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THE CONCEPT OF INVERSE α-CUTS IN MULTI Q-FUZZY SET

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ABSTRACT
In various mathematical theories such as fuzzy sets, fuzzy multisets, fuzzy soft sets, the concept of α-Cuts were applied together with their inverses. However, we noticed that in multi Q-fuzzy sets only α-Cuts were studied without their inverses. In this paper the concept of inverse α-Cuts and their properties in multi Q-fuzzy sets were introduced. Some distinctive features of α-Cuts and inverse α-Cuts were demonstrated. It is shown that as both first and second decomposition theorems hold in the former, it actually fails in the latter. Moreover, unlike $\alpha-$cuts, it was discovered that, a multi Q-fuzzy set cannot be uniquely represented as the family of all its weak inverse $\alpha-$cuts. Thus, both $\alpha-$cuts and inverse $\alpha-$cuts attract applications in many mathematical fields.
Key words: Fuzzy set, Fuzzy multiset, multi Q-fuzzy set, α-Cut, inverse α-Cut

INTRODUCTION

The concepts of fuzzy sets and α-cuts (α-level sets) together with their applications were first presented by (Zadeh, 1965). Subsequently, other researchers such as (Brown, 1971; Chutia et al., 2010; Dutta et al., 2011; Klir and Yuan, 1995; Kreinovich, 2013) studied the theory and its applications. In (Singh et al., 2014; Alkali and Isah, 2018; Isah, 2019a; Isah et al., 2019; Alkali and Isah, 2019), α-cuts and n-level sets and some of their properties in various contexts were studied. Inverse α-cuts in fuzzy set and their properties were introduced in (Sun and Han 2006), it is further extended to fuzzy multiset in (Singh et al., 2015). The theory of Multi Q-fuzzy set was introduced and studied in (Adam and Hassan, 2014; Adam and Hassan, 2014a; Adam and Hassan, 2015; Adam and Hassan, 2016). α-cuts and their properties in Multi Q-fuzzy sets were first presented in (Isah, 2019) where various theorems were established and proved. However, inverse α-cuts in multi Q-fuzzy sets were yet to be studied. In this paper, we intend to introduced inverse α-cuts and their properties in Multi Q-fuzzy sets and also investigate the distinctive feature of α-cuts and inverse α-cuts of multi Q-fuzzy sets.

MATERIALS AND METHODS

The concepts of multisets, Fuzzy multisets, multi Q-fuzzy sets and α-cuts of multi Q-fuzzy sets which are necessary tools for this study were presented in this section. First and second Decomposition theorems with regard to α-cuts were also studied extensively as that will enable us to come up with these theorems in terms of inverse α-cuts.

Preliminaries

Definition 2.1.1 (Jena et al., 2001)
Let X be a set, then a multiset M or an mset (for short) M over X is represented by a function *Count* M or C_M defined as $C_M: X \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers including zero.
One way of representing a multiset M from X with x_1 appearing k_1 times, x_2 appearing k_2 times and so on is $M = \{k_1/x_1, k_2/x_2, \dots, k_n/x_n\}$.

Let M and N be two msets drawn from a set X. Then
$M \subseteq N$ iff $C_M(x) \leq C_N(x), \forall x \in X$.
$M = N$ if $C_M(x) = C_N(x), \forall x \in X$.
$M \cup N = max\{C_M(x), C_N(x)\}, \forall x \in X$.
$M \cap N = min\{C_M(x), C_N(x)\}, \forall x \in X$.

Definition 2.1.2 (Syropoulos, 2012) A fuzzy multiset A is a multiset of pairs, where the first part of each pair is an element of a universe set X and the second part is the degree to which the first part

belongs to that fuzzy multiset. That is, $A: X \times I \longrightarrow \mathbb{N}$; where $I = [0,1]$ and $\mathbb{N}$ is the set of positive integers including 0.
Let A and B be fuzzy multisets. Then, Lengths $L(x;\ A)$ and $L(x;\ A,B)$ are respectively defined as $L(x;\ A) = \max\{j: \mu_A^j(x) \neq 0\}$; and $L(x;\ A,B) = \max\{L(x;\ A), L(x;\ B)\}$.
For brevity, $L\ (x)$ for $L\ (x;\ A)$ or $L\ (x; A, B)$ is also used if no confusion arises.
Note that for defining an operation between two fuzzy multisets A and B, the lengths of the membership sequences $\mu_A^1(x), \mu_A^2(x), \dots, \mu_A^p(x)$ and $\mu_B^1(x), \mu_B^2(x), \dots, \mu_B^p(x)$ need to be set equal.

Let A, B be fuzzy multisets. Then

$$\mu_{A\cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x), j = 1, \dots, L(x), \forall x \in X.$$
$$\mu_{A\cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x), j = 1, \dots, L(x), \forall x \in X.$$
$$A \subseteq B \Leftrightarrow \mu_A^j(x) \leq \mu_B^j(x), j = 1, \dots, L(x), \forall x \in X.$$

Thus, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Definition 2.1.3 (Adam and Hassan, 2014) Let I be a unit interval $[0,1]$, k be a positive integer, U be a universal set and Q be a non-empty set. A multi Q-fuzzy set A_Q in U and Q is a set of ordered sequences $A_Q = \{((u,q), (\mu_1(u,q),\ \mu_2(u,q), \dots, \mu_k(u,q))): u \in U, q \in Q\}$, where $\mu_i(u,q) \in I$ for all $i = 1, 2, \dots, k$.
The function $(\mu_1(u,q),\ \mu_2(u,q), \dots, \mu_k(u,q))$ is called the membership function of multi Q-fuzzy set A_Q and $\mu_1(u,q) + \mu_2(u,q) + \cdots + \mu_k(u,q) \leq 1$, k is called the dimension of A_Q. Thus, if the sequences of the membership functions have finite number of terms, say k-terms, the multi Q-fuzzy set is a function from $U \times Q$ to I^k such that for all $(u,q) \in U \times Q$, $\mu_{A_Q} = (\mu_1(u,q),\ \mu_2(u,q), \dots, \mu_k(u,q))$. The set of all multi Q-fuzzy sets of dimension k in U and Q is denoted by $M^kQF(U)$.

α-cuts in multi Q-fuzzy set

Definition 2.2.1 α-cuts in multi Q-fuzzy set (Isah, 2019)
Let $A_Q \in M^kQF(U)$ and $\alpha \in [0,1]$. Then the α –cut or α –level set of A_Q, denoted ${}^{\alpha}A_Q$ is defined as ${}^{\alpha}A_Q = \left\{(u,q)/A_m: \mu_{A_Q}(u,q) \geq \alpha\right\}$ where A_m is the cardinality of $\mu(u,q)$ such that $\mu(u,q) \geq \alpha$.
The strong α –cut or α –level set of A_Q, denoted ${}^{\alpha+}A_Q$ is defined as
${}^{\alpha+}A_Q = \left\{(u,q)/A_m: \mu_{A_Q}(u,q) > \alpha\right\}$ where A_m is the cardinality of $\mu(u,q)$ such that $\mu(u,q) > \alpha$.

Definition 2.2.2 Decomposition of Multi Q-fuzzy Set (Isah, 2019)
Let $U = \{u_1, u_2\}$, $Q = \{p, q, r\}$ and a multi Q-fuzzy set A_Q over U and Q be

$$A_Q = \{((u_1,p), 0.3{,}0.2{,}0.5), ((u_1,q), 0.2{,}0.8{,}0), ((u_1,r), 0.1{,}0.5{,}0.3), ((u_2,p), 0.3{,}0.1{,}0.2), ((u_2,q), 0{,}0.3{,}0.7), ((u_2,r), 0.2{,}0.3{,}0.1)\}.$$

Suppose, we have the following distinct α-cuts defined by characteristic functions viewed as special membership functions:

$${}^{0.1}A_Q = \{((u_1,p), 1{,}1{,}1), ((u_1,q), 1{,}1{,}0), ((u_1,r), 1{,}1{,}1), ((u_2,p), 1{,}1{,}1), ((u_2,q), 0{,}1{,}1), ((u_2,r), 1{,}1{,}1)\}, \alpha \geq 0.1.$$
$${}^{0.2}A_Q = \{((u_1,p), 1{,}1{,}1), ((u_1,q), 1{,}1{,}0), ((u_1,r), 0{,}1{,}1), ((u_2,p), 1{,}0{,}1), ((u_2,q), 0{,}1{,}1), ((u_2,r), 1{,}1{,}0)\}, \alpha \geq 0.2.$$
$${}^{0.3}A_Q = \{((u_1,p), 1{,}0{,}1), ((u_1,q), 0{,}1{,}0), ((u_1,r), 0{,}1{,}1), ((u_2,p), 1{,}0{,}0), ((u_2,q), 0{,}1{,}1), ((u_2,r), 0{,}1{,}0)\}, \alpha \geq 0.3.$$
$${}^{0.5}A_Q = \{((u_1,p), 0{,}0{,}1), ((u_1,q), 0{,}1{,}0), ((u_1,r), 0{,}1{,}0), ((u_2,p), 0{,}0{,}0), ((u_2,q), 0{,}0{,}1), ((u_2,r), 0{,}0{,}0)\}, \alpha \geq 0.5.$$
$${}^{0.7}A_Q = \{((u_1,p), 0{,}0{,}0), ((u_1,q), 0{,}1{,}0), ((u_1,r), 0{,}0{,}0), ((u_2,p), 0{,}0{,}0), ((u_2,q), 0{,}0{,}1), ((u_2,r), 0{,}0{,}0)\}, \alpha \geq 0.7.$$
$${}^{0.8}A_Q = \{\{((u_1,p), 0{,}0{,}0), ((u_1,q), 0{,}1{,}0), ((u_1,r), 0{,}0{,}0), ((u_2,p), 0{,}0{,}0), ((u_2,q), 0{,}0{,}0), ((u_2,r), 0{,}0{,}0)\}, \alpha \geq 0.8.$$

Thus, converting each of the above α-cuts to a special Multi Q-fuzzy Set ${}_{\alpha}A_Q$ defined for each $(u,q) \in A_Q$ as ${}_{\alpha}A_Q = \alpha.({}^{\alpha}A_Q)$
we get

$${}_{0.1}A_Q = \{((u_1,p), 0.1{,}0.1{,}0.1), ((u_1,q), 0.1{,}0.1{,}0), ((u_1,r), 0.1{,}0.1{,}0.1), ((u_2,p), 0.1{,}0.1{,}0.1), ((u_2,q), 0{,}0.1{,}0.1), ((u_2,r), 0.1{,}0.1{,}0.1)\},$$

$$_{0.2}A_Q = \{((u_1,p),0.2,0.2,0.2),((u_1,q),0.2,0.2,0),((u_1,r),0,0.2,0.2),((u_2,p),0.2,0,0.2),((u_2,q),0,0.2,0.2),((u_2,r),0.2,0.2,0)\},$$

$$_{0.3}A_Q = \{((u_1,p),0.3,0,0.3),((u_1,q),0,0.3,0),((u_1,r),0,0.3,0.3),((u_2,p),0.3,0,0),((u_2,q),0,0.3,0.3),((u_2,r),0,0.3,0)\},$$

$$_{0.5}A_Q = \{((u_1,p),0,0,0.5),((u_1,q),0,0.5,0),((u_1,r),0,0.5,0),((u_2,p),0,0,0),((u_2,q),0,0,0.5),((u_2,r),0,0,0)\},$$

$$_{0.7}A_Q = \{((u_1,p),0,0,0),((u_1,q),0,0.7,0),((u_1,r),0,0,0),((u_2,p),0,0,0),((u_2,q),0,0,0.7),((u_2,r),0,0,0)\},$$

$$_{0.8}A_Q = \{\{((u_1,p),0,0,0),((u_1,q),0,0.8,0),((u_1,r),0,0,0),((u_2,p),0,0,0),((u_2,q),0,0,0),((u_2,r),0,0,0)\}.$$

It is easy to see that $_{0.1}A_Q \cup {_{0.2}A_Q} \cup {_{0.3}A_Q} \cup {_{0.5}A_Q} \cup {_{0.7}A_Q} \cup {_{0.8}A_Q} = A_Q$.
In other words, any Multi Q-fuzzy Set A_Q can be represented as the union of its special α-cuts $_{\alpha}A_Q$, and this representation is usually referred to as Decomposition of A_Q.

Moreover, if for each $(u,q) \in A_Q$ we defined $_{\alpha+}A_Q$ as $_{\alpha+}A_Q = \alpha.(^{\alpha+}A_Q)$, we can see that by using a similar arguments, a Multi Q-fuzzy Set A_Q can be represented as the union of its strong α-cuts defined as above known as Decomposition of that Multi Q-fuzzy Set.

Theorem 2.2.3 First Decomposition Theorem (Isah, 2019)
Let $A_Q \in M^kQF(U)$, then $A_Q = \cup_{\alpha\in[0,1]}\ {_{\alpha}A_Q}$.
Proof
For each $(u,q) \in A_Q$, let $y = \mu^i_{A_Q}(u,q), i = 1,2,\dots,k$
Then for every $\alpha \in (y,1]$ we have $\mu^i_{A_Q}(u,q) = y < \alpha, i = 1,2,\dots,k$. Thus, $_{\alpha}A_Q = 0$.
On the other hand, for every $\alpha \in (0,y]$ we have $\mu^i_{A_Q}(u,q) = y \geq \alpha, i = 1,2,\dots,k$.
Thus, $_{\alpha}A_Q = \alpha$.
Hence, $(\cup_{\alpha\in[0,1]}\ {_{\alpha}A_Q})\ (u,q) = \sup_{\alpha\in(0,y]} \alpha = y = \mu^i_{A_Q}(u,q), i = 1,2,\dots,k$.
As the same argument is valid for each $(u,q) \in A_Q$, it follows that each multi Q-fuzzy set can be uniquely represented as the family of all its α −cuts.

Theorem 2.2.4 Second Decomposition Theorem (Isah, 2019)
Let $A_Q \in M^kQF(U)$, then $A_Q = \cup_{\alpha\in[0,1]}\ {_{\alpha+}A_Q}$.
Proof
For each $(u,q) \in A_Q$, let $y = \mu^i_{A_Q}(u,q), i = 1,2,\dots,k$. Then,
$(\cup_{\alpha\in[0,1]}\ {_{\alpha+}A_Q})\ (u,q) = \sup_{\alpha\in(0,1]} {_{\alpha+}A_Q} = \max[\sup_{\alpha\in(0,y]} {^{\alpha+}A_Q}, \sup_{\alpha\in(y,1]} {^{\alpha+}A_Q}]$.
Hence, $(\cup_{\alpha\in[0,1]}\ {_{\alpha+}A_Q})\ (u,q) = \sup_{\alpha\in(0,y]} \alpha = y = \mu^i_{A_Q}(u,q), i = 1,2,\dots,k$.
As the same argument is valid for each $(u,q) \in A_Q$, it follows that each multi Q-fuzzy set can be uniquely represented as the family of all its strong α −cuts.

Our presentions in the prelimineries and α-cuts in multi Q-fuzzy sets leads us to establish our results which is the introduction of inverse α −cuts and its distinctive features in multi Q-fuzzy sets.

RESULTS
In this section the concept of inverse α-Cuts and their properties in multi Q-fuzzy sets were introduced. It is shown that both first and second decomposition theorems of a multi Q-fuzzy set fails even though they hold in terms of α −cuts. Also, unlike α −cuts, it was presented that, a multi Q-fuzzy set cannot be uniquely represented as the family of all its weak inverse α −cuts.

Inverse α-cuts in Multi Q-fuzzy set
Based on the prelimineries and the ideas from α −cuts in multi Q-fuzzy sets the following Definition is presented.
Definition 3.1.1
Let $A_Q \in M^kQF(U)$ and $\alpha \in [0,1]$. Then the inverse α −cut of A_Q, denoted $^{\alpha}A_Q^{-1}$ is defined as
$^{\alpha}A_Q^{-1} = \left\{(u,q)/A_m : \mu_{A_Q}(u,q) < \alpha\right\}$ with A_m the cardinality of $\mu(u,q)$ such that $\mu(u,q) < \alpha$.
The weak inverse α −cut of A_Q, denoted $^{\alpha-}A_Q^{-1}$ is defined as

${}^{\alpha-}A_Q^{-1} = \left\{(u,q)/A_m \colon \mu_{A_Q}(u,q) \le \alpha\right\}$ with A_m the cardinality of $\mu(u,q)$ such that $\mu(u,q) \le \alpha$.
Exemplifying these we have.

Example 3.1.2

Let $U = \{u_1, u_2\}$, $Q = \{p, q, r\}$ and a multi Q fuzzy set A_Q over U and Q be

$$A_Q = \{((u_1,p),0.2,0.7,0.1),((u_1,q),0.6,0.2,0.1),((u_2,p),0.2,0.4,0.3),((u_2,q),0.2,0.8,0),((u_2,r),0.3,0.3,0.1)\}.$$

Then ${}^{0.1}A_Q^{-1} = \{(u_2,q)/1\}$, ${}^{0.2}A_Q^{-1} = \{(u_1,p)/1,(u_1,q)/1,(u_2,q)/1,(u_2,r)/1\}$, ${}^{0.5}A_Q^{-1} = \{(u_1,p)/2,(u_1,q)/2,(u_2,p)/3,(u_2,q)/2,(u_2,r)/3\}$, ${}^{0.7}A_Q^{-1} = \{(u_1,p)/2,(u_1,q)/3,(u_2,p)/3,(u_2,q)/2,(u_2,r)/3\}$, and ${}^{0.8}A_Q^{-1} = \{(u_1,p)/3,(u_1,q)/3,(u_2,p)/3,(u_2,q)/2,(u_2,r)/3\}$.

Moreover, ${}^{0.1-}A_Q^{-1} = \{(u_1,p)/1,(u_1,q)/1,(u_2,q)/1,(u_2,r)/1\}$, ${}^{0.3-}A_Q^{-1} = \{(u_1,p)/2,(u_1,q)/2,(u_2,p)/2,(u_2,q)/2,(u_2,r)/3\}$, ${}^{0.4-}A_Q^{-1} = \{(u_1,p)/2,(u_1,q)/2,(u_2,p)/3,(u_2,q)/2,(u_2,r)/3\}$ and ${}^{0.9-}A_Q^{-1} = \{(u_1,p)/3,(u_1,q)/3,(u_2,p)/3,(u_2,q)/3,(u_2,r)/3\}$.

Remark 3.1.3 First Decomposition theorem of a multi Q-fuzzy set fails.

Example 3.1.4

Let A_Q be as of example 3.1.2, then we have its distinct inverse α-cuts defined by characteristic functions considered as special membership functions as:

$${}^{0.1}A_Q^{-1} = \{((u_1,p),0,0,0),((u_1,q),0,0,0),((u_2,p),0,0,0),((u_2,q),0,0,1),((u_2,r),0,0,0)\}.$$
$${}^{0.2}A_Q^{-1} = \{((u_1,p),0,0,1),((u_1,q),0,0,1),((u_2,p),0,0,0),((u_2,q),0,0,1),((u_2,r),0,0,1)\}.$$
$${}^{0.3}A_Q^{-1} = \{((u_1,p),1,0,1),((u_1,q),0,1,1),((u_2,p),1,0,0),((u_2,q),1,0,1),((u_2,r),0,0,1)\}.$$
$${}^{0.4}A_Q^{-1} = \{((u_1,p),1,0,1),((u_1,q),0,1,1),((u_2,p),1,0,1),((u_2,q),1,0,1),((u_2,r),1,1,1)\}.$$
$${}^{0.5}A_Q^{-1} = \{((u_1,p),1,0,1),((u_1,q),0,1,1),((u_2,p),1,1,1),((u_2,q),1,0,1),((u_2,r),1,1,1)\}.$$
$${}^{0.6}A_Q^{-1} = \{((u_1,p),1,0,1),((u_1,q),0,1,1),((u_2,p),1,1,1),((u_2,q),1,0,1),((u_2,r),1,1,1)\}.$$
$${}^{0.7}A_Q^{-1} = \{((u_1,p),1,0,1),((u_1,q),1,1,1),((u_2,p),1,1,1),((u_2,q),1,0,1),((u_2,r),1,1,1)\}.$$
$${}^{0.8}A_Q^{-1} = \{((u_1,p),1,1,1),((u_1,q),1,1,1),((u_2,p),1,1,1),((u_2,q),1,0,1),((u_2,r),1,1,1)\}.$$
$${}^{0.9}A_Q^{-1} = \{((u_1,p),1,1,1),((u_1,q),1,1,1),((u_2,p),1,1,1),((u_2,q),1,1,1),((u_2,r),1,1,1)\}.$$
$${}^{1}A_Q^{-1} = \{((u_1,p),1,1,1),((u_1,q),1,1,1),((u_2,p),1,1,1),((u_2,q),1,1,1),((u_2,r),1,1,1)\}.$$

Thus, converting each of the above inverse α-cuts to a special Multi Q-fuzzy Set ${}_{\alpha}A_Q^{-1}$ defined for each $(u,q) \in A_Q$ as

${}_{\alpha}A_Q = \alpha.({}_{\alpha}A_Q^{-1})$

we get

$${}_{0.1}A_Q^{-1} = \{((u_1,p),0,0,0),((u_1,q),0,0,0),((u_2,p),0,0,0),((u_2,q),0,0,0.1),((u_2,r),0,0,0)\}.$$
$${}_{0.2}A_Q^{-1} = \{((u_1,p),0,0,0.2),((u_1,q),0,0,0.2),((u_2,p),0,0,0),((u_2,q),0,0,0.2),((u_2,r),0,0,0.2)\}.$$
$${}_{0.3}A_Q^{-1} = \{((u_1,p),0.3,0,0.3),((u_1,q),0,0.3,0.3),((u_2,p),0.3,0,0),((u_2,q),0.3,0,0.3),((u_2,r),0,0,0.3)\}.$$
$${}_{0.4}A_Q^{-1} = \left\{\begin{matrix}((u_1,p),0.4,0,0.4),((u_1,q),0,0.4,0.4),((u_2,p),0.4,0,0.4),((u_2,q),0.4,0,0.4),\\((u_2,r),0.4,0.4,0.4)\end{matrix}\right\}.$$
$${}_{0.6}A_Q^{-1} = \left\{\begin{matrix}((u_1,p),0.6,0,0.6),((u_1,q),0,0.6,0.6),((u_2,p),0.6,0.6,0.6),((u_2,q),0.6,0,0.6),\\((u_2,r),0.6,0.6,0.6)\end{matrix}\right\}.$$
$${}_{0.7}A_Q^{-1} = \left\{\begin{matrix}((u_1,p),0.7,0,0.7),((u_1,q),0.7,0.7,0.7),((u_2,p),0.7,0.7,0.7),((u_2,q),0.7,0,0.7),\\((u_2,r),0.7,0.7,0.7)\end{matrix}\right\}.$$
$${}_{0.8}A_Q^{-1} = \left\{\begin{matrix}((u_1,p),0.8,0.8,0.8),((u_1,q),0.8,0.8,0.8),((u_2,p),0.8,0.8,0.8),\\((u_2,q),0.8,0,0.8),((u_2,r),0.8,0.8,0.8)\end{matrix}\right\}.$$

Observed that ${}_{0.1}A_Q^{-1} \cup {}_{0.2}A_Q^{-1} \cup {}_{0.3}A_Q^{-1} \cup {}_{0.4}A_Q^{-1} \cup {}_{0.6}A_Q^{-1} \cup {}_{0.7}A_Q^{-1} \cup {}_{0.8}A_Q^{-1} \ne A_Q$.

In other words, contrary to the case of α-cuts, a Multi Q-fuzzy Set A_Q cannot be represented as the union of its special inverse α-cuts ${}_{\alpha}A_Q^{-1}$.

Remark 3.1.5 Second Decomposition theorem of a multi Q-fuzzy set fails.

Example 3.1.6

Let the following distinct weak inverse α −cut of A_Q in example 3.1.2 defined by characteristic functions be considered as special membership functions:

$$^{0.1-}A_Q^{-1} = \{((u_1,p),0,0,1),((u_1,q),0,0,1),((u_2,p),0,0,0),((u_2,q),0,0,1),((u_2,r),0,0,1)\}.$$
$$^{0.2-}A_Q^{-1} = \{((u_1,p),1,0,1),((u_1,q),0,1,1),((u_2,p),1,0,0),((u_2,q),1,0,1),((u_2,r),0,0,1)\}.$$
$$^{0.3-}A_Q^{-1} = \{((u_1,p),1,0,1),((u_1,q),0,1,1),((u_2,p),1,0,1),((u_2,q),1,0,1),((u_2,r),1,1,1)\}.$$
$$^{0.4-}A_Q^{-1} = \{((u_1,p),1,0,1),((u_1,q),0,1,1),((u_2,p),1,1,1),((u_2,q),1,0,1),((u_2,r),1,1,1)\}.$$
$$^{0.6-}A_Q^{-1} = \{((u_1,p),1,0,1),((u_1,q),1,1,1),((u_2,p),1,1,1),((u_2,q),1,0,1),((u_2,r),1,1,1)\}.$$
$$^{0.7-}A_Q^{-1} = \{((u_1,p),1,1,1),((u_1,q),1,1,1),((u_2,p),1,1,1),((u_2,q),1,0,1),((u_2,r),1,1,1)\}.$$
$$^{0.8-}A_Q^{-1} = \{((u_1,p),1,1,1),((u_1,q),1,1,1),((u_2,p),1,1,1),((u_2,q),1,1,1),((u_2,r),1,1,1)\}.$$

Thus, converting each of the above weak inverse α-cuts to a special Multi Q-fuzzy Set ${}_{\alpha-}A_Q^{-1}$ defined for each $(u,q) \in A_Q$ as ${}_{\alpha}A_Q = \alpha.({}_{\alpha-}A_Q^{-1})$

we get

$${}_{0.1-}A_Q^{-1} = \{((u_1,p),0,0,0.1),((u_1,q),0,0,0.1),((u_2,p),0,0,0),((u_2,q),0,0,0.1),((u_2,r),0,0,0.1)\}.$$
$${}_{0.2-}A_Q^{-1} = \{((u_1,p),0.2,0,0.2),((u_1,q),0,0.2,0.2),((u_2,p),0.2,0,0),((u_2,q),0.2,0,0.2),((u_2,r),0,0,0.2)\}.$$
$${}_{0.3-}A_Q^{-1} = \{((u_1,p),0.3,0,0.3),((u_1,q),0,0.3,0.3),((u_2,p),0.3,0,0.3),((u_2,q),0.3,0,0.3),((u_2,r),0.3,0.3,0.3)\}.$$
$${}_{0.4-}A_Q^{-1} = \left\{\begin{matrix}((u_1,p),0.4,0,0.4),((u_1,q),0,0.4,0.4),((u_2,p),0.4,0.4,0.4),\\ ((u_2,q),0.4,0,0.4),((u_2,r),0.4,0.4,0.4)\end{matrix}\right\}.$$
$${}_{0.6-}A_Q^{-1} = \left\{\begin{matrix}((u_1,p),0.6,0,0.6),((u_1,q),0.6,0.6,0.6),((u_2,p),0.6,0.6,0.6),\\ ((u_2,q),0.6,0,0.6),((u_2,r),0.6,0.6,0.6)\end{matrix}\right\}.$$
$${}_{0.7-}A_Q^{-1} = \left\{\begin{matrix}((u_1,p),0.7,0.7,0.7),((u_1,q),0.7,0.7,0.7),((u_2,p),0.7,0.7,0.7),\\ ((u_2,q),0.7,0,0.7),((u_2,r),0.7,0.7,0.7)\end{matrix}\right\}.$$
$${}_{0.8-}A_Q^{-1} = \left\{\begin{matrix}((u_1,p),0.8,0.8,0.8),((u_1,q),0.8,0.8,0.8),((u_2,p),0.8,0.8,0.8),\\ ((u_2,q),0.8,0.8,0.8),((u_2,r),0.8,0.8,0.8)\end{matrix}\right\}.$$

Clearly, ${}_{0.1-}A_Q^{-1} \cup {}_{0.2-}A_Q^{-1} \cup {}_{0.3-}A_Q^{-1} \cup {}_{0.4-}A_Q^{-1} \cup {}_{0.6-}A_Q^{-1} \cup {}_{0.7-}A_Q^{-1} \cup {}_{0.8-}A_Q^{-1} \neq A_Q$.

Thus, each multi Q-fuzzy set cannot be uniquely represented as the family of all its weak inverse α −cuts.

After establishing that both first and second decomposition theorems of a multi Q-fuzzy set fails, we then check the status of commutativity, associativity and containment in terms of inverse and weak inverse α −cuts, which we discovered they all hold by the following theorem.

Theorem 3.1.7 Let $A_Q, B_Q, C_Q \in M^kQF(U)$ and $\alpha \in [0,1]$. Then

(i) ${}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1} = {}^{\alpha}B_Q{}^{-1} \cup {}^{\alpha}A_Q{}^{-1}$,

(ii) ${}^{\alpha}A_Q{}^{-1} \cap {}^{\alpha}B_Q{}^{-1} = {}^{\alpha}B_Q{}^{-1} \cap {}^{\alpha}A_Q{}^{-1}$,

(iii) ${}^{\alpha-}A_Q{}^{-1} \cup {}^{\alpha-}B_Q{}^{-1} = {}^{\alpha-}B_Q{}^{-1} \cup {}^{\alpha-}A_Q{}^{-1}$,

(iv) ${}^{\alpha-}A_Q{}^{-1} \cap {}^{\alpha-}B_Q{}^{-1} = {}^{\alpha-}B_Q{}^{-1} \cap {}^{\alpha-}A_Q{}^{-1}$,

(v) ${}^{\alpha}A_Q{}^{-1} \cup ({}^{\alpha}B_Q{}^{-1} \cup {}^{\alpha}C_Q{}^{-1}) = ({}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1}) \cup {}^{\alpha}C_Q{}^{-1}$,

(vi) ${}^{\alpha}A_Q{}^{-1} \cap ({}^{\alpha}B_Q{}^{-1} \cap {}^{\alpha}C_Q{}^{-1}) = ({}^{\alpha}A_Q{}^{-1} \cap {}^{\alpha}B_Q{}^{-1}) \cap {}^{\alpha}C_Q{}^{-1}$,

(vii) ${}^{\alpha-}A_Q{}^{-1} \cup ({}^{\alpha-}B_Q{}^{-1} \cup {}^{\alpha-}C_Q{}^{-1}) = ({}^{\alpha-}A_Q{}^{-1} \cup {}^{\alpha-}B_Q{}^{-1}) \cup {}^{\alpha-}C_Q{}^{-1}$,

(viii) ${}^{\alpha-}A_Q{}^{-1} \cap ({}^{\alpha-}B_Q{}^{-1} \cap {}^{\alpha-}C_Q{}^{-1}) = ({}^{\alpha-}A_Q{}^{-1} \cap {}^{\alpha-}B_Q{}^{-1}) \cap {}^{\alpha-}C_Q{}^{-1}$,

(ix) ${}^{\alpha}A_Q{}^{-1} \subseteq {}^{\alpha-}A_Q{}^{-1}$,

(x) If $\alpha_1 \leq \alpha_2 \Longrightarrow {}^{\alpha_1}A_Q{}^{-1} \subseteq {}^{\alpha_2}A_Q{}^{-1}$,

(xi) If $\alpha_1 \leq \alpha_2 \Longrightarrow {}^{\alpha_1-}A_Q{}^{-1} \subseteq {}^{\alpha_2-}A_Q{}^{-1}$.

Proof
(i) Let ${}^{\alpha}C_Q{}^{-1} = {}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1}$ and let $(u,q)/C_m \in {}^{\alpha}C_Q{}^{-1}$
$\Rightarrow (u,q)/C_m \in {}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1}$
$\Rightarrow (u,q)/C_m \in {}^{\alpha}A_Q{}^{-1}$ or $(u,q)/C_m \in {}^{\alpha}B_Q{}^{-1}$
i.e., $(u,q)/A_m \in {}^{\alpha}A_Q{}^{-1}$ or $(u,q)/B_m \in {}^{\alpha}B_Q{}^{-1}$
$\Rightarrow \mu_{A_Q}(u,q) < \alpha,\ \forall\big((u,q),\mu_i(u,q)\big) \in A_Q$ or $\mu_{B_Q}(u,q) < \alpha,\ \forall((u,q),\mu_i(u,q)) \in B_Q$
$\Rightarrow \mu_{B_Q}(u,q) < \alpha,\ \forall\big((u,q),\mu_i(u,q)\big) \in B_Q$ or $\mu_{A_Q}(u,q) < \alpha,\ \forall((u,q),\mu_i(u,q)) \in A_Q$
$\Rightarrow (u,q)/B_m \in {}^{\alpha}B_Q{}^{-1}$ or $(u,q)/A_m \in {}^{\alpha}A_Q{}^{-1}$
$\Rightarrow (u,q)/C_m \in {}^{\alpha}B_Q{}^{-1}$ or $(u,q)/C_m \in {}^{\alpha}A_Q{}^{-1}$
i.e., $(u,q)/C_m \in {}^{\alpha}B_Q{}^{-1} \cup {}^{\alpha}A_Q{}^{-1}$
i.e., ${}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1} \subseteq {}^{\alpha}B_Q{}^{-1} \cup {}^{\alpha}A_Q{}^{-1}$
Similarly, ${}^{\alpha}B_Q{}^{-1} \cup {}^{\alpha}A_Q{}^{-1} \subseteq {}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1}$.

The proofs of (ii), (iii) and (iv) are analogous to that of (i).
(v) Let $\left({}^{\alpha}B_Q{}^{-1} \cup {}^{\alpha}C_Q{}^{-1}\right) = {}^{\alpha}D_Q{}^{-1}$ and $\left({}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1}\right) = {}^{\alpha}F_Q{}^{-1}$.
Let ${}^{\alpha}A_Q{}^{-1} \cup \left({}^{\alpha}B_Q{}^{-1} \cup {}^{\alpha}C_Q{}^{-1}\right) = {}^{\alpha}E_Q{}^{-1}$ and let $(u,q)/E_m \in {}^{\alpha}E_Q{}^{-1}$
$\Rightarrow (u,q)/A_m \in {}^{\alpha}A_Q{}^{-1}$ or $(u,q)/D_m \in \left({}^{\alpha}B_Q{}^{-1} \cup {}^{\alpha}C_Q{}^{-1}\right)$
$\Rightarrow (u,q)/A_m \in {}^{\alpha}A_Q{}^{-1}$ or $(u,q)/B_m \in {}^{\alpha}B_Q{}^{-1}$ or $(u,q)/C_m \in {}^{\alpha}C_Q{}^{-1}$
$\Rightarrow \mu_{A_Q}(u,q) < \alpha, \forall\big((u,q),\mu_i(u,q)\big) \in A_Q$ or $\mu_{B_Q}(u,q) < \alpha, \forall((u,q),\mu_i(u,q)) \in B_Q$ or
$\mu_{C_Q}(u,q) < \alpha, \forall((u,q),\mu_i(u,q)) \in C_Q$
$\Rightarrow \mu_{F_Q}(u,q) < \alpha, \forall\big((u,q),\mu_i(u,q)\big) \in F_Q$ or $\mu_{C_Q}(u,q) < \alpha, \forall((u,q),\mu_i(u,q)) \in C_Q$
$\Rightarrow (u,q)/F_m \in \left({}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1}\right)$ or $(u,q)/C_m \in {}^{\alpha}C_Q{}^{-1}$
$\Rightarrow (u,q)/E_m \in \left({}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1}\right) \cup {}^{\alpha}C_Q{}^{-1}$
$\Rightarrow {}^{\alpha}A_Q{}^{-1} \cup \left({}^{\alpha}B_Q{}^{-1} \cup {}^{\alpha}C_Q{}^{-1}\right) \subseteq \left({}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1}\right) \cup {}^{\alpha}C_Q{}^{-1}$
Similarly, $\left({}^{\alpha}A_Q{}^{-1} \cup {}^{\alpha}B_Q{}^{-1}\right) \cup {}^{\alpha}C_Q{}^{-1} \subseteq {}^{\alpha}A_Q{}^{-1} \cup \left({}^{\alpha}B_Q{}^{-1} \cup {}^{\alpha}C_Q{}^{-1}\right)$
The proofs of (vi) and (vii) are analogous to that of (v).
(viii) Let $\left({}^{\alpha-}B_Q{}^{-1} \cap {}^{\alpha-}C_Q{}^{-1}\right) = {}^{\alpha-}H_Q{}^{-1}$ and $\left({}^{\alpha-}A_Q{}^{-1} \cap {}^{\alpha-}B_Q{}^{-1}\right) = {}^{\alpha-}J_Q{}^{-1}$.
Let ${}^{\alpha-}K_Q{}^{-1} = {}^{\alpha-}A_Q{}^{-1} \cap \left({}^{\alpha-}B_Q{}^{-1} \cap {}^{\alpha-}C_Q{}^{-1}\right)$ and for any $(u,q)/K_m \in {}^{\alpha-}K_Q{}^{-1}$
we have $(u,q)/A_m \in {}^{\alpha-}A_Q{}^{-1}$ and $(u,q)/H_m \in \left({}^{\alpha-}B_Q{}^{-1} \cap {}^{\alpha-}C_Q{}^{-1}\right)$
$\Rightarrow (u,q)/A_m \in {}^{\alpha-}A_Q{}^{-1}$ and $(u,q)/B_m \in {}^{\alpha-}B_Q{}^{-1}$ and $(u,q)/C_m \in {}^{\alpha-}C_Q{}^{-1}$
$\Rightarrow \mu_{A_Q}(u,q) \le \alpha, \forall\big((u,q),\mu_i(u,q)\big) \in A_Q$ and $\mu_{B_Q}(u,q) \le \alpha, \forall((u,q),\mu_i(u,q)) \in B_Q$ and $\mu_{C_Q}(u,q) \le \alpha, \forall((u,q),\mu_i(u,q)) \in C_Q$
$\Rightarrow \mu_{J_Q}(u,q) \le \alpha, \forall\big((u,q),\mu_i(u,q)\big) \in J_Q$ and $\mu_{C_Q}(u,q) \le \alpha, \forall((u,q),\mu_i(u,q)) \in C_Q$
$\Rightarrow (u,q)/J_m \in \left({}^{\alpha-}A_Q{}^{-1} \cap {}^{\alpha-}B_Q{}^{-1}\right)$ and $(u,q)/C_m \in {}^{\alpha-}C_Q{}^{-1}$
$\Rightarrow (u,q)/K_m \in \left({}^{\alpha-}A_Q{}^{-1} \cap {}^{\alpha-}B_Q{}^{-1}\right) \cap {}^{\alpha-}C_Q{}^{-1}$
$\Rightarrow {}^{\alpha-}A_Q{}^{-1} \cap \left({}^{\alpha-}B_Q{}^{-1} \cap {}^{\alpha-}C_Q{}^{-1}\right) \subseteq \left({}^{\alpha-}A_Q{}^{-1} \cap {}^{\alpha-}B_Q{}^{-1}\right) \cap {}^{\alpha-}C_Q{}^{-1}$
Similarly, $\left({}^{\alpha-}A_Q{}^{-1} \cap {}^{\alpha-}B_Q{}^{-1}\right) \cap {}^{\alpha-}C_Q{}^{-1} \subseteq {}^{\alpha-}A_Q{}^{-1} \cap \left({}^{\alpha-}B_Q{}^{-1} \cap {}^{\alpha-}C_Q{}^{-1}\right)$.
(ix) Let $(u,q)/A_m \in {}^{\alpha}A_Q{}^{-1} \Rightarrow \mu_{A_Q}(u,q) < \alpha, \forall((u,q),\mu_i(u,q)) \in A_Q$.
$\Rightarrow \mu_{A_Q}(u,q) \le \alpha, \forall((u,q),\mu_i(u,q)) \in A_Q$
$\Rightarrow (u,q)/A_m \in {}^{\alpha-}A_Q{}^{-1}$
i.e., ${}^{\alpha}A_Q{}^{-1} \subseteq {}^{\alpha-}A_Q{}^{-1}$.
(x) $(u,q)/A_m \in {}^{\alpha_1}A_Q{}^{-1} \Rightarrow \mu_{A_Q}(u,q) < \alpha_1, \forall((u,q),\mu_i(u,q)) \in A_Q$,
$\Rightarrow \mu_{A_Q}(u,q) < \alpha_2, \forall((u,q),\mu_i(u,q)) \in A_Q$ since $\alpha_1 \le \alpha_2$
$\Rightarrow (u,q)/A_m \in {}^{\alpha_2}A_Q{}^{-1}$,
i.e., ${}^{\alpha_1}A_Q{}^{-1} \subseteq {}^{\alpha_2}A_Q{}^{-1}$.
(xi) $(u,q)/A_m \in {}^{\alpha_1-}A_Q{}^{-1} \Rightarrow \mu_{A_Q}(u,q) \le \alpha_1, \forall((u,q),\mu_i(u,q)) \in A_Q$,
$\Rightarrow \mu_{A_Q}(u,q) \le \alpha_2, \forall((u,q),\mu_i(u,q)) \in A_Q$ since $\alpha_1 \le \alpha_2$
$\Rightarrow (u,q)/A_m \in {}^{\alpha_2-}A_Q{}^{-1}$,
i.e., ${}^{\alpha_1-}A_Q{}^{-1} \subseteq {}^{\alpha_2-}A_Q{}^{-1}$.

DISCUSSION

The idea of α-cuts and their properties in Multi Q-fuzzy sets were presented in (Isah, 2019) where various theorems were established and proved. In this work we introduced inverse α-cut in multi Q-fuzzy set and compare it with α-cut in (Isah, 2019). We discovered that similar to α-cut; commutativity, associativity and containment in terms of inverse and weak inverse α −cuts hold. Also, some theorems such as First and second Decomposition theorems of a multi Q-fuzzy set which hold in α-cuts actually fails in the inverse α-cuts. Furthermore, contrary to the case of α-cuts, a Multi Q-fuzzy set cannot be uniquely represented either as the union of its inverse α-cuts nor as the family of all its weak inverse α −cuts. The implications of these results are that a multi Q-fuzzy set A_Q can be represented as the union of its α-cuts and strong α-cuts in a unique way while it cannot be uniquely represented as the union of its inverse α-cuts and weak inverse α-Cuts which shows that some properties holding in α-cuts fails in inverse α-cuts of a multi Q-fuzzy set.

CONCLUSION

Inverse α-Cuts and their properties in multi Q-fuzzy sets are described. It is shown that unlike α-Cuts, multi Q-fuzzy sets cannot be decomposed into its inverse α-cuts or into its weak inverse α-cuts.

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