BAJOPAS Volume 14 Number 2, December, 2021

http://dx.doi.org/10.4314/bajopas.v14i2.4

Bayero Journal of Pure and Applied Sciences, 14(2): 20 - 36
Received: October, 2021
Accepted: November, 2021
ISSN 2006 – 6996

GENERALIZED HUKUHARA DERIVATIVES OF SOLVING OF SECOND ORDER LINEAR HOMOGENEOUS ODES BY GENERALIZED TRIANGULAR FUZZY NUMBER USING FUZZY LAPLACE TRANSFORM

***[1]Tahir, A. and [2]Habib, H.**
[1]Department of Mathematics, Modibbo Adama University, Yola, Adamawa State, Nigeria
[2]Department of Mathematics, College of Education Waka-Biu, Borno State, Nigeria
*Corresponding Author's Email: atahir@mautech.edu.ng

ABSTRACT
Second order linear homogeneous ordinary differential equations are solved using fuzzy Laplace transform method with generalized Hukuhara differentiability concept. The results obtained is in this study is in the form of generalized triangular fuzzy number. Existence and uniqueness of solution are also obtained. However, based on the cases presented, the results have shown that relationship existed between the FLT of second order and its k^{th} derivative for $k \geq 1$.

Keywords: gH differentiability, Linear Homogeneous ODE, Generalized Triangular Fuzzy Number (GTFN), fuzzy Laplace transform (FLT)

INTRODUCTION

The concept of the fuzzy derivative was first introduced by Chang and Zadeh (1972). Later, Dubois and Prade (1982) presented a concept of the fuzzy derivative based on the extension principle. Buckley and Feuring (2000) compared various derivatives of fuzzy function that have been presented in the various literatures. Later, Bede and Gal (2004) introduced a concept for strongly generalized differentiability of fuzzy functions. Allahviranloo *et al.* (2009) used the concept of generalised differentiability and applied differential transformation method for solving fuzzy differential equations. Khastan *et al.* (2011) studied first order linear fuzzy differential equations by using the generalized differentiability concept.

Sankar and Tapan (2015) solved second order linear homogeneous ODEs in fuzzy environment based on the concept of generalized Hukuhara derivatives. They considered the linear homogeneous second order ODEs

$$\frac{d^2 x(t)}{dt^2} = k\, x(t)$$

with the fuzzy initial conditions

$$x(t_0) = \tilde{a} = (a_1, a_2, a_3, a_4, \omega),$$

$$\frac{d\, x(t_0)}{dt} = \tilde{b} = (b_1, b_2, b_3, b_4, \omega)$$

and solved with fuzzy number as generalized trapezoidal. However, limiting the work to generalized trapezoidal fuzzy numbers without considering the case when $\omega = 1$ or $b_2 = b_4$, $a_1 = a_3$ was not enough. Therefore, this needs to be addressed by applying generalized Hukuhara derivative concept with generalized triangular fuzzy number (GTFN). In addition to that, also establishments of the existence as well as the uniqueness of solutions to the given equations are deeming to be necessary.

MATERIAL AND METHOD

Consider second order linear homogeneous FDEs

$$\frac{d^2 x(t)}{dt^2} = k\, x(t) \qquad (1)$$

with fuzzy initial conditions

$$x(t_0) = \tilde{a} = (a_1, a_2, a_3, a_4, \omega) \qquad (2)$$

and

$$\frac{d\, x(t_0)}{dt} = \tilde{b} = (b_1, b_2, b_3, b_4, \omega) \qquad (3)$$

The procedures of solving second order linear homogeneous FODE are described as Type-I, Type-II and Type-III by taking the coefficients of equation (1) as positive and negative respectively. Here fuzzy numbers are taken as generalized trapezoidal fuzzy numbers (GTrFNs) and the solutions are described in four different cases by the concept of generalized Hukuhara differentiability as indicated below.

Case 1: When $x(t)$ and $\frac{dx(t)}{dt}$ are (i) gH-differentiable.

Case 2: When $x(t)$ is (i) gH-differentiable and $\frac{dx(t)}{dt}$ is (ii) gH-differentiable.

Case 3: When $x(t)$ is (ii) gH differentiable and $\frac{dx(t)}{dt}$ is (i) gH-differentiable.

Case 4: When $x(t)$ and $\frac{dx(t)}{dt}$ are (ii) gH-differentiable.

Using the concept of generalized Hukuhara differentiability cases 1 and 4 are treated in a similar way while cases 2 and 3 are similarly treated.

Cases 1 and 4:

In these cases, Sankar and Tapan (2015) obtained two sets of equations from equation (1), namely,

$$\frac{d^2 x_1(t,\alpha)}{dt^2} = k\, x_1(t,\alpha) \tag{4}$$

and

$$\frac{d^2 x_2(t,\alpha)}{dt^2} = k\, x_2(t,\alpha) \tag{5}$$

with initial conditions

$$x_1(t_0,\alpha) = a_1 + \frac{\alpha\, l_{\tilde{a}}}{\omega},\; x_2(t_0,\alpha) = a_4 - \frac{\alpha\, r_{\tilde{a}}}{\omega} \tag{6}$$

$$\frac{d x_1(t_0,\alpha)}{dt} = b_1 + \frac{\alpha\, l_{\tilde{b}}}{\omega},\; \frac{d x_2(t_0,\alpha)}{dt} = b_4 - \frac{\alpha\, r_{\tilde{b}}}{\omega} \tag{7}$$

The general solution of equation (4) is obtained using characteristic equation and presented below.

$$x_1(t,\alpha) = c_1\, e^{\sqrt{k}\,t} + c_2\, e^{-\sqrt{k}\,t} \tag{8}$$

Applying the initial condition (6) and (7) on equation (8) and solving, it is found that

$$c_1 = \frac{1}{2}\left\{\left(a_1 + \frac{\alpha\, l_{\tilde{a}}}{\omega}\right) + \frac{1}{\sqrt{k}}\left(b_1 + \frac{\alpha\, l_{\tilde{b}}}{\omega}\right)\right\} e^{-\sqrt{k}\,t_0}, \tag{9}$$

$$c_2 = \frac{1}{2}\left\{\left(a_1 + \frac{\alpha\, l_{\tilde{a}}}{\omega}\right) - \frac{1}{\sqrt{k}}\left(b_1 + \frac{\alpha\, l_{\tilde{b}}}{\omega}\right)\right\} e^{\sqrt{k}\,t_0}. \tag{10}$$

Substituting (9) and (10) in equation (8) and solving

$$x_1(t,\alpha) = \frac{1}{2}\left\{\left(a_1 + \frac{\alpha\, l_{\tilde{a}}}{\omega}\right) + \frac{1}{\sqrt{k}}\left(b_1 + \frac{\alpha\, l_{\tilde{b}}}{\omega}\right)\right\} e^{-\sqrt{k}(t-t_0)}$$
$$+ \frac{1}{2}\left\{\left(a_1 + \frac{\alpha\, l_{\tilde{a}}}{\omega}\right) - \frac{1}{\sqrt{k}}\left(b_1 - \frac{\alpha\, l_{\tilde{b}}}{\omega}\right)\right\} e^{-\sqrt{k}(t-t_0)} \tag{11}$$

Similarly, from equation (5), the general solution is obtained by using the characteristic equation and presented as

$$x_2(t,\alpha)=\frac{1}{2}\left\{\left(a_4-\frac{\alpha\, l_{\tilde{a}}}{\omega}\right)+\frac{1}{\sqrt{k}}\left(b_4-\frac{\alpha\, l_{\tilde{b}}}{\omega}\right)\right\}e^{-\sqrt{k}(t-t_0)}$$

$$-\frac{1}{2}\left\{\left(a_4+\frac{\alpha\, l_{\tilde{a}}}{\omega}\right)-\frac{1}{\sqrt{k}}\left(b_1-\frac{\alpha\, l_{\tilde{b}}}{\omega}\right)\right\}e^{-\sqrt{k}(t-t_0)}. \qquad (12)$$

Cases 2 and 3

In the above cases, equation (1) can be written as

$$\frac{d^2 x_2(t,\alpha)}{d t^2}=k\, x_1(t,\alpha) \qquad (13)$$

and

$$\frac{d^2 x_1(t,\alpha)}{d t^2}=k\, x_2(t,\alpha) \qquad (14)$$

Solving equation (13) and (14) using characteristic equation respectively, the general solutions is obtained as

$$x_1(t,\alpha)=d_1\, e^{\sqrt{k}\,t}+d_2\, e^{-\sqrt{k}\,t}+d_3\sin\sqrt{k}\,t+d_4\cos\sqrt{k}\,t, \qquad (15)$$

$$x_2(t,\alpha)=d_1\, e^{\sqrt{k}\,t}+d_2\, e^{-\sqrt{k}\,t}-d_3\sin\sqrt{k}\,t-d_4\cos\sqrt{k}\,t. \qquad (16)$$

Applying the initial conditions (13) and (14) on equations (15) and (16) and solving yield

$$d_1=\frac{1}{4}\left\{a_1+a_4+\frac{\alpha\left(l_{\tilde{a}}-r_{\tilde{a}}\right)}{\omega}+\frac{1}{\sqrt{k}}\left(b_1+b_4+\frac{\alpha\left(l_{\tilde{b}}-r_{\tilde{b}}\right)}{\omega}\right)\right\}e^{-\sqrt{k}\,t_0}, \qquad (17)$$

$$d_2=\frac{1}{4}\left\{a_1+a_4+\frac{\alpha\left(l_{\tilde{a}}-r_{\tilde{a}}\right)}{\omega}-\frac{1}{\sqrt{k}}\left(b_1+b_4+\frac{\alpha\left(l_{\tilde{b}}-r_{\tilde{b}}\right)}{\omega}\right)\right\}e^{-\sqrt{k}\,t_0}, \qquad (18)$$

$$d_3=\frac{1}{4}\frac{1}{\sin\sqrt{k}t_0}\left\{a_1-a_4+\frac{\alpha\left(l_{\tilde{a}}+r_{\tilde{a}}\right)}{\omega}+\frac{1}{\sqrt{k}}\left(b_1-b_4+\frac{\alpha\left(l_{\tilde{b}}+r_{\tilde{b}}\right)}{\omega}\right)\right\}, \qquad (19)$$

$$d_4=\frac{1}{4}\frac{1}{\sin\sqrt{k}\,t_0}\left\{a_1-a_4+\frac{\alpha\left(l_{\tilde{a}}+r_{\tilde{a}}\right)}{\omega}-\frac{1}{\sqrt{k}}\left(b_1-b_4+\frac{\alpha\left(l_{\tilde{b}}+r_{\tilde{b}}\right)}{\omega}\right)\right\}. \qquad (20)$$

The process used in solving equations (13) and (14) is repeated when k is negative in equation (1). The results obtained are tested using numerical examples on FDE, which indicated that it is a strong solution.

Concept of gH Differentiability for Second Order FODE

The second order gH-derivative of a fuzzy valued function $f:[a,b]\to\Re_F$ at t_0 is defined as $f''(t_0)=\lim_{h\to 0}\frac{f'(t_0+h)-_{g}H\, f'(t_0)}{h}$. If $f''(t_0)\in\Re_F$ then $f'(t_0)$ is gH-differentiable at t_0. Also, $f'(t_0)$ is:

(i) gH-differentiable at t_0 if $f''(t_0,\alpha)=\left(f_1'(t_0,\alpha),\, f_2'(t_0,\alpha)\right)$, then f is gH- differentiable on (a,b) and if $f''(t_0,\alpha)=\left(f_2'(t_0,\alpha),\, f_1'(t_0,\alpha)\right)$, then f is gH- differentiable on (a,b) for all $\alpha\in[0,1]$; and

(ii) gH-differentiable at t_0 on (a,b) and if $f''(t_0,\alpha)=\left(f_2'(t_0,\alpha),\, f_1'(t_0,\alpha)\right)$, then f is gH-differentiable on (a,b) and if $f''(t_0,\alpha)=\left(f_1'(t_0,\alpha),\, f_2'(t_0,\alpha)\right)$, then f is gH-differentiable on (a,b), for all $\alpha\in[0,1]$.

RESULTS AND DISCUSSION

Established Relationship between FLT and its k^{th} Derivative for $k \geq 1$

Consider equation (1) and applying FLT we have

$$L[\underline{f}(t, y(t), y'(t))] = p^2 L[\underline{y}(t,\alpha)] - p\underline{y}_0(\alpha) - \underline{z}_0(\alpha) \quad (21)$$

$$L[\overline{f}(t, y(t), y'(t))] = p^2 L[\overline{y}(t,\alpha)] - p\overline{y}_0(\alpha) - \overline{z}_0(\alpha). \quad (22)$$

Consider the fuzzy linear function

$$f\left(t,\ y(t),\ y'(t)\right) = ay'(t) + by''(t) \quad (23)$$

where a , b are real constants.

In order to see the relation between the FLT and its k^{th} derivative, we will apply the following cases: (a) Case 1a: If $a \geq 0$ and $b \geq 0$; (b) Case 2a: If $a \geq 0$ and $b < 0$; (c) Case 3a: If $a < 0$ and $b \geq 0$; (d) Case 4a: If $a < 0$ and $b < 0$. Below, we consider them case by case.

(a) Equation (21), (22) and (23) are subcategorized under four different cases and is presented below.

Case 1a: If $a \geq 0$ and $b \geq 0$. Taking the Fuzzy Laplace transform of right hand sides of equation (23) and simplify we have

$$L[f(t,\ y(t),\ y'(t))] = (a+b)sL\left[\underline{y}(t,\alpha)\right] - (a-bs)\underline{y}_0(\alpha) - b\overline{y}_0(\alpha) \quad (24)$$

From equation (21), (22) and (24) we have

$$(a+b)pL\left[\underline{y}(t,\alpha)\right] - (a-bp)\underline{y}_0(\alpha) - b\overline{y}_0(\alpha) = p^2 L\left[\underline{y}(t,\alpha)\right] - p\underline{y}_0(\alpha) - \underline{z}_0(\alpha). \quad (25)$$

Solving equation (25) we have,

$$(a+b)pL\left[\underline{y}(t,\alpha)\right] - p^2 L\left[\underline{y}(t,\alpha)\right] = b\overline{y}_0(\alpha) + (a-bp)\underline{y}_0(\alpha) - p\underline{y}_0(\alpha) - \underline{z}_0(\alpha). \quad (26)$$

Rearranging equation (26)

$$L\left[\underline{y}(t,\alpha)\right] = \frac{b\overline{y}(\alpha) + (a-bp-p)\underline{y}_0(\alpha) - z_0(\alpha)}{ap+bp-p^2}. \quad (27)$$

Also,

$$(a+b)pL\left[\overline{y}(t,\alpha)\right] - (a-b)p\overline{y}_0(\alpha) - b\underline{y}_0(\alpha) = p^2 L\left[\overline{y}(t,\alpha)\right] - p\overline{y}_0(\alpha) - \overline{z}_0(\alpha). \quad (28)$$

Solving equation (28)

$$L\left[\overline{y}(t,\alpha)\right] = \frac{b\underline{y}_0(\alpha) + (a-bp-p)\overline{y}_0(\alpha) - \overline{z}_0(\alpha)}{ap+bp-p^2}. \quad (29)$$

Therefore,

$$H_{a_1}(t,\alpha) = \frac{b\overline{y}_0(\alpha) + (a-bp-p)\underline{y}_0(\alpha) - z_0(\alpha)}{ap+bp-p^2}. \quad (30)$$

$$K_{a_1}(t,\alpha) = \frac{b\underline{y}_0(\alpha) + (a-bp-p)\overline{y}_0(\alpha) - \overline{z}_0(\alpha)}{ap+bp-p^2}. \quad (31)$$

$H_{a_1}(t,\alpha)$ and $K_{a_1}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a \geq 0$ and $b \geq 0$.

Case 2a: If $a \geq 0$ and $b < 0$, then from equations (21), (22) and (24) we have

$$(a+b)pL\left[\underline{y}(t,\alpha)\right] - (a-b)p\underline{y}_0(\alpha) - b\overline{y}_0(\alpha) = p^2 L\left[\underline{y}(t,\alpha)\right] - p\underline{y}_0(\alpha) - \underline{z}_0(\alpha). \quad (32)$$

$$(a+b)pL\left[\overline{y}(t,\alpha)\right] - (a-b)p\overline{y}_0(\alpha) - b\underline{y}_0(\alpha) = p^2 L\left[\underline{y}(t,\alpha)\right] - p\overline{y}_0(\alpha) - \overline{z}_0(\alpha). \quad (33)$$

Solving equation (32) we have

$$\begin{aligned} &bpL\left[\underline{y}(t,\alpha)\right] - p^2 L\left[\underline{y}(t,\alpha)\right] + apL\left[\overline{y}(t,\alpha)\right] \\ &= ap\overline{y}_0(\alpha) + b\overline{y}_0(\alpha) - bp\underline{y}_0(\alpha) - \underline{z}_0(\alpha) - p\underline{y}_0(\alpha). \end{aligned} \quad (34)$$

Rearranging equation (24)

$$(bp - p^2)L\left[\underline{y}(t,\alpha)\right] + apL\left[\overline{y}(t,\alpha)\right] = (ap+b)\overline{y}_0(\alpha) - (bp+p)\underline{y}_0(\alpha) - \underline{z}_0(\alpha). \quad (35)$$

Also, from equation (33)

$$apL\left[\overline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]-ap\overline{y}_0(\alpha)+bp\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)$$
$$= p^2L\left[\overline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (36)$$

Rearranging equation (36)

$$(ap-p^2)pL\left[\overline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]=(ap+b)\overline{y}_0(\alpha)-(bp+p)\underline{y}_0(\alpha)-\overline{z}_0(\alpha) \quad (37)$$

Considering equations (35) and (37), denote

$$A_1(p,\alpha)=(ap+b)\overline{y}_0(\alpha)-(bp+p)\underline{y}_0(\alpha)-\underline{z}_0(\alpha), \quad (38)$$

$$B_1(p,\alpha)=(ap+b)\overline{y}_0(\alpha)-(bp+p)\underline{y}_0(\alpha)-\overline{z}_0(\alpha). \quad (39)$$

Solving equation (38) and (39) we have

$$L\left[\underline{y}(t,\alpha)\right]=\frac{(ap-p^2)A_1(p,\alpha)-apB_1(p,\alpha)}{(ap-p^2)(ap-p^2)}. \quad (40)$$

Also,

$$L\left[\overline{y}(t,\alpha)\right]=\frac{bpA_1(p,\alpha)-(bp-p^2)B_1(p,\alpha)}{abp^2-(bp-p^2)(ap-p^2)}. \quad (41)$$

Therefore,

$$H_{a_2}(p,\alpha)=\frac{(ap-p^2)A_1(p,\alpha)-apB_1(p,\alpha)}{(ap-p^2)(ap-p^2)}. \quad (42)$$

$$K_{a_2}(p,\alpha)=\frac{bpA_1(p,\alpha)-(bp-p^2)B_1(p,\alpha)}{abp^2-(bp-p^2)(ap-p^2)}. \quad (43)$$

$H_{a_2}(t,\alpha)$ and $K_{a_2}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a\geq 0$ and $b<0$.

Case 3a: If $a<0$ and $b\geq 0$, then from equation (1), (2) and (24) we have

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-(a-b)p\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)=p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha), \quad (44)$$

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-(a-b)p\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)=p^2L\left[\underline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\overline{z}_0(\alpha). \quad (45)$$

Utilizing the relation in case 3a, equation (44) is in the form

$$apL\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]-ap\underline{y}_0(\alpha)+bp\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)$$
$$= p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (46)$$

Rearranging equation (46)

$$(ap-p^2)pL\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]=b\underline{y}_0(\alpha)+bp\overline{y}_0(\alpha)+(ap-p)\underline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (47)$$

Also, from equation (45)

$$apL\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]-ap\underline{y}_0(\alpha)+bp\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)$$
$$= p^2L\left[\overline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\overline{z}_0(\alpha). \quad (48)$$

Rearranging equation (48)

$$(bp-p^2)pL\left[\overline{y}(t,\alpha)\right]+apL\left[\underline{y}(t,\alpha)\right]=b\underline{y}_0(\alpha)+ap\underline{y}_0(\alpha)-(bp+p)\overline{y}_0(\alpha)-\overline{z}_0(\alpha). \quad (49)$$

Considering equations (47) and (48), denote

$$A_2(p,\alpha)=b\underline{y}_0(\alpha)+bp\overline{y}_0(\alpha)+(ap-p)\underline{y}_0(\alpha)-\underline{z}_0(\alpha), \quad (50)$$

$$B_2(p,\alpha)=b\underline{y}_0(\alpha)+bp\underline{y}_0(\alpha)-(bp+p)\overline{y}_0(\alpha)-\overline{z}_0(\alpha). \quad (51)$$

Solving equation (50) and (51) we have

$$L\left[\overline{y}(t,\alpha)\right]=\frac{apA_2(p,\alpha)-(ap-p^2)B_2(p,\alpha)}{abp^2-(ap-p^2)(bp-p^2)}. \quad (52)$$

Also,

$$L\left[\underline{y}(t,\alpha)\right]=\frac{(bp-p^2)A_2(p,\alpha)-bpB_2(p,\alpha)}{(bp-p^2)(ap-p^2)-abp^2}\ . \qquad (53)$$

Therefore,

$$H_{a_3}(p,\alpha)=\frac{(ap-p^2)B_2(p,\alpha)-apA_2(p,\alpha)}{(ap-p^2)(bp-p^2)-abp^2}\ . \qquad (54)$$

$$K_{a_3}(p,\alpha)=\frac{(bp-p^2)A_2(p,\alpha)-bpB_2(p,\alpha)}{(bp-p^2)(ap-p^2)-abp^2}\ . \qquad (55)$$

$H_{a_3}(t,\alpha)$ and $K_{a_3}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a<0$ and $b\geq 0$.

Case 4a: If $a<0$ and $b<0$, then from equation (21), (22) and (24) we have

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-(a-b)p\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)=p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha)\,. \qquad (56)$$

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-(a-b)p\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)=p^2L\left[\underline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\overline{z}_0(\alpha)\,. \qquad (57)$$

Utilizing the relation in case 4, equation (56) is in form of

$$apL\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]-ap\underline{y}_0(\alpha)+bp\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)$$
$$=p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha). \qquad (58)$$

Rearranging equation (58)

$$(ap+bp-p^2)L\left[\underline{y}(t,\alpha)\right]=(ap-bp-p)\underline{y}_0(\alpha)+b\overline{y}_0(\alpha)-\underline{z}_0(\alpha)\,. \qquad (59)$$

$$L\left[\underline{y}(t,\alpha)\right]=\frac{(ap-bp-p)\underline{y}_0(\alpha)+b\overline{y}-\underline{z}_0(\alpha)}{ap+bp-p^2}\ . \qquad (60)$$

Similarly,

$$L\left[\overline{y}(t,\alpha)\right]=\frac{(ap-bp-p)\underline{y}_0(\alpha)+b\overline{y}_0-\underline{z}_0(\alpha)}{ap+bp-p^2}\ . \qquad (61)$$

Therefore,

$$H_{a_4}(p,\alpha)=\frac{(ap-bp-p)\underline{y}_0(\alpha)+b\overline{y}_0-\underline{z}_0(\alpha)}{ap+bp-p^2}\ . \qquad (62)$$

$$K_{a_4}(p,\alpha)=\frac{(ap-bp-p)\underline{y}_0(\alpha)+b\overline{y}_0-\underline{z}_0(\alpha)}{ap+bp-p^2}\ . \qquad (63)$$

$H_{a_4}(t,\alpha)$ and $K_{a_4}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a<0$ and $b<0$.

(b) Consider equation (21), (22) and (24) from which we obtained the following:

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-(a-bp)\overline{y}_0(\alpha)-b\overline{y}_0(\alpha)=p^2L\left[\overline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha)\,, \qquad (64)$$

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-(a-bp)\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)=p^2L\left[\underline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\overline{z}_0(\alpha)\,. \qquad (65)$$

We discuss equation (64) and (65) using the conditions below.

Case 1b: If $a\geq 0$ and $b\geq 0$

$$apL\left[\underline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]-a\overline{y}_0(\alpha)+bp\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)$$
$$=p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha). \qquad (66)$$

Rearranging equation (66)

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-p^2L\left[\underline{y}(t,\alpha)\right]=b\underline{y}_0(\alpha)+a\overline{y}_0(\alpha)-bp\overline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha)\,, \qquad (66)$$

$$(ap+bp-p^2)L\left[\underline{y}(t,\alpha)\right]=(a-bp-p)\underline{y}_0(\alpha)+b\underline{y}(\alpha)-\underline{z}_0(\alpha)\,, \qquad (67)$$

$$L\left[\overline{y}(t,\alpha)\right]=\frac{(a-bp-p)\underline{y}_0(\alpha)+b\underline{y}_0-\underline{z}_0(\alpha)}{ap+bp-p^2}\ . \tag{68}$$

Similarly,

$$L\left[\underline{y}(t,\alpha)\right]=\frac{(a-bp-p)\overline{y}_0(\alpha)+b\underline{y}_0-\overline{z}_0(\alpha)}{ap+bp-p^2}\ . \tag{69}$$

Therefore,

$$H_{b_1}(t,\alpha)=\frac{(a-bp-p)\underline{y}_0(\alpha)+b\underline{y}_0-\underline{z}_0(\alpha)}{ap+bp-p^2}\ . \tag{70}$$

$$K_{b_1}(t,\alpha)=\frac{(a-bp-p)\overline{y}_0(\alpha)+b\underline{y}_0-\overline{z}_0(\alpha)}{ap+bp-p^2}\ . \tag{71}$$

$H_{b_1}(t,\alpha)$ and $K_{b_1}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a\geq 0$ and $b\geq 0$.

Case 2b: If $a\geq 0$ and $b<0$. Applying these conditions on equation (64) and (65)

$$\begin{aligned}&apL\left[\overline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]-a\overline{y}_0(\alpha)+bp\underline{y}_0(\alpha)-b\underline{y}_0(\alpha)\\&=p^2L\left[\overline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha).\end{aligned} \tag{72}$$

Rearranging equation (72)

$$(ap-p^2)L\left[\overline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]=(2b-p)\underline{y}_0(\alpha)+a\overline{y}_0(\alpha)-\underline{z}_0(\alpha)\ . \tag{73}$$

Also,

$$\begin{aligned}&apL\left[\overline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]-a\overline{y}_0(\alpha)+bp\underline{y}_0(\alpha)-b\underline{y}_0(\alpha)\\&=p^2L\left[\underline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\overline{z}_0(\alpha).\end{aligned} \tag{74}$$

Rearranging equation (74)

$$(bp-p^2)L\left[\underline{y}(t,\alpha)\right]+apL\left[\overline{y}(t,\alpha)\right]=(a-p)\overline{y}_0(\alpha)+(b-p)\underline{y}_0(\alpha)-\overline{z}_0(\alpha)\ . \tag{75}$$

Solving equations (73) and (75)

$$(ap-p^2)L\left[\overline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]=R_1(\alpha), \tag{76}$$

$$(bp-p^2)L\left[\underline{y}(t,\alpha)\right]+apL\left[\overline{y}(t,\alpha)\right]=R_2(\alpha)\ . \tag{77}$$

where $R_1(\alpha)=(2b-p)\underline{y}_0(\alpha)+a\overline{y}_0(\alpha)-\underline{z}_0(\alpha)$ and

$R_2(\alpha)=(a-p)\overline{y}_0(\alpha)+(b-p)\underline{y}_0(\alpha)-\overline{z}_0(\alpha)$.

Therefore,

$$L\left[\underline{y}(t,\alpha)\right]=\frac{apR_1(\alpha)-(ap-p^2)R_2(\alpha)}{abp^2-(ap-p^2)^2} \tag{78}$$

and

$$L\left[\overline{y}(t,\alpha)\right]=\frac{(ap-p^2)R_1(\alpha)-bpR_2(\alpha)}{(bp-p^2)^2-abp^2}\ . \tag{79}$$

Therefore,

$$H_{b_2}(t,\alpha)=\frac{(ap-p^2)R_2(\alpha)-apR_1(\alpha)}{abp^2-(ap-p^2)^2}\ . \tag{80}$$

$$K_{b_2}(t,\alpha)=\frac{bpR_2(\alpha)-(ap-p^2)R_1(\alpha)}{(bp-p^2)^2-abp^2}\ . \tag{81}$$

$H_{b_2}(t,\alpha)$ and $K_{b_2}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a\geq 0$ and $b<0$.

Case 3b: If $a<0$ and $b\geq 0$, then applying these conditions on equation (64) and (65)

$$apL\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]-a\underline{y}_0(\alpha)+bp\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)$$
$$=p^2L\left[\overline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (82)$$

Rearranging equation (82)

$$(bp-p^2)L\left[\overline{y}(t,\alpha)\right]+apL\left[\underline{y}(t,\alpha)\right]$$
$$=b\underline{y}_0(\alpha)+bp\overline{y}_0(\alpha)-(a-p)\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (83)$$

Also,

$$apL\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]-a\underline{y}_0(\alpha)+bp\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)$$
$$=p^2L\left[\underline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\overline{z}_0(\alpha). \quad (84)$$

Rearranging equation (84)

$$(ap-p^2)L\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]$$
$$=b\underline{y}_0(\alpha)-bp\underline{y}_0(\alpha)-(bp+p)\overline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\overline{z}_0(\alpha). \quad (85)$$

Solving equations (83) and (85)

$$(bp-p^2)L\left[\overline{y}(t,\alpha)\right]+apL\left[\underline{y}(t,\alpha)\right]=R_3(\alpha), \quad (86)$$

$$(ap-p^2)L\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]=R_4(\alpha) \quad (87)$$

where

$$R_3(\alpha)=b\underline{y}_0(\alpha)-bp\overline{y}_0(\alpha)+(a-p)\underline{y}_0(\alpha)-\underline{z}_0(\alpha)$$

and

$$R_4(\alpha)L=b\underline{y}_0(\alpha)-a\underline{y}_0(\alpha)-(bp+p)\overline{y}_0(\alpha)-\underline{z}_0(\alpha).$$

Therefore,

$$L\left[\underline{y}(t,\alpha)\right]=\frac{bpR_3(\alpha)-(bp-p^2)R_4(\alpha)}{abp^2-(bp-p^2)(ap-p^2)} \quad (88)$$

and

$$L\left[\overline{y}(t,\alpha)\right]=\frac{(ap-p^2)R_3(\alpha)-apR_4(\alpha)}{(ap-p^2)(bp-p^2)-abp^2}. \quad (89)$$

Therefore,

$$H_{b_3}(t,\alpha)=\frac{(bp-p^2)R_4(\alpha)-bpR_3(\alpha)}{abp^2-(bp-p^2)(ap-p^2)}. \quad (90)$$

$$K_{b_3}(t,\alpha)=\frac{apR_4(\alpha)-(ap-p^2)R_3(\alpha)}{(ap-p^2)(bp-p^2)-abp^2}. \quad (91)$$

$H_{b_3}(t,\alpha)$ and $K_{b_3}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a<0$ and $b\geq 0$.

Case 4b: If $a<0$ and $b<0$, then applying the conditions on equation (64) and (65)

$$\left[(a+b)p-p^2\right]L\left[\overline{y}(t,\alpha)\right]-a\underline{y}_0(\alpha)+bp\underline{y}(\alpha)-b\overline{y}_0=p\underline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (92)$$

Rearranging equation (92)

$$L\left[\overline{y}(t,\alpha)\right]=\frac{(a-bp+p)\underline{y}_0(\alpha)+b\overline{y}_0-\underline{z}_0(\alpha)}{ap+bp-p^2} \quad (93)$$

Also,

$$\left[(a+b)p-p^2\right]L\left[\underline{y}(t,\alpha)\right]-a\underline{y}_0(\alpha)+bp\underline{y}(\alpha)-b\overline{y}_0=-p\overline{y}_0(\alpha)-\overline{z}_0(\alpha). \quad (94)$$

Rearranging equation (94)

$$L\left[\overline{y}(t,\alpha)\right]=\frac{(a-bp)\underline{y}_0(\alpha)+b\overline{y}_0-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)}{ap+bp-p^2}. \quad (95)$$

Therefore,

$$H_{b_4}(t,\alpha)=\frac{(a-bp+p)\underline{y}_0(\alpha)+b\overline{y}_0-\underline{z}_0(\alpha)}{ap+bp-p^2}\ . \quad (96)$$

$$K_{b_4}(t,\alpha)=\frac{(a-bp)\underline{y}_0(\alpha)+b\overline{y}_0-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)}{ap+bp-p^2}\ . \quad (97)$$

$H_{b_4}(t,\alpha)$ and $K_{b_4}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a<0$ and $b<0$.

(c) Consider equation (21), (22) and (24) we have the results below.

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-(a-bp)\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)=p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha)\,, \quad (98)$$

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-(a-bp)\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)=p^2L\left[\overline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)\,. \quad (99)$$

Using following conditions on equation (98) and (99) we have the following cases.

Case 1c: If $a\geq 0$ and $b\geq 0$, then

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-(a-bp)\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)-p^2L\left[\underline{y}(t,\alpha)\right]=-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha)\,. \quad (100)$$

Rearranging equation (100)

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-p^2L\left[\underline{y}(t,\alpha)\right]=(a-bp)\overline{y}_0(\alpha)$$
$$+b\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha)\,. \quad (101)$$

Also, from equation (99)

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-p^2L\left[\overline{y}(t,\alpha)\right]=(a-bp)\underline{y}_0(\alpha)$$
$$+b\overline{y}_0(\alpha)-p\overline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha)\,. \quad (102)$$

From equations (101) and (102)

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-p^2L\left[\underline{y}(t,\alpha)\right]=B_1(\alpha)\,, \quad (103)$$

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-p^2L\left[\overline{y}(t,\alpha)\right]=B_2(\alpha) \quad (104)$$

where

$$B_1(\alpha)=(a-bp)\overline{y}_0(\alpha)+b\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha)$$

and

$$B_2(\alpha)=(a-bp)\underline{y}_0(\alpha)+b\overline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)\,.$$

Therefore, solving equation (103) and (104) we have

$$\left[(a+b)^2p^2+p^4\right]L\left[\underline{y}(t,\alpha)\right]=(a+b)pB_2(\alpha)-p^2B_1(\alpha)\,, \quad (105)$$

$$L\left[\underline{y}(t,\alpha)\right]=\frac{(a+b)pB_2(\alpha)-p^2B_1(\alpha)}{(a+b)^2p^2+p^4}\ , \quad (106)$$

$$\left[(a+b)^2p^2+p^4\right]L\left[\overline{y}(t,\alpha)\right]=(a+b)pB_1(\alpha)-p^2B_2(\alpha)\,, \quad (107)$$

and

$$L\left[\overline{y}(t,\alpha)\right]=\frac{(a+b)pB_1(\alpha)-p^2B_2(\alpha)}{(a+b)^2p^2+p^4}\ . \quad (108)$$

Therefore,

$$H_{c_1}(t,\alpha)=\frac{p^2B_1(\alpha)-(a+b)pB_2(\alpha)}{(a+b)^2p^2+p^4}\ . \quad (109)$$

$$K_{c_1}(t,\alpha)=\frac{p^2B_2(\alpha)-(a+b)pB_1(\alpha)}{(a+b)^2p^2+p^4}\ . \quad (110)$$

$H_{c_1}(t,\alpha)$ and $K_{c_1}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a\geq 0$ and $b\geq 0$.

Case 2c: If $a \geq 0$ and $b < 0$, applying these conditions on equations (64) and (65)

$$apL\left[\overline{y}(t,\alpha)\right] + bpL\left[\underline{y}(t,\alpha)\right] - a\overline{y}_0(\alpha) + bp\underline{y}_0(\alpha) - b\overline{y}_0(\alpha) = p^2L\left[\underline{y}(t,\alpha)\right] - p\underline{y}_0(\alpha) - \overline{z}_0(\alpha) \quad (111)$$

and

$$apL\left[\overline{y}(t,\alpha)\right] + bpL\left[\underline{y}(t,\alpha)\right] - a\overline{y}_0(\alpha) + bp\underline{y}_0(\alpha) - b\overline{y}_0(\alpha) = p^2L\left[\overline{y}(t,\alpha)\right] - p\overline{y}_0(\alpha) - \underline{z}_0(\alpha). \quad (112)$$

From equation (4.91)

$$(bp - p^2)L\left[\underline{y}(t,\alpha)\right] + apL\left[\overline{y}(t,\alpha)\right] = b\overline{y}_0(\alpha) + a\overline{y}_0(\alpha) - (b+p)\underline{y}_0(\alpha) - \overline{z}_0(\alpha), \quad (113)$$

$$(bp - p^2)L\left[\underline{y}(t,\alpha)\right] + apL\left[\overline{y}(t,\alpha)\right] = D_1(\alpha). \quad (114)$$

where $D_1(\alpha) = b\overline{y}_0(\alpha) + a\overline{y}_0(\alpha) - (b+p)\underline{y}_0(\alpha) - \overline{z}_0(\alpha)$.

Also, from equation (112)

$$(ap - p^2)L\left[\overline{y}(t,\alpha)\right] + bpL\left[\underline{y}(t,\alpha)\right] = b\overline{y}(\alpha) + a\overline{y}_0(\alpha) - bp\underline{y}_0(\alpha) - p\overline{y}_0(\alpha) - \underline{z}_0(\alpha) \quad (115)$$

$$(ap - p^2)L\left[\overline{y}(t,\alpha)\right] + bpL\left[\underline{y}(t,\alpha)\right] = D_2(\alpha) \quad (116)$$

where $D_2(\alpha) = b\overline{y}_0(\alpha) + a\overline{y}_0(\alpha) - bp\underline{y}_0(\alpha) - p\overline{y}_0(\alpha) - \underline{z}_0(\alpha)$.

Therefore, solving equation (114) and (116) we have

$$\left[abp^2 - (ap - p^2)(bp - p^2)\right]L\left[\overline{y}(t,\alpha)\right] = bpD_1(\alpha) - (bp - p^2)D_2(\alpha), \quad (117)$$

$$L\left[\overline{y}(t,\alpha)\right] = \frac{bpD_1(\alpha) - (bp - p^2)D_2(\alpha)}{abp^2 - (ap - p^2)(bp - p^2)}, \quad (118)$$

and

$$\left[(ap - p^2)(bp - p^2) - abp^2\right]L\left[\underline{y}(t,\alpha)\right] = (ap - p^2)D_1(\alpha) - apD_2(\alpha), \quad (119)$$

$$L\left[\underline{y}(t,\alpha)\right] = \frac{(ap - p^2)D_1(\alpha) - apD_2(\alpha)}{(ap - p^2)(bp - p^2) - abp^2}. \quad (120)$$

Therefore,

$$H_{c_2}(p,\alpha) = \frac{(bp - p^2)D_2(\alpha) - bpD_1(\alpha)}{abp^2 - (ap - p^2)(bp - p^2)} \quad (121)$$

$$K_{c_2}(p,\alpha) = \frac{apD_2(\alpha) - (ap - p^2)D_1(\alpha)}{(ap - p^2)(bp - p^2) - abp^2} \quad (122)$$

$H_{c_2}(t,\alpha)$ and $K_{c_2}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a \geq 0$ and $b < 0$.

Case 3c: If $a < 0$. $b \geq 0$. Applying these conditions in equations (64) and (65)

$$apL\left[\underline{y}(t,\alpha)\right] + bpL\left[\overline{y}(t,\alpha)\right] - a\underline{y}_0(\alpha) + bp\overline{y}_0(\alpha) - b\underline{y}_0(\alpha) = p^2L\left[\underline{y}(t,\alpha)\right] - p\underline{y}_0(\alpha) - \overline{z}_0(\alpha) \quad (123)$$

and

$$apL\left[\underline{y}(t,\alpha)\right] + bpL\left[\overline{y}(t,\alpha)\right] - a\underline{y}_0(\alpha) + bp\overline{y}_0(\alpha) - b\underline{y}(\alpha) = p^2L\left[\overline{y}(t,\alpha)\right] - p\overline{y}_0(\alpha) - \underline{z}_0(\alpha). \quad (124)$$

From equation (123)

$$(ap - p^2)L\left[\underline{y}(t,\alpha)\right] + bpL\left[\overline{y}(t,\alpha)\right] = b\underline{y}_0(\alpha) + a\underline{y}_0(\alpha) - bp\overline{y}_0(\alpha) - p\underline{y}_0(\alpha) - \overline{z}_0(\alpha) \quad (125)$$

$$(ap - p^2)L\left[\underline{y}(t,\alpha)\right] + bpL\left[\overline{y}(t,\alpha)\right] = M_1(\alpha) \quad (126)$$

where $M_1(\alpha) = b\underline{y}(\alpha) + a\underline{y}_0(\alpha) - bp\overline{y}_0(\alpha) - p\underline{y}_0(\alpha) - \overline{z}_0(\alpha)$.

Also, from equation (124)

$$(bp-p^2)L\left[\overline{y}(t,\alpha)\right]+apL\left[\underline{y}(t,\alpha)\right]=b\underline{y}_0(\alpha)+a\underline{y}_0(\alpha)-bp\overline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha) \quad (127)$$

$$(bp-p^2)L\left[\overline{y}(t,\alpha)\right]+apL\left[\underline{y}(t,\alpha)\right]=M_2(\alpha) \quad (128)$$

where $M_2(\alpha)=b\underline{y}_0(\alpha)+a\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)$.

Therefore, solving equation (126) and (128) we have

$$\left[abp^2-(ap-p^2)(bp-p^2)\right]L\left[\overline{y}(t,\alpha)\right] = apM_1(\alpha)-(ap-p^2)M_2(\alpha). \quad (129)$$

$$L\left[\overline{y}(t,\alpha)\right]=\frac{apM_1(\alpha)-(ap-p^2)M_2(\alpha)}{abp^2-(ap-p^2)(bp-p^2)} \quad (130)$$

and

$$\left[(ap-p^2)(bp-p^2)-abp^2\right]L\left[\underline{y}(t,\alpha)\right] =(bp-p^2)M_1(\alpha)-bpM_2(\alpha) \quad (131)$$

Also,

$$L\left[\underline{y}(t,\alpha)\right]=\frac{(bp-p^2)M_1(\alpha)-bpM_2(\alpha)}{(ap-p^2)(bp-p^2)-abp^2}. \quad (132)$$

Therefore,

$$H_{c_3}(t,\alpha)=\frac{(bp-p^2)M_2(\alpha)-bpM_1(\alpha)}{abp^2-(ap-p^2)(bp-p^2)} \quad (133)$$

$$K_{c_3}(t,\alpha)=\frac{apM_2(\alpha)-(ap-p^2)M_1(\alpha)}{(ap-p^2)(bp-p^2)-abp^2}. \quad (134)$$

$H_{c_3}(t,\alpha)$ and $K_{c_3}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a<0$ and $b\geq 0$.

Case 4c: If $a<0$ and $b<0$, applying these conditions on equations (64) and (65)

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-(a-bp)\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)=p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha), \quad (135)$$

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-(a-bp)\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)=p^2L\left[\overline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (136)$$

Rearranging equation (135) and (136)

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-p^2L\left[\underline{y}(t,\alpha)\right] =b\overline{y}'(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha) \quad (137)$$

Also,

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-p^2L\left[\overline{y}(t,\alpha)\right] =b\overline{y}(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (138)$$

Solving equations (137) and (138)

$$\left[(a+b)p-p^2\right]pL\left[\overline{y}(t,\alpha)\right]=b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha), \quad (139)$$

$$L\left[\overline{y}(t,\alpha)\right]=\frac{b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha)}{(a+b)p-p^2}, \quad (140)$$

$$\left[(a+b)p-p^2\right]pL\left[\underline{y}(t,\alpha)\right]=b\overline{y}(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha) \quad (141)$$

$$L\left[\underline{y}(t,\alpha)\right]=\frac{b\overline{y}(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)}{(a+b)p-p^2} \quad (142)$$

Therefore,

$$H_{c_4}(t,\alpha)=\frac{b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha)}{(a+b)p-p^2}. \quad (143)$$

$$K_{c_4}(t,\alpha)=\frac{b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)}{(a+b)p-p^2}. \quad (144)$$

$H_{c_4}(t,\alpha)$ and $K_{c_4}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a<0$ and $b<0$.

(d) Consider equation (21), (22) and (24) the following are obtained.

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-(a-bp)\underline{y}_0(\alpha)-b\overline{y}(\alpha)=p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha), \quad (145)$$

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-(a-bp)\overline{y}_0(\alpha)-b\underline{y}(\alpha)=p^2L\left[\overline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (146)$$

We discuss equation (145) and (146) using the conditions below.

Case 1d: If $a\geq 0$, $b\geq 0$. Applying these conditions on equations (145) and (146)

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-(a-bp)\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)=p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha), \quad (147)$$

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-(a-bp)\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)=p^2L\left[\overline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (148)$$

Therefore,

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-p^2L\left[\underline{y}(t,\alpha)\right]=(a-bp)\overline{y}_0(\alpha)+b\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha), \quad (149)$$

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-p^2L\left[\overline{y}(t,\alpha)\right]=(a-bp)\overline{y}_0(\alpha)+b\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha), \quad (150)$$

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-p^2L\left[\underline{y}(t,\alpha)\right]=R_1(\alpha), \quad (151)$$

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-p^2L\left[\overline{y}(t,\alpha)\right]=R_2(\alpha). \quad (152)$$

where

$R_1(\alpha)=(a-bp)\overline{y}_0(\alpha)+b\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha)$,

$R_2(\alpha)=(a-bp)\overline{y}_0(\alpha)+b\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)$.

Therefore, solving equation (150) and (151) we have

$$\left[(a+b)^2p^2+p^4\right]L\left[\underline{y}(t,\alpha)\right]=(a+b)pR_2(\alpha)-p^2R_1(\alpha), \quad (153)$$

$$L\left[\underline{y}(t,\alpha)\right]=\frac{(a+b)pR_2(\alpha)-p^2R_1(\alpha)}{(a+b)^2p^2+p^4} \quad (154)$$

and

$$\left[(a+b)^2p^2+p^4\right]L\left[\overline{y}(t,\alpha)\right]=(a+b)pR_1(\alpha)-p^2R_2(\alpha). \quad (155)$$

$$L\left[\overline{y}(t,\alpha)\right]=\frac{(a+b)pR_1(\alpha)-p^2R_2(\alpha)}{(a+b)^2p^2+p^4}. \quad (156)$$

Therefore,

$$H_{d_1}(t,\alpha)=\frac{p^2R_1(\alpha)-(a+b)pR_2(\alpha)}{(a+b)^2p^2+p^4}. \quad (157)$$

$$K_{d_1}(t,\alpha)=\frac{p^2R_2(\alpha)-(a+b)pR_1(\alpha)}{(a+b)^2p^2+p^4}. \quad (158)$$

$H_{d_1}(t,\alpha)$ and $K_{d_1}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a\geq 0$ and $b\geq 0$.

Case 2d: If $a\geq 0$, $b<0$. Applying these conditions on equations (144) and (145)

$$\begin{aligned}&apL\left[\overline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]-a\overline{y}_0(\alpha)+b\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)\\&=p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha)\end{aligned} \quad (159)$$

and

$$\begin{aligned}&apL\left[\overline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]-a\overline{y}_0(\alpha)+bp\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)\\&=p^2L\left[\overline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)\end{aligned}. \quad (160)$$

From equation (158),

$$(bp-p^2)L\left[\underline{y}(t,\alpha)\right]+apL\left[\overline{y}(t,\alpha)\right]=b\overline{y}_0(\alpha)+a\underline{y}_0(\alpha)-b\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha) \quad (161)$$

$$(bp-p^2)L\left[\underline{y}(t,\alpha)\right]+apL\left[\overline{y}(t,\alpha)\right]=C_1(\alpha) \quad (162)$$

where $C_1(\alpha)=b\overline{y}_0(\alpha)+a\overline{y}_0(\alpha)-b\underline{y}_0(\alpha)-b\underline{y}_0(\alpha)-\overline{z}_0(\alpha)$.

Also, from equation (159),

$$(ap-p^2)L\left[\overline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]=b\overline{y}_0(\alpha)+a\overline{y}_0(\alpha)-bp\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha) \quad (163)$$

$$(ap-p^2)L\left[\overline{y}(t,\alpha)\right]+bpL\left[\underline{y}(t,\alpha)\right]=C_2(\alpha) \quad (164)$$

where $C_2(\alpha)=b\overline{y}_0(\alpha)+a\overline{y}_0(\alpha)-bp\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)$.

Therefore, solving equation (161) and (164) we have

$$\left[(ap-p^2)(bp-p^2)-abp^2\right]L\left[\underline{y}(t,\alpha)\right]=(ap-p^2)C_2(\alpha)-apC_1(\alpha) \quad (165)$$

$$L\left[\overline{y}(t,\alpha)\right]=\frac{(ap-p^2)C_2(\alpha)-apC_1(\alpha)}{(ap-p^2)(bp-p^2)-abp^2}\ . \quad (166)$$

and

$$\left[abp^2-(ap-p^2)(bp-p^2)\right]L\left[\overline{y}(t,\alpha)\right]=bpC_1(\alpha)-(bp-p^2)C_2(\alpha) \quad (167)$$

$$L\left[\overline{y}(t,\alpha)\right]=\frac{bpC_1(\alpha)-(bp-p^2)C_2(\alpha)}{abp^2-(ap-p^2)(bp-p^2)}\ . \quad (168)$$

Therefore,

$$H_{d_2}(t,\alpha)=\frac{(bp-p^2)C_2(\alpha)-bpC_1(\alpha)}{abp^2-(ap-p^2)(bp-p^2)}\ . \quad (169)$$

$$K_{d_2}(t,\alpha)=\frac{(bp-p^2)C_2(\alpha)-bpC_1(\alpha)}{abp^2-(ap-p^2)(bp-p^2)}\ . \quad (170)$$

$H_{d_2}(t,\alpha)$ and $K_{d_2}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a\geq 0$ and $b<0$.

Case 3d: If $a<0$, $b\geq 0$. Applying these conditions on equations (144) and (145)

$$apL\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]-a\underline{y}_0(\alpha)+bp\overline{y}_0(\alpha)-b\underline{y}_0(\alpha) \\ =p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha) \quad (171)$$

and

$$apL\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]-a\underline{y}_0(\alpha)+bp\overline{y}_0(\alpha)-b\underline{y}_0(\alpha) \\ =p^2L\left[\overline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\overline{z}_0(\alpha)\ . \quad (172)$$

From equation (170)

$$(ap-p^2)L\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right] \\ =b\underline{y}_0(\alpha)+(a-p)\underline{y}_0(\alpha)-bp\overline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha) \quad (173)$$

$$(ap-p^2)L\left[\underline{y}(t,\alpha)\right]+bpL\left[\overline{y}(t,\alpha)\right]=L_1(\alpha) \quad (174)$$

where $L_1(\alpha)=b\underline{y}_0(\alpha)+(a-p)\underline{y}_0(\alpha)-bp\overline{y}_0(\alpha)-\overline{z}_0(\alpha)$. Also, from equation (121)

$$(bp-p^2)L\left[\overline{y}(t,\alpha)\right]+apL\left[\underline{y}(t,\alpha)\right] \\ =b\underline{y}_0(\alpha)+a\underline{y}_0(\alpha)-(b+p)p\overline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\overline{z}_0(\alpha)\ . \quad (175)$$

$$(bp-p^2)L\left[\overline{y}(t,\alpha)\right]+apL\left[\underline{y}(t,\alpha)\right]=L_2(\alpha) \quad (176)$$

where $L_2(\alpha)=b\underline{y}_0(\alpha)+ay_0(\alpha)-(b+p)\overline{y}_0(\alpha)-\overline{z}_0(\alpha)$. Therefore, solving equation (174) and (176) we have

$$\left[(ap-p^2)(bp-p^2)-abp^2\right]L\left[\underline{y}(t,\alpha)\right]=(bp-p^2)L_1(\alpha)-bpL_2(\alpha), \quad (177)$$

$$L\left[\underline{y}(t,\alpha)\right]=\frac{(bp-p^2)L_1(\alpha)-bpL_2(\alpha)}{(ap-p^2)(bp-p^2)-abp^2}, \quad (178)$$

and

$$\left[abp^2-(ap-p^2)(bp-p^2)\right]L\left[\overline{y}(t,\alpha)\right]=apL_1(\alpha)-(ap-p^2)L_2(\alpha). \quad (179)$$

$$L\left[\overline{y}(t,\alpha)\right]=\frac{apL_1(\alpha)-(ap-p^2)L_2(\alpha)}{abp^2-(ap-p^2)(bp-p^2)}. \quad (180)$$

Therefore,

$$H_{d_3}(t,\alpha)=\frac{bpL_2(\alpha)-(bp-p^2)L_1(\alpha)}{(ap-p^2)(bp-p^2)-abp^2}. \quad (181)$$

$$K_{d_3}(t,\alpha)=\frac{(ap-p^2)L_2(\alpha)-apL_1(\alpha)}{abp^2-(ap-p^2)(bp-p^2)}. \quad (182)$$

$H_{d_3}(t,\alpha)$ and $K_{d_3}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a<0$ and $b\geq 0$

Case 4d: If $a<0$ and $b<0$. Applying these conditions on equations (144) and (146)

$$(a+b)pL\left[\underline{y}(t,\alpha)\right]-(a-bp)\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)=p^2L\left[\underline{y}(t,\alpha)\right]-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha), \quad (183)$$

$$(a+b)pL\left[\overline{y}(t,\alpha)\right]-(a-bp)\underline{y}_0(\alpha)-b\overline{y}_0(\alpha)=p^2L\left[\overline{y}(t,\alpha)\right]-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha). \quad (184)$$

Rearranging equation (183) and (184)

$$\begin{aligned}&(a+b)pL\left[\underline{y}(t,\alpha)\right]-p^2L\left[\underline{y}(t,\alpha)\right]\\&=b\underline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha).\end{aligned} \quad (185)$$

Also,

$$\begin{aligned}&(a+b)pL\left[\overline{y}(t,\alpha)\right]-p^2L\left[\overline{y}(t,\alpha)\right]\\&=b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\underline{z}_0(\alpha).\end{aligned} \quad (186)$$

Solving equations (185) and (186)

$$\left[(a+b)p-p^2\right]pL\left[\underline{y}(t,\alpha)\right]=b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha), \quad (187)$$

$$L\left[\underline{y}(t,\alpha)\right]=\frac{b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha)}{(a+b)p-p^2}, \quad (188)$$

$$\left[(a+b)p-p^2\right]pL\left[\overline{y}(t,\alpha)\right]=b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha), \quad (189)$$

$$L\left[\underline{y}(t,\alpha)\right]=\frac{b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)}{(a+b)p-p^2}. \quad (190)$$

Therefore,

$$H_{d_4}(t,\alpha)=\frac{b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\underline{y}_0(\alpha)-\overline{z}_0(\alpha)}{(a+b)p-p^2}. \quad (191)$$

$$K_{d_4}(t,\alpha)=\frac{b\overline{y}_0(\alpha)+(a-bp)\underline{y}_0(\alpha)-p\overline{y}_0(\alpha)-\underline{z}_0(\alpha)}{(a+b)p-p^2}. \quad (192)$$

$H_{d_4}(t,\alpha)$ and $K_{d_4}(t,\alpha)$ are the relation between the FLT and its k^{th} derivative when $a<0$ and $b<0$.

The results presented as (a), (b), (c) and (d) above, show that $H_{a_1}K_{a_1}$ to $H_{a_4}K_{a_4}$ and $H_{b_1}K_{b_1}$ to $H_{b_4}K_{b_4}$, $H_{c_1}K_{c_1}$ to $H_{c_4}K_{c_4}$ and $H_{d_1}K_{d_1}$ to $H_{d_4}K_{d_4}$ are algebraically equivalent respectively.

Constructed Examples

Existence of Second Order Differential Equations

Consider the following equations

$$y_0'(t) = y^0 + \int_{t_0}^{t} f\left(s, y_0(s), y_0'(s)\right) ds \quad (193)$$

$$y_1'(t) = y^1 + \int_{t_0}^{t} f\left(s, y_1(s), y_1'(s)\right) ds \quad (194)$$

Let $y_0'(t) = y_0(t)$, $y_1'(t) = y_1(t)$, $|t - t_0| \le \alpha$

$$\|y_1 - y_0\| = \left\| \int_{t_0}^{t} f\left(s, y_0(s), y_0'(s)\right) ds \right\| \le M\,|t - t_0| \le \alpha M \le b .$$

thus $\|y_1(s) - y_0(s)\| \le b$. Then $\int f\left(s, y_1(s), y_1'(s)\right) ds$ is defined as $|t - t_0| \le \alpha$.

Hence,

$$\|y_2(t) - y_0\| = \left\| \int_{t_0}^{t} f\left(s, y_1(s), y_1'(s) ds\right) \right\| \le \int_{t_0}^{t} \left\| f\left(s, y_1(s), y_1'(s)\right) \right\| ds \le \alpha M \le b .$$

Also,

$$|t - t_0| \le \|y_k(t) - y_0\| \le \alpha M \le b, k = 1, \ldots, n .$$

Now, for $|t - t_0| \le \alpha$,

$$\|y_{k+1}(t) - y_k(t)\| = \left\| y^0 + \int_{t_0}^{t} f\left(s, y_k(s), y_k'(s)\right) ds - y^0 - \int_{t_0}^{t} f\left(s, y_{k-1}(s),\right) y_{k-1}'(s) ds \right\|$$

$$= \left\| \int_{t_0}^{t} f\left(s, y_k(s), y_k'(s)\right) - f\left(s, y_{k-1}(s),\right) y_{k-1}'(s) ds \right\| \le L \int_{t_0}^{t} \left\| \left(y_k(s), y_k'(s)\right) - \left(y_{k-1}(s), y_{k-1}'(s)\right) \right\| ds$$

where the inequality above results to the fact that f is Lipschitz.

Next is to prove that, for all k

$$\|y_{k+1} - y_k\| \le b \frac{\left(L|t - t_0|\right)^k}{k!} , \quad |t - t_0| \le \alpha . \quad (195)$$

Indeed, equation (195) holds for $k = 1$ as previously established. Now assume that equation (195) holds for $k = n$, then

$$\|y_{k+1}(t) - y_k(t)\| = \left\| y^0 + \int_{t_0}^{t} f\left(s, y_k(s), y_k'(s)\right) ds - y^0 - \int_{t_0}^{t} f\left(s, y_{k-1}(s), y_{k-1}'(s)\right) ds \right\|$$

$$\|y_{n+2} - y_{n+1}\| = \left\| \int_{t_0}^{t} f\left(s, y_{n+1}(s), y_{n+1}'(s)\right) - f\left(s, y_n(s), y_n'(s)\right) ds \right\|$$

$$\le \int_{t_0}^{t} L \left\| y_{n+1}(s), y_{n+1}'(s) - y_n(s) y_n'(s), \right\| ds \le \int_{t_0}^{t} Lb \frac{\left(L|t - t_0|\right)^n}{n!} ds \le b \frac{L^{n+1}}{n!} \frac{|t - t_0|^{n+1}}{n+1} \Bigg|_{s=t_0}^{s=t}$$

$$\le b \frac{\left(L|t - t_0|\right)^{n+1}}{(n+1)!} \quad |t - t_0| \le \alpha$$

therefore, equation (195) holds for $k = 1, \ldots$. Thus, for $N > n$, we have

$$\|y_N(t) - y_n(t)\| \le \sum_{k=n}^{N-1} \left\| y_{k+1}(t) y_{k+1}'(t), - y_k(t) y_k'(t), \right\| . \quad (196)$$

$$\|y_N(t)-y_n(t)\| \le \sum_{k=n}^{N-1} b\frac{\left(L|t-t_0|\right)^k}{k!} \le b\sum_{k=n}^{N-1}\frac{\left(L\alpha\right)^k}{k!}. \qquad (197)$$

Equation (197) tends to zero an $n \to \infty$. Therefore, $\{y_k(t)\}$ converges uniformly to a function $y(t)$ on the interval $|t-t_0| \le \alpha$. As the convergence is uniform, the limit function is continuous, moreover, $y(t_0) = y_0$. Indeed,

$$y_N(t) = y_0(t) + \sum_{k=1}^{N}\left(y_k(t)y_k'(t), -y_{k-1}(t), y_{k-1}'(t)\right).$$

Therefore,

$$y(t) = y_0(t) + \sum_{k=1}^{\infty}\left(y_k(t) - y_{k-1}(t)\right).$$

The fact that $y(t)$ is a solution of fuzzy differential equations follows from the following results. If a sequence of functions $\{y_k(t)\}$ converges uniformly and that $y_k(t)$ are continuous on the interval $|t-t_0| \le \alpha$, then

$$\lim_{n\to\infty}\int_{t_0}^{t} y_n(s)y_n'(s)ds = \int_{t_0}^{t}\lim_{n\to\infty} y_n(s)y_n'(s)\,ds.$$

Hence

$$y(t) = \lim_{n\to\infty} y_n(t)y_n'(t) = y^0 + \lim_{n\to\infty}\int_{t_0}^{t} f\left(s, y_{n-1}(s), y_{n-1}'(s)\right)ds = y^0 + \int_{t_0}^{t}\lim_{n\to\infty} f\left(s, y_{n-1}(s), y_{n-1}'(s)\right)ds$$

$$= y^0 + \int_{t_0}^{t} f\left(s, y(s)y'(s),\right)ds.$$

This is to say that

$$y(t) = y^0 + \int_{t_0}^{t} f\left(s, y(s), y'(s)\right)ds \text{ for } |t-t_0| \le \alpha,$$

as the integrand $f(t, y)$ is a continuous function, $y(t)$ is differentiable with respect to t, and $y'(t) = f(t, y(t))$, so $y(t)$ is a solution of the second order fuzzy differential equations (1). This shows that there exists a solution $y(t)$ to equation (1).

Uniqueness of Second Order Differential Equations

Consider the second order linear ordinary differential equation

$$y''(t) = f\left(t, y(t), y'\right), y(0) = y_0 = \left(\underline{y}_0, \overline{y}_0\right), y'(0) = z_0 = \left(\underline{z}_0, \overline{z}_0\right)$$

$$y'(t) = y^0 + \int_{t_0}^{t} f\left(s, y(s), y'(s)\right)ds. \qquad (198)$$

$$x'(t) = x^0 + \int_{t_0}^{t} f\left(s, x(s), x'(s)\right)ds. \qquad (199)$$

Now, substracting equation (198) and (199) we have

$y'(t) = y(t), x'(t) = x(t)$

where $x(t), y(t) \in D$. Then,

$$y(t) - x(t) = y^0 - x^0 + \int_{t_0}^{t}\left[f(s, y(s), y'(s)) - f(s, x(s), x'(s))\right]ds. \qquad (200)$$

Taking the normed of both sides of equation (199) and applying the Lipschitz condition, indicates that

$$0 \le \|y(t) - x(t)\| \le \|y^0 - x^0\| + \left\|\int_{t_0}^{t} L\|y(s), y'(s) - x(s), x'(s)\|ds\right\|. \qquad (201)$$

We then apply Gronwall to inequality (200) for $k=0$ and $r(t)=\|y(t)-x(t)\|$.

For $t_0 \le t \le t_0+\alpha$, we get $0 \le \|y(t)-x(t)\| \le 0$, that is, $\|y(t)-x(t)\|=0$, thus $y(t)=x(t)$.

For $t_0 \le t \le t_0+\alpha$. Similarly, for $t_0-\alpha \le t \le t_0$, $\|y(t)-x(t)\|=0$. Therefore, $y(t)=x(t)$ for $|t-t_0| \le \alpha$. This shows that the solution $y(t)=x(t)$ to equation (1) is unique.

DISCUSSION

A GTFN was applied to second order FODEs and **i**n that case, results were obtained for the solutions of equations (1), (2) and (3) categorized mainly as (a), (b), (c) and (d). Each category was subcategorized into four cases and solved using FLTM. Case 1 dealt with the situation when $a \ge 0$ and $b \ge 0$. The results obtained in that case were indicated as equations (10) and (11). Case 2 dealt with the situation when $a \ge 0$ and $b < 0$, which yielded the results in equations (22) and (23). Case 3 dealt with the situation when $a \le 0$ and $b \ge 0$, which yielded the results in equations (34) and (35). Also, case 4 dealt with the situation when $a < 0$ and $b < 0$ which yielded the results in equations (42) and (43). Similarly other results were further obtained such as equations (50), (51); (60), (61); (70), (71); and (76), (77) respectively for those categories mentioned above. See also additional results in equations (89), (90); (101), (102); (113), (114); (123); (4.124); (137), (138); (149), (150); (161), (162) and (171), (172) respectively. Those results presented above have shown the relationship that existed between the FLT of second order and its k^{th} derivative for $k \ge 1$. Also existence and uniqueness of solution were also presented in this work.

Moreover, obtained results for all the cases indicated above have further shown that GTFN is more convenient to use in solving second order FODEs compared to what one used in Sankar and Tapan (2015) and further proved that FLT and its k^{th} derivative are algebraically related.

CONCLUSION

In this study, a generalized triangular fuzzy number is used and fuzzy Laplace transform method is modified and applied on second order linear homogeneous ODE (1). Examples were constructed using the existence and uniqueness. The results obtained in this study showed that generalized triangular fuzzy number is more convenient in obtaining the solution of second order FODEs and its k^{th} derivative.

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