BAJOPAS Volume 14 Number 2, December, 2021

http://dx.doi.org/10.4314/bajopas.v14i2.5

Bayero Journal of Pure and Applied Sciences, 14(2): 37 - 51

Received: October, 2021

Accepted: December, 2021

ISSN 2006 – 6996

SOLUTION OF FIRST ORDER FUZZY PARTIAL DIFFERENTIAL EQUATIONS BY FUZZY LAPLACE TRANSFORM METHOD

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ABSTRACT

In this study, first order fuzzy partial differential equations with negative coefficient on one hand and with both positive and negative coefficients on the other hand are solved using fuzzy Laplace transform method. The results obtained indicate that all the solutions exist and the examples presented illustrate the applicability of the method.

INTRODUCTION

Fuzzy differential equations (FDEs) appeared as a natural way to model the propagation of epistemic (relating to) uncertainty in a dynamical environment (Bede, 2013). The idea of fuzzy number and fuzzy arithmetic was first introduced by Zadeh (1965) followed by Dubois and Parade (1978). Work that involved fuzzy derivative was first introduced by Chang and Zadeh (1972). Kaleva (1987) was first to formulate FDEs and subsequently, the idea was extended to include fuzzy partial derivatives by Buckley and Feuring (1999). However, Hukuhara derivative, Zadehs extension principle and fuzzy differential inclusions are the several ways that FDEs can be interpreted (Bede, 2013).

Laplace transform is a widely used integral transform in mathematics with many applications in science and engineering (Sawant, 2018). The Laplace transform of an expression $f(t)$ denoted by $L\{f(t)\}$ and defined as a semi-infinite integral is in the form

$$L\{f(t)\} = \int_0^\infty e^{-pt} f(t)dt.$$

The parameter p assumed to be positive and large enough to ensure that the integral converges. The Laplace transform $f(t)$ is said to exist if the integral $\int_0^\infty f(t)e^{-pt}\,dt$ converges for all values of p, otherwise it does not exist (Gomes, *et al.*, 2015). The fuzzy Laplace transform method (FLTM) for solving fuzzy partial differential equations (FPDEs) has been a subject of investigation by researchers like Eljaoui and Melliani (2016) and Ullah *et al.* (2018) where they initiated FLTM for the solution of FPDEs.

Eijaoui and Melliani (2016) proposed some theorems for the continuity and differentiability of a fuzzy valued function defined through a fuzzy improper Riemann integral which was used to prove some results concerning FLTM in fuzzy environment. They also generalized the LTM for first order FPDEs with positive constant coefficients $c \geq 0$ under the strongly generalized Hukuhara differentiability concept where the the solutions of $y(x,t)$ for some cases were obtained. Their work did not consider negative coefficients i.e. $c < 0$. This study addresses the condition for $c < 0$, which gives results for negative coefficients only, it is also discovered that gap may still exist if we consider only $c < 0$ or only $c \geq 0$ which will be taken care of if we consider $c \geq 0$ and $c < 0$ together for both positive and negative coefficients respectively.

MATERIAL AND METHODS

Existing Theorems on Results of Some FPDEs

Theorem 2.1 (Eljaoui & Melliani, 2016): Let $f;\Re \to F(\Re)$ be a continuous fuzzy function and $f(t) = \left[\underline{f}(t,r), \overline{f}(t,r)\right]$ for every $r \in [0,1]$, the following hold.

(i) If the fuzzy function $f(t)$ is (i)-differentiable then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are both differentiable and $f'(t) = \left[\underline{f}'(t,r), \overline{f}'(t,r)\right]$.

(ii) If the fuzzy function $f(t)$ is (ii)-differentiable then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are both differentiable and $f'(t)=\left[\overline{f}'(t,r),\underline{f}'(t,r)\right]$.

Theorem 2.2 (Ullah *et al,* 2018): Let $y:(a,b)\times(a,b)\to E$ be a fuzzy valued function such that its derivatives up to the n^{th} order with respect to t are continuous for all $t>0$ and y^n exists, then

$$L\left[\frac{\partial^n}{\partial t^n}y(x,t)\right]=p^n y(x,p)\ominus p^{n-1}y(x,0)\ominus p^{n-2}y'(x,0)\ominus\ldots\ominus y^{n-1}(x,0)$$

and

$$L\left[y_{x^n}(x,t)\right]=\frac{\partial^n}{\partial x^n}L\left[y(x,t)\right]=\frac{d^n}{dx^n}Y(x,p).$$

Existing Method for Solution of First Order FPDEs by FLTM due to Eijaoui and Melliani (2016)

Eijaoui and Melliani (2016) developed a method for solving first order FPDEs. Therefore, consider the equation below

$$y_x(x,t)+cy_t(x,t)=f(x,t,y(x,t)) \quad (2.1)$$

with initial condition

$$y(x,0;r)=\left(\underline{g}(x,r),\overline{g}(x,r)\right)$$

and boundary condition

$$y(0,t,r)=\left(\underline{h}(t,r),\overline{h}(t,r)\right)$$

where $y_x(x,t)$ and $y_t(x,t)$ are fuzzy valued functions for $x\geq 0,\ t\geq 0$, c is a real constant and $f(x,t,y(x,t))$, is a fuzzy valued function, $\underline{g}(x,r)$, $\underline{h}(x,r)$ are the lower cases for the initial and the boundary condition and $\overline{g}(t,r)$, $\overline{h}(t,r)$ are the upper cases for the initial and boundary conditions and $f(x,t,y(x,t))$ is linear with respect to y. Assume $c\geq 0$ and applying LTM on equation (2.1) we have

$$L\left[y_x(x,t)\right]+cL\left[y_t(x,t)\right]=L\left[f(x,t,y(x,t))\right]. \quad (2.2)$$

Four cases aroused in respect of equation (2.2) and used to obtain the solutions of equation (2.1).

Case 1: If y is (i)-differentiable with respect to both x and t, then by Theorem (2.1) equation (2.2) become

$$L\left[\underline{y}_x(x,t,r)\right]+cL\left[\underline{y}_t(x,t,r)\right]=L\left[\underline{f}(x,t,y(x,t),r)\right], \quad (2.3)$$

$$L\left[\overline{y}_x(x,t,r)\right]+cL\left[\overline{y}_t(x,t,r)\right]=L\left[\overline{f}(x,t,y(x,t),r)\right] \quad (2.4)$$

where $\underline{f}(x,t,y(x,t),r)$ = min$\left\{\underline{f}(x,t,v)\backslash v\in\left[\underline{y}(x,t,r),\overline{y}(x,t,r)\right]\right\}$

and

$\overline{f}(x,t,y(x,t),r)$= max$\left\{\overline{f}(x,t,v)\backslash v\in\left[\overline{y}(x,t,r),\overline{y}(x,t,r)\right]\right\}$.

By Theorem (2.2), equations (2.3) and (2.4) respectively become

$$\frac{\partial}{\partial x}\left(L\left[\underline{y}(x,t,r)\right]\right)+cpL\left[\underline{y}(x,t,r)\right]=c\underline{g}(x,r)+L\left[\underline{f}(x,t,y(x,t),r)\right] \quad (2.5)$$

and

$$\frac{\partial}{\partial x}\left(L\left[\overline{y}(x,t,r)\right]\right)+cpL\left[\overline{y}(x,t,r)\right]=c\overline{g}(x,r)+L\left[\overline{f}(x,t,y(x,t),r)\right]. \quad (2.6)$$

Equations (2.5) and (2.6) satisfy the following boundary conditions;

$$L\left[\underline{y}(0,t,r)\right]=L\left[\underline{h}(t,r)\right] \quad (2.7)$$

and

$$L\left[\overline{y}(0,t,r)\right]=L\left[\overline{h}(t,r)\right] \quad (2.8)$$

respectively. Assuming that the solutions of equations (2.5) and (2.6) are given as

$$L\left[\underline{y}(x,t,r)\right]=H_1(p,r) \quad (2.9)$$

and

$$L\left[\overline{y}(x,t,r)\right]=K_1(p,r). \quad (2.10)$$

Now, taking the inverse Laplace transform of (2.9) and (2.10), we have

$$\underline{y}(x,t,r)=L^{-1}[H_1(p,r)], \quad (2.11)$$

and

$$\overline{y}(x,t,r)=L^{-1}[K_1(p,r)]. \quad (2.12)$$

Case 2: if y is (i)-differentiable with respect to x and (ii) differentiable with respect to t. Theorem (2.1) and (2.2) hold for equation (2.2) with equations (2.7) and (2.8) satisfied respectively. Consequently, the equations obtained below are thus

$$\frac{\partial}{\partial x}\left(L\left[\underline{y}(x,t,r)\right]\right)+cpL\left[\overline{y}(x,t,r)\right]=c\overline{g}(x,r)+L\left[\underline{f}(x,t,y(x,t),r)\right] \quad (2.13)$$

and

$$\frac{\partial}{\partial x}\left(L\left[\overline{y}(x,t,r)\right]\right)+cpL\left[\underline{y}(x,t,r)\right]=c\underline{g}(x,r)+L\left[\overline{f}(x,t,y(x,t),r)\right]. \quad (2.14)$$

Assuming that the solutions of equations (2.13) and (2.14) are given as

$$L\left[\underline{y}(x,t,r)\right]=H_2(p,r) \quad (2.15)$$

and

$$L\left[\overline{y}(x,t,r)\right]=K_2(p,r) \quad (2.16)$$

respectively. Taking the inverse Laplace transform of equations (2.15) and (2.16), the results obtained are

$$\underline{y}(x,t,r)=L^{-1}[H_2(p,r)], \quad (2.17)$$

and

$$\overline{y}(x,t,r)=L^{-1}[K_2(p,r)]. \quad (2.18)$$

Case 3: If y is (ii)-differentiable with respect to x and (i)-differentiable with respect to t. As indicated in the last two cases considered above, the following results are also true.

$$\frac{\partial}{\partial x}\left(L\left[\overline{y}(x,t,r)\right]\right)+cpL\left[\underline{y}(x,t,r)\right]=c\underline{g}(x,r)+L\left[\underline{f}(x,t,y(x,t),r)\right] \quad (2.19)$$

and

$$\frac{\partial}{\partial x}\left(L\left[\underline{y}(x,t,r)\right]\right)+cpL\left[\overline{y}(x,t,r)\right]=c\overline{g}(x,r)+L\left[\overline{f}(x,t,y(x,t),r)\right] \quad (2.20)$$

respectively. Assuming that the solutions of equations (2.19) and (2.20) are given as

$$L\left[\underline{y}(x,t,r)\right]=H_3(p,r) \quad (2.21)$$

and

$$L\left[\overline{y}(x,t,r)\right]=K_3(p,r) \quad (2.22)$$

Next is to take the inverse Laplace transform of equations (2.21) and (2.22), the following are arrived at;

$$\underline{y}(x,t,r)=L^{-1}[H_3(p,r)] \quad (2.23)$$

and

$$\overline{y}(x,t,r)=L^{-1}[K_3(p,r)]. \quad (2.24)$$

Case 4: If y is (ii)-differentiable with respect to x and t. After considering Theorem (2.1), (2.2) and the boundary conditions therein, the results below are therefore

$$\frac{\partial}{\partial x}\left(L\left[\overline{y}(x,t,r)\right]\right)+cpL\left[\overline{y}(x,t,r)\right]=c\overline{g}(x,r)+L\left[\underline{f}(x,t,y(x,t),r)\right] \quad (2.25)$$

and

$$\frac{\partial}{\partial x}\left(L\left[\underline{y}(x,t,r)\right]\right)+cpL\left[\underline{y}(x,t,r)\right]=c\underline{g}(x,r)+L\left[\overline{f}\left(x,t,y(x,t),r\right)\right]. \quad (2.26)$$

Assuming that the solutions of equations (3.25) and (3.26) are given as

$$L\left[\underline{y}(x,t,r)\right]=H_4(p,r), \quad (2.27)$$

and

$$L\left[\overline{y}(x,t,r)\right]=K_4(p,r). \quad (2.28)$$

Hence taking the inverse Laplace transform of equations (2.27) and (2.28) respectively, we have

$$\underline{y}(x,t,r)=L^{-1}[H_4(p,r)] \quad (2.29)$$

and

$$\overline{y}(x,t,r)=L^{-1}[K_4(p,r)]. \quad (2.30)$$

RESULTS AND DISCUSSION

Obtaining Solutions of First Order FPDEs for $c<0$ by FLTM

Eljaoui and Melliani (2016) established the result for first order FPDEs with only positive coefficients i.e. $c\geq 0$, to bridge the gaps in their work, we considered a case of negative coefficients $c<0$, presented below.

Consider equation (2.1) but in this case with negative coefficient c. Assume $c=-a$ and $f\left(x,t,y(x,t)\right)=f(x,r)$ we have

$$y_x(x,t)=-ay_t(x,t)+f(x,r) \quad (3.4)$$

with fuzzy initial conditions

$$y(x,0,r)=\left(\underline{s}(x,r),\overline{s}(x,r)\right) \quad (3.5)$$

and fuzzy boundary conditions

$$y(0,t,r)=\left(\underline{u}(t,r),\overline{u}(t,r)\right) \quad (3.6)$$

where $f(x,r)$ is a fuzzy valued function, $\underline{s}(x,r)$ and $\overline{s}(x,r)$ are the lower and upper cases for the appropriate initial conditions respectively, also $\underline{u}(t,r)$ and $\overline{u}(t,r)$ are the lower and upper cases of the appropriate boundary condition respectively, for $t\geq 0$, $x\geq 0$ and $r\in[0,1]$.

Taking Laplace transform of equation (3.4) as expressed in equation (2.2), we have

$$L\left[y_x(x,t)\right]=-aL\left[y_t(x,t)\right]+L\left[f(x,r)\right]. \quad (3.7)$$

Four cases arise as consequence of equation (3.7) which is similar to the cases related to equation (2.3)

Case 1: When y is (i)-differentiable with respect to both x and t. We apply Theorem (2.1) on equation (3.7), to have

$$L\left[\underline{y}_x(x,t)\right]=-aL\left[\underline{y}_t(x,t)\right]+L\left[\underline{f}(x,r)\right] \quad (3.8)$$

and

$$L\left[\overline{y}_x(x,t)\right]=-aL\left[\overline{y}_t(x,t)\right]+L\left[\overline{f}(x,r)\right]. \quad (3.9)$$

Applying theorem (2.2) on equations (3.8) and (3.9) after which we substituting the initial condition (3.5) into it respectively, gives

$$\frac{d}{dx}\left(\underline{Y}(x,p,r)\right)=-ap\underline{Y}(x,p,r)+a\underline{s}(x,r)+\underline{F}(x,p,r) \quad (3.10)$$

and

$$\frac{d}{dx}\left(\overline{Y}(x,p,r)\right)=-ap\overline{Y}(x,p,r)+a\overline{s}(x,r)+\overline{F}(x,p,r). \quad (3.11)$$

Taking the Laplace transform of equation (3.6) gives results similar to that of equations (2.9) and (2.10) respectively. Therefore, we have

$$L\left[\underline{y}(0,t,r)\right]=L\left[\underline{u}(t,r)\right]=\underline{U}(p,r) \tag{3.12}$$

and

$$L\left[\bar{y}(0,t,r)\right]=L\left[\bar{u}(t,r)\right]=\bar{U}(p,r). \tag{3.13}$$

Solving equation (3.10) and (3.11) together with equation (3.12) and (3.13) respectively, we have

$$\underline{Y}(x,p,r)=\frac{\underline{s}(x,r)}{p}-\frac{\underline{s}(r)}{ap^2}+\frac{\underline{F}(x,p,r)}{ap}-\frac{\underline{F}(p,r)}{a^2p^2}+\underline{U}(p,r)e^{-apx}+\frac{\underline{s}(r)e^{-apx}}{ap^2}+\frac{\underline{F}(p,r)e^{-apx}}{a^2p^2} \tag{3.14}$$

and

$$\bar{Y}(x,p,r)=\frac{\bar{s}(x,r)}{p}-\frac{\bar{s}(r)}{ap^2}+\frac{\bar{F}(x,p,r)}{ap}-\frac{\bar{F}(p,r)}{a^2p^2}+\bar{U}(p,r)e^{-apx}+\frac{\bar{s}(r)e^{-apx}}{ap^2}+\frac{\bar{F}(p,r)e^{-apx}}{a^2p^2}, \tag{3.15}$$

Taking the inverse Laplace transform of the equations (3.14) and (3.15) we have

$$\underline{y}(x,t,r)=\underline{s}(x,r)-\frac{\underline{s}(r)t}{a}+\frac{\underline{f}(x,\mathrm{t},r)}{a}-\frac{\underline{F}(\mathrm{t},r)t}{a^2}+\underline{u}(r)(t-ax)H(t-ax)+\frac{\underline{s}(r)}{a}(t-ax)H(t-ax)+\frac{\underline{f}(t,r)}{a^2}(t-ax)H(t-ax)$$

and

$$\bar{y}(x,t,r)=\bar{s}(x,r)-\frac{\bar{s}(r)t}{a}+\frac{\bar{f}(x,\mathrm{t},r)}{a}-\frac{\bar{f}(\mathrm{t},r)t}{a^2}+\bar{u}(r)(t-ax)H(t-ax)+\frac{\bar{s}(r)}{a}(t-ax)H(t-ax)+\frac{\bar{f}(t,r)}{a^2}(t-ax)\mathrm{H}(t-ax),$$

Case 2: When y is (i)-differentiable with respect to x and (ii)-differentiable with respect to t. We apply Theorem (2.1) and (2.2) to equation (3.7) and Substituting the initial condition (4.5) into it, after which we solve it together with equation (3.12) and (3.13), we arrive at

$$\underline{Y}(x,p,r)=\frac{\bar{s}(x,r)}{p}-\frac{\bar{s}(r)}{ap^2}+\frac{\underline{F}(x,p,r)}{ap}-\frac{\underline{F}(p,r)}{a^2p^2}+\underline{U}(p,r)e^{-apx}+\frac{\bar{s}(r)e^{-apx}}{ap^2}+\frac{\underline{F}(p,r)e^{-apx}}{a^2p^2} \tag{3.16}$$

and

$$\bar{Y}(x,p,r)=\frac{\underline{s}(x,r)}{p}-\frac{\underline{s}(r)}{ap^2}+\frac{\bar{F}(x,p,r)}{ap}-\frac{\underline{F}(p,r)}{a^2p^2}+\bar{U}(p,r)e^{-apx}+\frac{\underline{s}(r)e^{-apx}}{ap^2}+\frac{\bar{F}(p,r)e^{-apx}}{a^2p^2}, \tag{3.17}$$

Again, taking the inverse Laplace transform of the equations (3.16) and (3.17) we have

$$\underline{y}(x,t,r)=\bar{s}(x,r)-\frac{\bar{s}(r)t}{a}+\frac{\underline{f}(x,\mathrm{t},r)}{a}-\frac{\underline{f}(t,r)t}{a^2}+\underline{u}(r)(t-ax)H(t-ax)+\frac{\bar{s}(r)}{a}(t-ax)H(t-ax)+\frac{\underline{f}(t,r)}{a^2}(t-ax)H(t-ax)$$

and

$$\overline{y}(x,t,r)=\underline{s}(x,r)-\frac{\underline{s}(r)t}{a}+\frac{\overline{f}(x,\mathrm{t},r)}{a}-\frac{\overline{f}(\mathrm{t},r)t}{a^2}+\overline{u}(r)(t-ax)H(t-ax)+\frac{\underline{s}(r)}{a}(t-ax)H(t-ax)+\frac{\overline{f}(r)}{a^2}(t-ax)H(t-ax).$$

Case 3: When y is (ii)-differentiable with respect to x and (i)-differentiable with respect to t. We apply Theorem (2.1) and (2.2) to equation (3.7) and substituting the initial condition (4.5) into it, after which we solve it together with equation (3.12) and (3.13), we arrive at

$$\overline{Y}(x,p,r)=\frac{\underline{s}(x,r)}{p}-\frac{\underline{s}(r)}{ap^2}+\frac{\underline{F}(x,p,r)}{ap}-\frac{\underline{F}(p,r)}{a^2p^2}+\overline{U}(p,r)e^{-apx}+\frac{\underline{s}(r)e^{-apx}}{ap^2}+\frac{\underline{F}(p,r)e^{-apx}}{a^2p^2} \quad (3.18)$$

and

$$\underline{Y}(x,p,r)=\frac{\overline{s}(x,r)}{p}-\frac{\overline{s}(r)}{ap^2}+\frac{\overline{F}(x,p,r)}{ap}-\frac{\overline{F}(p,r)}{a^2p^2}+\underline{U}(p,r)e^{-apx}+\frac{\overline{s}(r)e^{-apx}}{ap^2}+\frac{\overline{F}(p,r)e^{-apx}}{a^2p^2} \quad (3.19)$$

Also, taking the inverse Laplace transform of the equations (3.18) and (3.19), we have

$$\overline{y}(x,t,r)=\underline{s}(x,r)-\frac{\underline{s}(r)t}{a}+\frac{\underline{f}(x,\mathrm{t},r)}{a}-\frac{\underline{f}(\mathrm{t},r)t}{a^2}+\overline{u}(r)(t-ax)H(t-ax)+\frac{\underline{s}(r)}{a}(t-ax)H(t-ax)-\frac{\underline{f}(r)}{a^2}(t-ax)H(t-ax)$$

and

$$\underline{y}(x,t,r)=\overline{s}(x,r)-\frac{\overline{s}(r)t}{a}+\frac{\overline{f}(x,\mathrm{t},r)}{a}-\frac{\overline{f}(t,r)t}{a^2}+\underline{u}(r)(t-ax)H(t-ax)+\frac{\overline{s}(r)}{a}(t-ax)H(t-ax)+\frac{\overline{f}(r)}{a^2}(t-ax)H(t-ax).$$

Case 4: When y is (ii)-differentiable with respect to both x and t. We apply Theorem (2.1) and (2.2) to equation (3.7) and substituting the initial condition (3.5) into it, after which we solve it together with equation (3.12) and (3.13) we have

$$\overline{Y}(x,p,r)=\frac{\overline{s}(x,r)}{p}-\frac{\overline{s}(r)}{ap^2}+\frac{\underline{F}(x,p,r)}{ap}-\frac{\underline{F}(p,r)}{a^2p^2}+\overline{U}(p,r)e^{-apx}+\frac{\overline{s}(r)e^{-apx}}{ap^2}+\frac{\underline{F}(p,r)e^{-apx}}{a^2p^2} \quad (3.20)$$

and

$$\underline{Y}(x,p,r)=\frac{\underline{s}(x,r)}{p}-\frac{\underline{s}(r)}{ap^2}+\frac{\overline{F}(x,p,r)}{ap}-\frac{\overline{F}(p,r)}{a^2p^2}+\underline{U}(p,r)e^{-apx}+\frac{\underline{s}(r)e^{-apx}}{ap^2}+\frac{\overline{F}(p,r)e^{-apx}}{a^2p^2}, \quad (3.21)$$

Therefore, taking the inverse Laplace transform of the equations (3.20) and (3.21), we have

$$\bar{y}(x,t,r)=\bar{s}(x,r)-\frac{\bar{s}(r)t}{a}+\frac{\underline{f}(x,\mathrm{t},r)}{a}-\frac{\underline{f}(\mathrm{t},r)t}{a^2}+\bar{u}(r)(t-ax)H(t-ax)+$$
$$\frac{\bar{s}(r)}{a}(t-ax)H(t-ax)+\frac{\underline{f}(r)}{a^2}(t-ax)H(t-ax)$$

and

$$\underline{y}(x,t,r)=\underline{s}(x,r)-\frac{\underline{s}(r)t}{a}+\frac{\bar{f}(x,\mathrm{t},r)}{a}+\underline{u}(r)(t-ax)H(t-ax)+$$
$$\frac{\underline{s}(r)}{a}(t-ax)H(t-ax)+\frac{\bar{f}(r)}{a^2}(t-ax)H(t-ax)$$

Basically, all the cases discussed and the results obtained above have addressed the identified gap (for $c<0$) pointed out in the work of Eljaoui and Melliani (2016). Therefore, it is also achievable when $c<0$ is set for negative coefficients of first order FPDEs.

Obtaining solutions for first order FPDEs for $c\geq 0$ and $c<0$ by FLTM

Having established the results for $c<0$ we discovered that gaps may still exist if we ignore results for both $c\geq 0$ and $c<0$, so we considered a case for both positive and negative coefficients as presented below. Consider equation (2.1) but in this case c is positive. Assume $c=a$ and $f(x,t,y(x,t))=f(x,r)$, gives the equation below

$$y_x(x,t)=ay_t(x,t)+f(x,r), \tag{3.22}$$

with fuzzy initial conditions

$$y(x,0,r)=(\underline{s}(x,r),\bar{s}(x,r)) \tag{3.23}$$

and fuzzy boundary conditions

$$y(0,t,r)=(\underline{u}(t,r),\bar{u}(t,r)) \tag{3.24}$$

where $f(x,r)$ is a fuzzy valued function, $\underline{s}(x,r)$ and $\bar{s}(x,r)$ are the appropriate lower and upper cases of the initial condition (3.29) respectively also, $\underline{u}(t,r)$ and $\bar{u}(x,r)$ are the appropriate lower and upper cases for the boundary condition (3.30) respectively, with $t\geq 0$, $x\geq 0$ and $r\in[0,1]$.

Taking the Laplace transform of equation (3.22) as expressed in equation (2.2), we have

$$L[y_x(x,t)]=aL[y_t(x,t)]+L[f(x,r)]. \tag{3.25}$$

Four cases arise as a result of equation (3.25) which is similar to cases related to equation (2.3).

Case 1: When y is (i)-differentiable with respect to both x and t.

Taking the Laplace transform of equation (3.24), the result is similar to that of equation (2.9) and (2.10) respectively. Therefore, we arrive at

$$L[\underline{y}(0,t,r)]=L[\underline{u}(t,r)]=\underline{U}(p,r) \tag{3.26}$$

and

$$L[\bar{y}(0,t,r)]=L[\bar{u}(t,r)]=\bar{U}(p,r). \tag{3.27}$$

We apply Theorem (2.1) and (2.2) on equation (3.25) after which we substitute the initial condition (3.23) into it, and solve it together with equation (3.26) and (3.27), we have

$$\underline{Y}(x,p,r)=\frac{\underline{s}(x,r)}{p}+\frac{\underline{s}(r)}{ap^2}-\frac{\underline{F}(x,p,r)}{ap}-\frac{\underline{F}(p,r)}{a^2p^2}+\underline{U}(p,r)e^{apx}-\frac{\underline{s}(r)e^{apx}}{ap^2}+\frac{\underline{F}(p,r)e^{apx}}{a^2p^2} \tag{3.28}$$

and

$$\overline{Y}(x,p,r)=\frac{\overline{s}(x,r)}{p}+\frac{\overline{s}(r)}{ap^2}-\frac{\overline{F}(x,p,r)}{ap}-\frac{\overline{F}(p,r)}{a^2p^2}+\overline{U}(p,r)e^{apx}-\frac{\overline{s}(r)e^{apx}}{ap^2}+\frac{\overline{F}(p,r)e^{apx}}{a^2p^2}, \quad (3.29)$$

However, to get the result for both $c\geq 0$ and $c<0$, we add equation (3.14) to (3.28) and equation (3.15) to (3.29) and get

$$\underline{Y}(x,p,r)=\frac{2\underline{s}(x,r)}{p}-\frac{2\underline{F}(p,r)}{a^2p^2}+\left(\underline{U}(p,r)+\frac{\underline{F}(p,r)}{a^2p^2}\right)\left(e^{-apx}+e^{apx}\right) \quad (3.30)$$

and

$$\overline{Y}(x,p,r)=\frac{2\overline{s}(x,r)}{p}-\frac{2\overline{F}(p,r)}{a^2p^2}+\left(\overline{U}(p,r)+\frac{\overline{F}(p,r)}{a^2p^2}\right)\left(e^{-apx}+e^{apx}\right). \quad (3.31)$$

Now, taking the inverse Laplace transform of the equations (3.30) and (3.31), we have

$$\underline{y}(x,t,r)=2\underline{s}(x,r)-\frac{2\underline{f}(t,r)t}{a^2}+\underline{u}(r)(t-ax)H(t-ax)+\underline{u}(r)(t+ax)H(t+ax)+\frac{\underline{f}(r)}{a^2}(t-ax)H(t-ax)+\frac{\underline{f}(r)}{a^2}(t+ax)H(t+ax)$$

and

$$\overline{y}(x,t,r)=2\overline{s}(x,r)-\frac{2\overline{f}(t,r)t}{a^2}+\overline{u}(r)(t-ax)H(t-ax)+\overline{u}(r)(t+ax)H(t+ax)+\frac{\overline{f}(r)}{a^2}(t-ax)H(t-ax)+\frac{\overline{f}(r)}{a^2}(t+ax)H(t+ax).$$

Case 2: When y is (i)-differentiable with respect to x and (ii)-differentiable with respect to t. We apply theorem (2.1) and (2.2) on equation (3.25), substituting equation (3.23) to it, and solving it together with equation (3.26) and (3.27), we arrive at

$$\underline{Y}(x,p,r)=\frac{\overline{s}(x,r)}{p}+\frac{\overline{s}(r)}{ap^2}-\frac{\underline{F}(x,p,r)}{ap}-\frac{\underline{F}(p,r)}{a^2p^2}+\underline{U}(p,r)e^{apx}-\frac{\overline{s}(r)e^{apx}}{ap^2}+\frac{\underline{F}(p,r)e^{apx}}{a^2p^2} \quad (3.32)$$

and

$$\overline{Y}(x,p,r)=\frac{\underline{s}(x,r)}{p}+\frac{\underline{s}(r)}{ap^2}-\frac{\overline{F}(x,p,r)}{ap}-\frac{\underline{F}(p,r)}{a^2p^2}+\overline{U}(p,r)e^{apx}-\frac{\underline{s}(r)e^{apx}}{ap^2}+\frac{\overline{F}(p,r)e^{apx}}{a^2p^2}, \quad (3.33)$$

However, to get the result for both $c\geq 0$ and $c<0$, we add equation (3.16) to (3.32) and equation (3.17) to (3.33) and have

$$\underline{Y}(x,p,r)=\frac{2\overline{s}(x,r)}{p}-\frac{2\underline{F}(p,r)}{a^2p^2}+\left(\underline{U}(p,r)+\frac{\underline{F}(p,r)}{a^2p^2}\right)\left(e^{-apx}+e^{apx}\right) \quad (3.34)$$

and

$$\overline{Y}(x,p,r)=\frac{2\underline{s}(x,r)}{p}-\frac{2\overline{F}(p,r)}{a^2p^2}+\left(\overline{U}(p,r)+\frac{\overline{F}(p,r)}{a^2p^2}\right)\left(e^{-apx}+e^{apx}\right). \quad (3.35)$$

Also, taking the inverse Laplace transform of the equations (3.34) and (3.35), we have

$$\underline{y}(x,t,r)=2\overline{s}(x,r)-\frac{2\underline{f}(t,r)t}{a^2}+\underline{u}(r)(t-ax)H(t-ax)+\underline{u}(r)(t+ax)H(t+ax)+\frac{\underline{f}(r)}{a^2}(t-ax)H(t-ax)+\frac{\underline{f}(r)}{a^2}(t+ax)H(t+ax)$$

and

$$\overline{y}(x,t,r)=2\underline{s}(x,r)-\frac{2\overline{f}(t,r)t}{a^2}+\overline{u}(r)(t-ax)H(t-ax)+\overline{u}(r)(t+ax)H(t+ax)+\frac{\overline{f}(r)}{a^2}(t-ax)H(t-ax)+\frac{\overline{f}(r)}{a^2}(t+ax)H(t+ax).$$

Case 3: When y is (ii)-differentiable with respect to x and (i)-differentiable with respect to t. We apply Theorem (2.1) and (2.2) on equation (3.25) and substituting the equation (3.23), after which we solve it together with equation (3.26) and (3.27), we have

$$\overline{Y}(x,p,r)=\frac{\underline{s}(x,r)}{p}+\frac{\underline{s}(r)}{ap^2}-\frac{\underline{F}(x,p,r)}{ap}-\frac{\underline{F}(p,r)}{a^2p^2}+\overline{U}(p,r)e^{apx}-\frac{\underline{s}(r)e^{apx}}{ap^2}+\frac{\underline{F}(p,r)e^{apx}}{a^2p^2} \quad (3.36)$$

and

$$\underline{Y}(x,p,r)=\frac{\overline{s}(x,r)}{p}+\frac{\overline{s}(r)}{ap^2}-\frac{\overline{F}(x,p,r)}{ap}-\frac{\overline{F}(p,r)}{a^2p^2}+\underline{U}(p,r)e^{apx}-\frac{\overline{s}(r)e^{apx}}{ap^2}+\frac{\overline{F}(p,r)e^{apx}}{a^2p^2}, \quad (3.37)$$

However, to get the result for both $c\geq 0$ and $c<0$, we add equation (3.18) to (3.32) and equation (3.19) to (3.37) which gives

$$\overline{Y}(x,p,r)=\frac{2\underline{s}(x,r)}{p}-\frac{2\underline{F}(p,r)}{a^2p^2}+\left(\overline{U}(p,r)+\frac{\underline{F}(p,r)}{a^2p^2}\right)\left(e^{-apx}+e^{apx}\right) \quad (3.38)$$

and

$$\underline{Y}(x,p,r)=\frac{2\overline{s}(x,r)}{p}-\frac{2\overline{F}(p,r)}{a^2p^2}+\left(\underline{U}(p,r)+\frac{\overline{F}(p,r)}{a^2p^2}\right)\left(e^{-apx}+e^{apx}\right), \quad (3.39)$$

Again, taking the inverse Laplace transform of the equations (3.38) and (3.39), we have

$$\overline{y}(x,t,r)=2\underline{s}(x,r)-\frac{2\underline{f}(t,r)t}{a^2}+\underline{u}(r)(t-ax)H(t-ax)+\underline{u}(r)(t+ax)H(t+ax)+\frac{\underline{f}(r)}{a^2}(t-ax)H(t-ax)+\frac{\underline{f}(r)}{a^2}(t+ax)H(t+ax)$$

and

$$\underline{y}(x,t,r)=2\overline{s}(x,r)-\frac{2\overline{f}(t,r)t}{a^2}+\overline{u}(r)(t-ax)H(t-ax)+\overline{u}(r)(t+ax)H(t+ax)+\frac{\overline{f}(r)}{a^2}(t-ax)H(t-ax)+\frac{\overline{f}(r)}{a^2}(t+ax)H(t+ax).$$

Case 4: When y is (ii)-differentiable with respect to both x and t. We apply Theorem (2.1) and (2.2) on equation (3.25) and substituting equation (3.23) into it, after which we solve it together with equation (3.26) and (3.27), we have

$$\overline{Y}(x,p,r)=\frac{\overline{s}(x,r)}{p}+\frac{\overline{s}(r)}{ap^2}-\frac{\underline{F}(x,p,r)}{ap}-\frac{\underline{F}(p,r)}{a^2p^2}+\overline{U}(p,r)e^{apx}-\frac{\overline{s}(r)e^{apx}}{ap^2}+\frac{\underline{F}(p,r)e^{apx}}{a^2p^2} \tag{3.40}$$

and

$$\underline{Y}(x,p,r)=\frac{\underline{s}(x,r)}{p}+\frac{\underline{s}(r)}{ap^2}-\frac{\overline{F}(x,p,r)}{ap}-\frac{\overline{F}(p,r)}{a^2p^2}+\underline{U}(p,r)e^{apx}-\frac{\underline{s}(r)e^{apx}}{ap^2}+\frac{\overline{F}(p,r)e^{apx}}{a^2p^2}, \tag{3.41}$$

However, to get the result for both. $c\geq 0$ and $c<0$, we add equation (3.20) to (3.40) and equation (3.21) to (3.41) and get

$$\overline{Y}(x,p,r)=\frac{2\overline{s}(x,r)}{p}-\frac{2\underline{F}(p,r)}{a^2p^2}+\left(\overline{U}(p,r)+\frac{\underline{F}(p,r)}{a^2p^2}\right)\left(e^{-apx}+e^{apx}\right) \tag{3.42}$$

and

$$\underline{Y}(x,p,r)=\frac{2\underline{s}(x,r)}{p}-\frac{2\overline{F}(p,r)}{a^2p^2}+\left(\underline{U}(p,r)+\frac{\overline{F}(p,r)}{a^2p^2}\right)\left(e^{-apx}+e^{apx}\right), \tag{3.43}$$

Therefore, taking the inverse Laplace transform of the equations (3.42) and (3.43), we have

$$\overline{y}(x,t,r)=2\overline{s}(x,r)-\frac{2\underline{f}(t,r)t}{a^2}+\overline{u}(r)(t-ax)H(t-ax)+\overline{u}(r)(t+ax)H(t+ax)+\frac{\underline{f}(r)}{a^2}(t-ax)H(t-ax)+\frac{\underline{f}(r)}{a^2}(t+ax)H(t+ax)$$

and

$$\underline{y}(x,t,r)=2\underline{s}(x,r)-\frac{2\overline{f}(t,r)t}{a^2}+\underline{u}(r)(t-ax)H(t-ax)+\underline{u}(r)(t+ax)H(t+ax)+\frac{\overline{f}(r)}{a^2}(t-ax)H(t-ax)+\frac{\overline{f}(r)}{a^2}(t+ax)H(t+ax).$$

Basically, all the cases discussed and the results obtained above have addressed the identified gap (for both $c<0$ and $c\geq 0$) pointed out in the work of Eljaoui and Melliani (2016). Therefore, it is also achievable for both $c<0$ and $c\geq 0$ is set for both negative and positive coefficients of first order FPDEs.

Constructed Examples

Example 3.1

Consider $y_x(x,t)=-3y_t(x,t)+x(r,2-r)$ (3.44)

with initial conditions

$$y(x,0,r)=-3x(r,2-r)+\frac{x^2}{2}(r,2-r) \tag{3.45}$$

and boundary conditions

$$y(0,t,r)=t(r,2-r)\ . \tag{3.46}$$

Example (3.1) is similar to equation (3.4), where $c=-3$ which is negative and $x(r,2-r)$ is a fuzzy valued function. $t\geq 0$, $x\geq 0$ and $r\in[0,1]$.

Solution:

Applying Laplace transform on equation (3.44), we get

$$L\left[y_x(x,t)\right] = -3L\left[y_t(x,t)\right] + x(r,2-r)L[1]. \quad (3.47)$$

Four cases arises as a result of equation (3.47).
Case 1: When y is (i)-differentiable with respect to both x and t.
Taking the Laplace transform of the boundary conditions (3.46), it becomes

$$L\left[\underline{y}(0,t,r)\right] = L[tr] = \frac{r}{p^2} \quad (3.48)$$

and

$$L\left[\overline{y}(0,t,r)\right] = L\left[t(2-r)\right] = \frac{(2-r)}{p^2} \quad (3.49)$$

Applying Theorem (2.1) and (2.2) on equation (3.47), substituting the initial conditions (3.45) into it and solving it together with equation (3.48) and (3.49), we have

$$\underline{Y}(x,p,r) = \frac{-3xr}{p} - \frac{r}{p^2} + \frac{x^2 r}{2p} + \frac{2r}{p^2}e^{-3px} \quad (3.50)$$

and

$$\overline{Y}(x,p,r) = \frac{-3x(2-r)}{p} - \frac{(2-r)}{p^2} + \frac{x^2(2-r)}{2p} + \frac{2(2-r)}{p^2}e^{-3px} \quad (3.51)$$

Taking the inverse Laplace transform of the equations (3.50) and (3.51), we have

$$\underline{y}(x,t,r) = -3xr - rt + \frac{x^2 r}{2} + 2r(t-3x)H(t-3x)$$

and

$$\overline{y}(x,t,r) = -3x(2-r) - (2-r)t + \frac{x^2(2-r)}{2} + 2(2-r)(t-3x)H(t-3x).$$

Case 2: When y is (i)-differentiable with respect to x and (ii)-differentiable with respect to t. Applying Theorem (2.1) and (2.2) on equation (3.47) substituting the initial conditions (3.45) into it and solving it together with equation (3.48) and (3.49) we have

$$\underline{Y}(x,p,r) = \frac{-3x(2-r)}{p} - \frac{(2-r)}{p^2} + \frac{x^2(2-r)}{2p} + \frac{2x(r-1)}{3p^2} + \frac{2(1-r)}{9p^3} + \frac{2(r-1)}{p^2}e^{-3px} - \frac{2(1-r)}{9p^3}e^{-3px} \quad (3.52)$$

and

$$\overline{Y}(x,p,r) = \frac{-3xr}{p} - \frac{r}{p^2} + \frac{x^2 r}{2p} + \frac{2x(1-r)}{3p^2} + \frac{2(r-1)}{9p^3} + \frac{2(1-r)}{p^2}e^{-3px} - \frac{2(r-1)}{9p^3}e^{-3px} \quad (3.53)$$

Taking the inverse Laplace transform of the equations (3.52) and (3.53), we have

$$\underline{y}(x,t,r) = -3x(2-r) - (2-r)t + \frac{x^2(2-r)}{2} + \frac{2x(r-1)}{3} + \frac{(1-r)t^2}{9} + 2(2-r)(t-3x)H(t-3x) - \frac{(1-r)}{9}H(t-3x)(t-3x)^2$$

and

$$\overline{y}(x,t,r) = -3xr - rt + \frac{x^2 r}{2} + \frac{2x(1-r)t}{3} + \frac{(r-1)t^2}{9} + 2(1-r)H(t-3x)(t-3x) - \frac{(r-1)}{9}H(t-3x)(t-3x)^2.$$

Case 3: When y is (ii)-differentiable with respect to x and (i)-differentiable with respect to t. When we apply Theorem (2.1) and (2.2) on equation (3.47), substituting the initial conditions (3.45) into it and solving it together with equation (3.48) and (3.49) we have

$$\bar{Y}(x,p,r)=\frac{-3xr}{p}-\frac{r}{p^2}+\frac{x^2r}{2p}+\frac{2}{p^2}e^{-3px} \tag{3.54}$$

and

$$\underline{Y}(x,p,r)=\frac{-3x(2-r)}{p}-\frac{(2-r)}{p^2}+\frac{x^2(2-r)}{2p}+\frac{2}{p^2}e^{-3px} \tag{3.55}$$

Taking the inverse Laplace transform of the equations (3.54) and (3.55), we have

$$\bar{y}(x,t,r)=-3xr-rt+\frac{x^2r}{2}+2H(t-3x)(t-3x)$$

and

$$\underline{y}(x,t,r)=-3x(2-r)-(2-r)t+\frac{x^2(2-r)}{2}+2(t-3x)H(t-3x).$$

Case 4: When y is (ii)-differentiable with respect to both x and t. Applying Theorem (2.1) and (2.2) on equation (3.47), substituting the initial conditions (3.45) into it and solving it together with equation (3.48) and (3.49) we have

$$\bar{Y}(x,p,r)=\frac{-3x(2-r)}{p}-\frac{(2-r)}{p^2}+\frac{x^2(2-r)}{2p}+\frac{2x(r-1)}{3p^2}+\frac{2(1-r)}{9p^3}+ \frac{2(2-r)}{p^2}e^{-3px}-\frac{2(1-r)}{9p^3}e^{-3px} \tag{3.56}$$

and

$$\underline{Y}(x,p,r)=\frac{-3xr}{p}-\frac{r}{p^2}+\frac{x^2r}{2p}+\frac{2x(1-r)}{3p^2}+\frac{2(r-1)}{9p^3}+\frac{2r}{p^2}e^{-3px}-\frac{2(r-1)}{9p^3}e^{-3px} \tag{3.57}$$

Taking the inverse Laplace transform of the equations (3.56) and (3.57), we have

$$\bar{y}(x,t,r)=-3x(2-r)-(2-r)t+\frac{x^2(2-r)}{2}+\frac{2x(r-1)}{3}+\frac{(1-r)t^2}{9}+ 2(2-r)(t-3x)H(t-3x)-\frac{(1-r)}{9}H(t-3x)(t-3x)^2$$

and

$$\underline{y}(x,t,r)=-3xr-rt+\frac{x^2r}{2}+\frac{2x(1-r)t}{3}+\frac{(r-1)t^2}{9}+2rH(t-3x)(t-3x)- \frac{(r-1)}{9}H(t-3x)(t-3x)^2.$$

To ascertain the applicability of the FLTM on first order FPDE with both positive and negative coefficient, we consider example 4.2 as seen bellow.

Example 4.2

Consider $y_x(x,t)=ay_t(x,t)+x(r,2-r)$ (3.58)

with initial conditions

$$y(x,0,r)=ax(r,2-r)+\frac{x^2}{2}(r,2-r) \tag{3.59}$$

and boundary conditions

$$y(0,t,r)=t(r,2-r) \tag{3.60}$$

Example (4.2) is similar to equation (3.28) where a can be either positive or negative and that $x(r,2-r)$ is a fuzzy valued function. Assume that $a=3$ in equation (3.28). for $t\geq 0$, $x\geq 0$ and $r\in[0,1]$.

Solution:

Applying Laplace transform on equation (3.58), we have

$$L\left[y_x(x,t)\right]=3L\left[y_t(x,t)\right]+x(r,2-r)L[1]. \qquad (3.61)$$

Four cases arises as a result of equation (3.61)

Case 1: When y is (i) differentiable with respect to both x and t.

Taking the Laplace transform of the equation (3.60), we have

$$L\left[\underline{y}(0,t,r)\right]=L[tr]=\frac{r}{p^2} \qquad (3.62)$$

and

$$L\left[\overline{y}(0,t,r)\right]=L\left[t(2-r)\right]=\frac{(2-r)}{p^2}\ . \qquad (3.63)$$

Applying theorem (2.1) and (2.2) on equation (3.61) then substitute the initial conditions (3.59) into it and solving it together with equation (3.62) and (3.63) after which we add equation (3.50) and (3.51) to its solution respectively gives

$$\underline{Y}(x,p,r)=-\frac{2r}{p^2}+\frac{x^2r}{2p}+\frac{2r}{p^2}e^{-3px} \qquad (3.64)$$

and

$$\overline{Y}(x,p,r)=-\frac{2(2-r)}{p^2}+\frac{x^2(2-r)}{p}+\frac{2(2-r)}{p^2}e^{-3px}. \qquad (3.65)$$

Taking the inverse Laplace transform of the equations (3.64) and (3.65), we have

$$\underline{y}(x,t,r)=\frac{x^2r}{2}-2rt+2rH(t-3x)(t-3x)$$

and

$$\overline{y}(x,t,r)=x^2(2-r)-2(2-r)t+2(2-r)H(t-3x)(t-3x).$$

Case 2: When y is (i)-differentiable with respect to x and (ii)-differentiable with respect to t. Applying theorem (2.1) and (2.2) on equation (3.61), then we substitute the initial conditions (3.59) it and solving it together with equation (3.62) and (3.63) after which we add the equation (3.52) and (3.53) to its solution respectively gives

$$\underline{Y}(x,p,r)=-\frac{2(2-r)}{p^2}+\frac{x^2(2-r)}{p}+\frac{4(1-r)}{9p^3}+\frac{2(r-1)}{p^2}e^{3px}+\frac{2(r-1)}{p^2}e^{-3px}-\frac{2(1-r)}{9p^3}e^{3px}-\frac{2(1-r)}{9p^3}e^{-3px} \qquad (3.66)$$

and

$$\overline{Y}(x,p,r)=-\frac{2r}{p^2}+\frac{x^2r}{p}+\frac{4(r-1)}{9p^3}+\frac{2(1-r)}{p^2}e^{3px}+\frac{2(1-r)}{p^2}e^{-3px}-\frac{2(r-1)}{9p^3}e^{3px}-\frac{2(r-1)}{9p^3}e^{-3px}. \qquad (3.67)$$

Taking the inverse Laplace transform of the equations (3.66) and (3.67), we have

$$\underline{y}(x,t,r)=-2(2-r)t+x^2(2-r)+\frac{2(1-r)t^2}{9}+2(r-1)H(t+3x)(t+3x)+2(r-1)H(t-3x)(t-3x)-\frac{(1-r)}{9}H(t+3x)(t+3x)^2-\frac{(1-r)}{9}H(t-3x)(t-3x)^2$$

and

$$\bar{y}(x,t,r) = -2rt + x^2r + \frac{2(r-1)t^2}{9} + 2(1-r)H(t+3x)(t+3x) + 2(1-r)H(t-3x)(t-3x) - \frac{(r-1)}{9}H(t+3x)(t+3x)^2 - \frac{(r-1)}{9}H(t-3x)(t-3x)^2.$$

Case 3: When y is (ii)-differentiable with respect to x and (i)-differentiable with respect to t. Applying theorem (2.1) and (2.2) on equation (3.61), then we substitute the initial conditions (3.59) into it after which we solve it together with equation (3.62) and (3.63) after which we add equation (3.54) and (3.55) to its solution respectively gives

$$\bar{Y}(x,p,r) = -\frac{2r}{p^2} + \frac{x^2r}{p} + \frac{2(1-r)}{p^2}e^{3px} + \frac{2}{p^2}e^{-3px} \quad (3.68)$$

and

$$\underline{Y}(x,p,r) = -\frac{2(2-r)}{p^2} + \frac{x^2(2-r)}{p} + \frac{2(r-1)}{p^2}e^{3px} + \frac{2}{p^2}e^{-3px}. \quad (3.69)$$

Taking the inverse Laplace transform of the equations (3.68) and (3.69), gives

$$\bar{y}(x,t,r) = -2rt + x^2r + 2(1-r)H(t+3x)(t+3x) + 2H(t-3x)(t-3x)$$

and

$$\underline{y}(x,t,r) = -2(2-r)t + x^2(2-r) + 2(r-1)H(t+3x)(t+3x) + 2H(t-3x)(t-3x).$$

Case 4: When y is (ii)-differentiable with respect to both x and t. Applying theorem (2.1) and (2.2) on equation (3.61), then we substitute the initial conditions (3.59) into it and solving it together with equation (3.62) and (3.63) after which we add the equation (3.56) and (3.57) to its solution gives

$$\bar{Y}(x,p,r) = -\frac{2(2-r)}{p^2} + \frac{x^2(2-r)}{p} + \frac{4(1-r)}{9p^3} + \frac{2(2-r)}{p^2}e^{-3px} - \frac{2(1-r)}{9p^3}e^{3px} - \frac{2(1-r)}{9p^3}e^{-3px} \quad (3.70)$$

and

$$\underline{Y}(x,p,r) = -\frac{2r}{p^2} + \frac{x^2r}{p} + \frac{4x(1-r)}{3p^2} + \frac{2r}{p^2}e^{-3px} - \frac{2(r-1)}{9p^3}e^{-3px} - \frac{2(r-1)}{9p^3}e^{3px}. \quad (3.71)$$

Taking the inverse Laplace transform of the equations (3.70) and (3.71), we have

$$\bar{y}(x,t,r) = -2(2-r)t + x^2(2-r) + \frac{2(1-r)t^2}{9} + 2(2-r)H(t-3x)(t-3x) - \frac{(1-r)}{9}H(t-3x)(t-3x)^2 - \frac{(1-r)}{9}H(t+3x)(t+3x)^2$$

and

$$\underline{y}(x,t,r) = -2rt + x^2r + \frac{4x(r-1)t}{3} + 2rH(t-3x)(t-3x) - \frac{2(r-1)}{9}H(t-3x)(t-3x)^2 - \frac{2(r-1)}{9}H(t+3x)(t+3x)^2.$$

DISCUSSION

It is observe that certain gaps existed for solution of first order FPDEs where negative coefficients are not mention. This study considered as its mandate to address the above problem. However the subsection 3.1 has rightfully taken care of the above problem by obtaining the solution of first order FPDE for $c < 0$. In that case equation (3.4), (3.5) and (3.6) were considered and their results after transformation were categorized into four cases. Case 1 deals with a situation whereby y is (i)-differentiable with respect to both x and t. The result in this case includes a unit step function was obtained. Moreover, other cases included a case where y is (i)-differentiable with respect to x and (ii)-differentiable with respect to t.

Again applying the Laplace transform to equation (3.4) subject to equations (3.5) and (3.6) yielded the result in equations (3.14) and (3.15), which after taking the inverse Laplace transform gave a result, which also included a unit step function. Also, the solution for the case of y being (ii)-differentiable with respect to x and (i)-differentiable with respect to t was also presented and finally the case when y is (ii)-differentiable with respect to both x and t was also solved and their results included a unit step function were also presented as well. In all the above cases, one can conclude that it is quite possible that you can conveniently obtain solutions when $c < 0$ is considered a case may be.

After obtaining results for $c < 0$, it is observed that a problem may still occur where both positive and negative coefficients are involved. In this regards, this study extend the case to include a situation when the coefficient can be either of $c < 0$ or $c \geq 0$.

To obtain the result for both $c < 0$ and $c \geq 0$, we first considered equation (3.22) where results for positive coefficients were established. Recall that section 3.1 established results for negative coefficients, therefore the combine results we considered both equation (3.4) and (3.22) that gave rise to four cases discussed in subsection 3.2. It is establish that situations involving both positive and negative coefficients of first order FPDEs can be addressed using the results we obtained in subsection 3.2.

CONCLUSION

In this paper, first order FPDEs with negative coefficients and that of but positive and negative coefficient is been solved by FLTM. Their results is rightfully established which will pave way for researchers which will encounter problems relation in FPDEs with both positive and negative coefficients.

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