbf{T}(t,k)) = s^\alpha\,\hat{\mathbf{T}}(s,k) - s^{\alpha-1}\mathbf{T}(0,k).$$

For the right-hand-side, we write the Laplace transform of the result (12)

$$-k^2\nu_2 \int_0^\infty e^{-st}\mathbf{T}(t,k)\mathrm{d}t = -k^2\nu_2\hat{\mathbf{T}}(s,k). \tag{18}$$

Therefore, the equation (1), using the result of equation (18), we have, in the Laplace domain:

$$s^\alpha\,\hat{\mathbf{T}}(s,k) - s^{\alpha-1}\mathbf{T}(0,k) = -k^2\nu_2\hat{\mathbf{T}}(s,k).$$

This can further be simplified to

$$\hat{\mathbf{T}}(s,k) = \left\{\frac{s^{\alpha-1}}{s^\alpha + (k^2\nu_2)}\right\}\mathbf{T}(0,k) \tag{19}$$

Next, we recall the relationship between the Mittag-Leffler function $E_{\alpha,\beta}(x)$ and Laplace transform $\mathcal{L}$:

$$\frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda} = \mathcal{L}\left(x^{\beta-1}E_{\alpha,\beta}(\pm\lambda x^\alpha)\right), \qquad \lambda \in \mathbb{C}:\ |\lambda s^{-\alpha}| < 1, \tag{20}$$

and the Mittag-Leffler function $E_{\alpha,\beta}(x)$ is defined as

$$E_{\alpha,\beta}(x) = \sum_{\ell=0}^{\infty} \frac{x^\ell}{\Gamma(\alpha\ell + \beta\,)} \tag{21}$$

There, in comparison of the relation (20) with the term in the right hand-side of (19) we have $\beta = 1$. Thus, the Laplace inverse transform of (19) w.r.t. t is

$$\mathbf{T}(t,k) = E_{\alpha,1}(-(k^2\nu_2)t^\alpha)\mathbf{T}(0,k).$$

The inverse Fourier-transform of the above equation defined by

$$\begin{aligned} T(t,x) &= \mathcal{F}^{-1}\left(E_{\alpha,1}(-(k^2\nu_2)t^\alpha)\right) \star \mathcal{F}^{-1}(\mathbf{T}(0,k)) = \mathcal{F}^{-1}\left(E_{\alpha,1}(-(k^2\nu_2)t^\alpha)\right) \star g(x) \\ &= \int_{-\infty}^{\infty} G(t,x-y)g(y)\mathrm{d}y \end{aligned} \tag{22}$$

where by definition $G(t,x)$ denotes a Green-function (aka fundamental solutions), such that

$$G(t,x) = \mathcal{F}^{-1}\left(E_{\alpha,1}(-(k^2\nu_2)t^\alpha)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{\alpha,1}(-(k^2\nu_2)t^\alpha)e^{ikx}\mathrm{d}k, \tag{23}$$

where $\boldsymbol{u}(0,k) = \int_{-\infty}^{\infty} f(x)\,e^{-ikx}\mathrm{d}k$ corresponds to $u(0,x)$ in the Fourier space and $\star$ represents convolution operation. The equation (23) is expressible in terms of W-function as found in (Mainardi et. al. (2010), Sec. 4.5) and Mclean (2011):

$$M_\alpha(x) := W_{-\alpha,-1-\alpha}(-x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\,\Gamma(1-(n+1)\alpha)} = \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(-x)^n}{(n-1)!}\Gamma(n\alpha)\sin(\alpha n\pi), \tag{24}$$

since, originally, on the whole $\mathbb{C}$ plane

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\alpha n + \beta)}, \qquad \alpha > -1,\ \beta \in \mathbb{C}, \qquad z \in \mathbb{C}, \tag{25}$$

using the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ showing that the two series are equal. Moreover,

$$\mathcal{F}(M_\alpha(|x|)) = 2E_{2\alpha,1}(-k^2), \qquad 0 < \alpha < 1,$$

yielding

$$G(t,x) = \frac{1}{2\sqrt{\nu_2 t^\alpha}} M_{\frac{\alpha}{2}}\left(\frac{|x|}{\sqrt{\nu_2 t^\alpha}}\right). \tag{26}$$

The solution is, now,

$$T(t,x) = \frac{1}{2\pi}\cdot\frac{1}{2\sqrt{\nu_2 t^\alpha}}\int_{-\infty}^{\infty} M_{\frac{\alpha}{2}}\left(\frac{|x-y|}{\sqrt{\nu_2 t^\alpha}}\right)g(y)\mathrm{d}y = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{\nu_2 t^\alpha}}\int_0^{\infty} M_{\frac{\alpha}{2}}\left(\frac{|(x-y)|}{\sqrt{\nu_2 t^\alpha}}\right)g(y)\mathrm{d}y \tag{27}$$

for some reasonable function $g(x)$ over $(-\infty, \infty)$. Note that, in the limit $\alpha \to 1$, the function $M_{\frac{1}{2}}$ is a Gaussian function. More explicitly, the solution for T is equivalently written as

$$T(t,x) = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{\nu_2 t^{\alpha}}} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{\left(-\frac{(x-y)}{\sqrt{\nu_2\, t^{\alpha}}}\right)^n}{n!\,\Gamma\left(1-(n+1)\frac{\alpha}{2}\right)} g(y)\mathrm{d}y, \tag{28}$$

Now, with a specific choice of initial data $g(x)$ in equation (28), one equivalently writes the exact solution to the equations (1) and (2).

Applying the Fourier-Laplace transforms to the second part of the heat problem (3) and (4), to write the solution. First take the Fourier transform to get

$$\frac{\partial^{\alpha}\boldsymbol{u}(t,k)}{\partial t^{\alpha}} = (-ik)^2\nu_1\boldsymbol{u}(t,k) + \mu\mathbf{T}(t,k) = -k^2\nu_1\boldsymbol{u}(t,k) + \mu\mathbf{T}(t,k). \tag{29}$$

By Laplace transform of the equation (29)

$$s^{\alpha}\,\hat{\boldsymbol{u}}(s,k) - s^{\alpha-1}\boldsymbol{u}(0,k) = -k^2\nu_1\hat{\boldsymbol{u}}(t,k) + \mu\hat{\mathbf{T}}(t,k). \tag{30}$$

Solving for $\hat{\boldsymbol{u}}$ in the equation (30), we get

$$\hat{\boldsymbol{u}}(t,k) = \frac{s^{\alpha-1}}{s^{\alpha}+\nu_1 k^2}\boldsymbol{u}(0,k) + \frac{\mu}{s^{\alpha}+\nu_1 k^2}\,\hat{\mathbf{T}}(s,k). \tag{31}$$

On taking the inverse Laplace transform and by using the equation (19), then (31) becomes

$$\boldsymbol{u}(t,k) = E_{\alpha,1}(-\nu_1 k^2 t^{\alpha})\,\boldsymbol{u}(0,k) + \mu\mathbf{T}(0,k)\mathcal{L}^{-1}\left[\frac{s^{\alpha-1}}{(s^{\alpha}+\nu_1 k^2)(s^{\alpha}+\nu_2 k^2)}\right] \tag{32}$$

whereas

$$\frac{1}{(s^{\alpha}+\nu_1 k^2)(s^{\alpha}+\nu_2 k^2)} = \frac{1}{k^2(\nu_1-\nu_2)}\left[\frac{1}{(s^{\alpha}+\nu_2 k^2)} - \frac{1}{(s^{\alpha}+\nu_1 k^2)}\right], \tag{33}$$

so that

$$\mathcal{L}^{-1}\left[\frac{s^{\alpha-1}}{(s^{\alpha}+\nu_1 k^2)(s^{\alpha}+\nu_2 k^2)}\right] = \frac{1}{k^2(\nu_1-\nu_2)}\left[\mathcal{L}^{-1}\left[\frac{1}{(s^{\alpha}+\nu_2 k^2)}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s^{\alpha}+\nu_1 k^2)}\right]\right]$$

$$= \frac{1}{k^2(\nu_1-\nu_2)}\left[E_{\alpha,1}(-(\nu_2 k^2)t^{\alpha}) - E_{\alpha,1}(-(\nu_1 k^2)t^{\alpha})\right] \tag{34}$$

Next, to express more explicitly, the solution in the equation (32), with the result (34), takes the form

$$\boldsymbol{u}(t,k) = E_{\alpha,1}(-(\nu_1 k^2)t^{\alpha})\boldsymbol{u}(0,k) + \frac{\mu\mathbf{T}(0,k)}{k^2(\nu_1-\nu_2)}\left[E_{\alpha,1}(-(\nu_2 k^2)t^{\alpha}) - E_{\alpha,1}(-(\nu_1 k^2)t^{\alpha})\right]$$

It is left to transform into the physical space by taking the inverse Fourier transform

$$u(t,x) = \mathcal{F}^{-1}\left(E_{\alpha,1}(-(\nu_1 k^2)t^{\alpha})\boldsymbol{u}(0,k)\right)$$

$$+ \frac{\mu}{(\nu_1-\nu_2)}\mathcal{F}^{-1}\left[\frac{\mathbf{T}(0,k)}{k^2}E_{\alpha,1}(-(\nu_2 k^2)t^{\alpha}) - \frac{\mathbf{T}(0,k)}{k^2}E_{\alpha,1}(-(\nu_1 k^2)t^{\alpha})\right] \tag{35}$$

Then, the first part of the equation (35) can be further simplified as follows:

$$\mathcal{F}^{-1}\left(\frac{\mathbf{T}(0,k)}{k^2}E_{\alpha,1}(-\nu_1 k^2 t^{\alpha})\right) = \frac{1}{2\pi}\mathcal{F}^{-1}\left(\frac{1}{k^2}\right) \star \mathcal{F}^{-1}\left(E_{\alpha,1}(-\nu_1 k^2 t^{\alpha})\mathbf{T}(0,k)\right)$$

$$= \frac{1}{2\pi}\left(-\sqrt{\frac{\pi}{2}}x\,\mathrm{sgn}(x)\right) \star \frac{1}{2\sqrt{\nu_1 t^{\alpha}}}M_{\frac{\alpha}{2}}\left(\frac{|x|}{\sqrt{\nu_1 t^{\alpha}}}\right) \star T(0,x)$$

$$= -\frac{1}{4\sqrt{2\pi\nu_1 t^{\alpha}}}\int_{-\infty}^{\infty}(x-y)\mathrm{sgn}(x-y)M_{\frac{\alpha}{2}}\left(\frac{|x-y|}{\sqrt{\nu_1 t^{\alpha}}}\right)T(0,y)\mathrm{d}y, \tag{36}$$

where the $\mathrm{sgn}(x)$ is a sign function which is 0 at $x=0$, but 1 for $x>1$ and $-1$for $x<0$. Therefore, using the result in the equation (36) above, the solution to the main field $u(t,x)$ is

$$u(t,x) = \frac{1}{2\pi} \cdot \frac{1}{2\sqrt{\nu_1 t^\alpha}} \left[\int_{-\infty}^{\infty} M_{\frac{\alpha}{2}}\left(\frac{|x-y|}{\sqrt{\nu_1 t^\alpha}}\right) u(0,y)\mathrm{d}y - \frac{\mu}{(\nu_1 - \nu_2)} \cdot \frac{1}{4\sqrt{2\pi\nu_2 t^\alpha}} \int_{-\infty}^{\infty} (x-y)\mathrm{sgn}(x-y)\, M_{\frac{\alpha}{2}}\left(\frac{|x-y|}{\sqrt{\nu_2 t^\alpha}}\right) T(0,y)\mathrm{d}y + \frac{\mu}{(\nu_1 - \nu_2)} \cdot \frac{1}{4\sqrt{2\pi\nu_1 t^\alpha}} \int_{-\infty}^{\infty} (x-y)\mathrm{sgn}(x-y)\, M_{\frac{\alpha}{2}}\left(\frac{|x-y|}{\sqrt{\nu_1 t^\alpha}}\right) T(0,y)\mathrm{d}y \right].$$

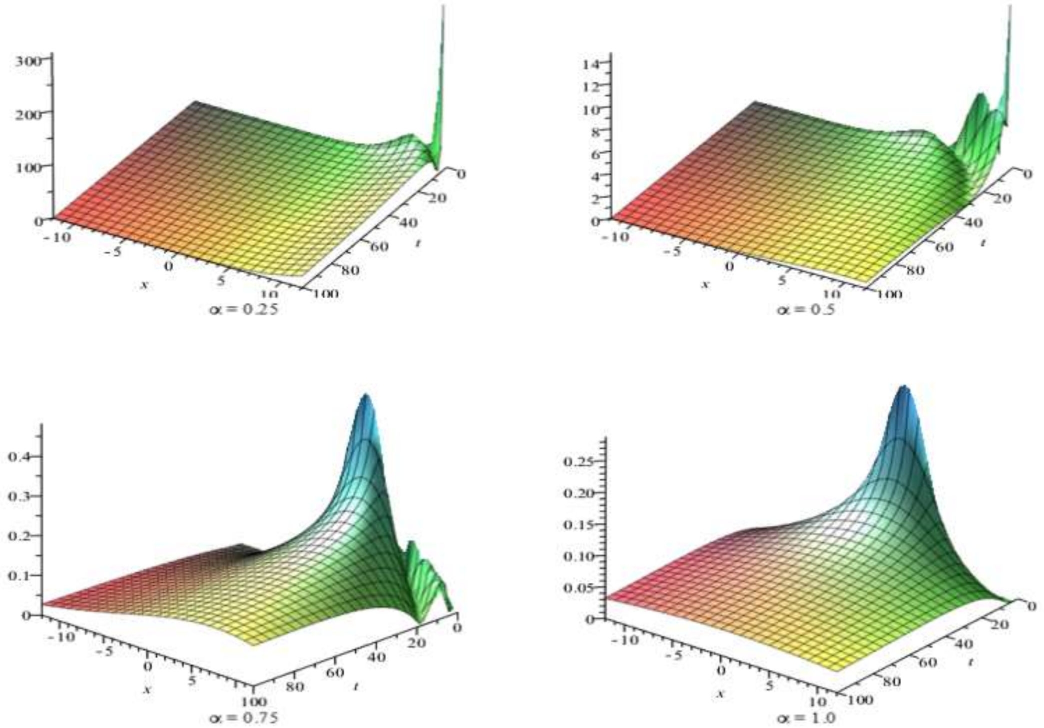

FIGURE 1. (Top left-bottom right): The profiles of $T(t,x)$ for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ resp. with $\nu_2 = 1.4$.

NUMERICAL EXAMPLE

If $g(x)$ is a Gaussian function e^{-x^2}, the profiles $T(t,x)$ for different values of α and the fixed values of $\nu_2 = 1.4$ and 1.8 are shown in the figures Fig.1 and Fig.2. It shows no much difference is detected for different values of ν_i only that the profile can change for slightly larger value of ν_i for smaller value of α, where $i = 1,2$. For the main variable $u(t,x)$ we use $u_0(x) = f(x) = \mathrm{sech}(x)$ using the same initial condition for $T_0(x) = g(x)$. The solutions $u(t, x)$ for different values of ν_1 and ν_2 are shown in the Fig 3. We also take into account the definition of signum function $\mathrm{sgn}(x) = |x|/x$ so that $|x| = x\,\mathrm{sgn}(x)$.

Now, based on the difficulty we may encounter in integrating the expressions in the respective equations (28) and (36), the use of numerical integration in space becomes necessary. We use series expansions of the Mittag-Lefler's function $M_\alpha(x)$ and taking the first 100 terms to obtain the plots in the figures shown in Fig.1, Fig. 2 and Fig.3. Within the domain of consideration, the number of terms of the series used here is optimal, the higher number of terms does not add up any significant difference.

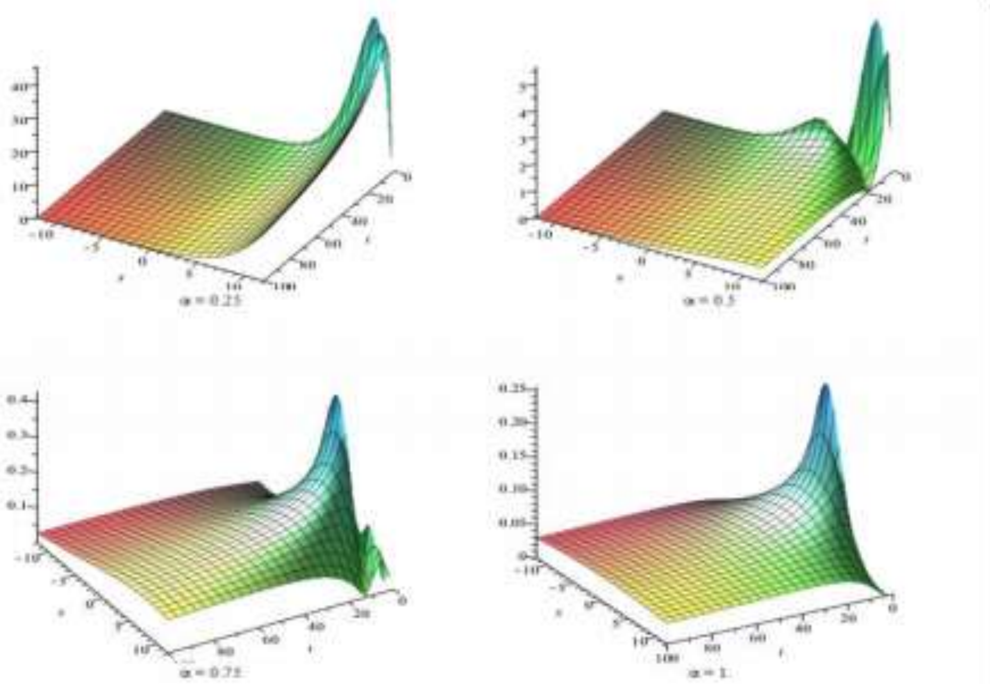

FIGURE 2. (Top left-bottom right): The profiles of $T(t,x)$ for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ resp. with $\nu_2 = 1.8$.

In Table 1, it is shown that for fractional derivatives, as shown in particular α = 0.75, solutions dissipate as time increases. This proves directly the dissipation property of solutions to the linear heat diffusion problem of integer kind.

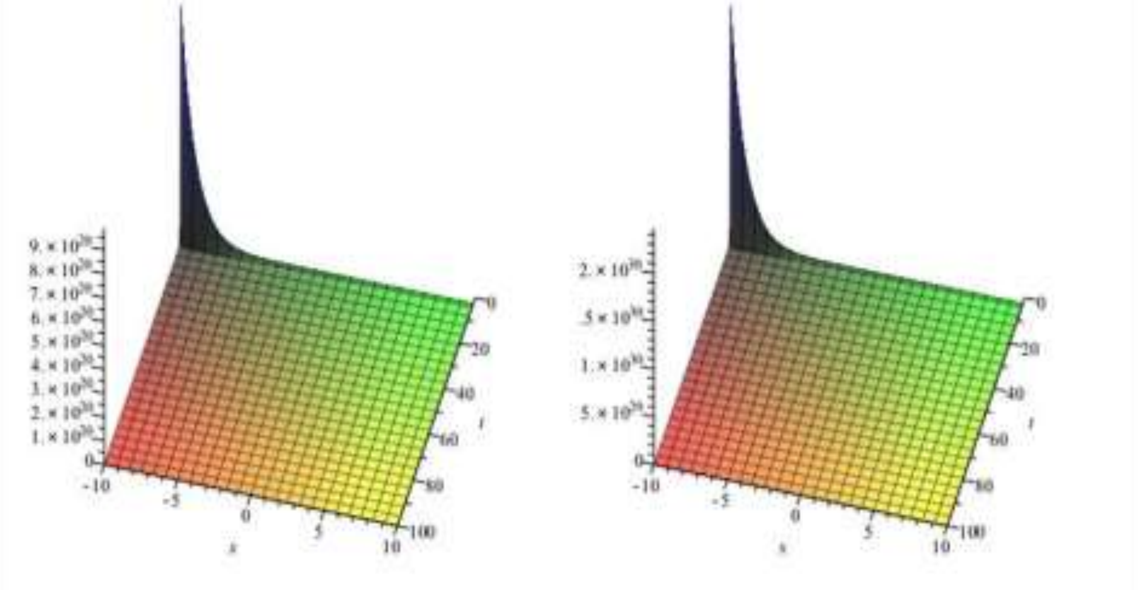

FIGURE 3. (Left-right) The profiles of $u(t,x)$ for $\alpha = \frac{1}{4}$ and $\frac{3}{4}$ respectively with $\nu_1 = 0.25$ and $\nu_2 = 1.25$.

Since, we are dealing with coupled linear equations, in which the main variable's solution depends on the auxiliary variable, the solution is expected to be taller in amplitude. This is the consequence of the superposition principle which is expected from the integer-time derivative heat equations.

t	x	$\|u(t,x)\|$
1	−12.00	$2.771698386 \times 10^{55}$
10	−10.00	$2.080090259 \times 10^{15}$
20	−8.00	$7.038164178 \times 10^{3}$
30	−6.00	$8.01492119 \times 10^{-2}$
40	−4.00	$2.46644089 \times 10^{-2}$
50	−2.00	$9.16240664 \times 10^{-2}$
60	0.00	$8.11493029 \times 10^{-2}$
70	2.00	$8.75087400 \times 10^{-4}$
80	6.00	$2.080022302 \times 10^{-1}$
90	8.00	$2.23592003 \times 10^{-1}$
100	12.00	$1.33718218 \times 10^{-1}$

TABLE 1. The evolution of $|u|$ at different positions x and time t for $\alpha = 0.75$, $\nu_1 = 0.75$, $\nu_2 = 1.25$, $\mu = 1$, with the functions $f(x) = \exp(-x^2)$ and $g(x) = \text{sech}(y)$.

CONCLUSION

It is observed that, based on the simulations provided, the solution behaviour of both the main and auxiliary variable follows the heat conduction properties. It is well known that, the heat equation has solutions that decays (dissipation), which is indicating the loss of heat energy in a material over a time as indicated in the Figure Fig.1 for $\alpha = 1$. However, considering the fact that the two equations are linear, the solutions for the main variable reflects similar behaviour except that it has greater amplitude at the start. That's the reflection of linearity property of the superposition principle. Apart from the well-posedness of the solutions, the evolution time of the solution $u(x, t)$ can be elongated or contracted through the scaling parameter λ by taking λ to be sufficiently large or small.

In the future, it would be interesting to know if there is a parameter dependence on the bounds of the solution for each choice of the fractional parameter α. Furthermore, analytic solutions for non-Schwartzian initial data are worthy of consideration.

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