BAJOPAS Volume 11 Number 2 December, 2018 http://dx.doi.org/10.4314/bajopas.v11i2.29

Bayero Journal of Pure and Applied Sciences, 11(2): 219 - 224
Received: November, 2018
Accepted: December, 2018
ISSN 2006 – 6996

ON CLASSIFYING GROUPS OF SMALL ORDER

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ABSTRACT
Finite group theory is one of the most delightful areas in mathematics. The ideal aim of finite group theory is to find all finite groups up to isomorphism. In this paper, we reconsider the classification of groups of order less than or equal to 17 by using the presentation techniques of each group with the help of Lagrange's Theorem.
Keywords: Finite groups, Maximal order element, Isomorphism, Generator, Cyclic group, Group presentation

INTRODUCTION

In this paper, we will denote a group by G, its identity element by e, its order by $|G|$and a group generated by an element a by $\langle a\rangle$.

The finite group theory is concern with constructing finite groups of every possible type, and establishing sufficient procedures which will determine whether two finite groups are the same (up to isomorphism). Arthur Cayley (1859) was the first to classify all groups of order eight and found that there are exactly five non-isomorphic groups of order 8. (Cayley, 1859). The attainment of classifying all finite groups in general is of course per beyond the reach of present techniques since we have no general result which classified all groups of order n, $\forall\, n \in \mathbb{N}$, though the corresponding procedure for finite abelian groups was achieved a hundred years ago. (Wehrfritz, 1999). Similarly, for a prime numbers p and q, groups of order p, p^2,p^3 for $p \geq 2$ and pq are also well classified. (Grabowski, 2008). Pyber (1993) showed that, the number of non-isomorphic groups of order *n* generated by *d* elements is $f(n,d) \leq n^{c(d)\log n}$, for all positive integer *n*, where *n* is the order of the group, *c(d)* is a constant and *d* is a minimum number of generators. (Pyber, 1993). Avinoam Mann (2006) presented a result that, the number of non-isomorphic groups of order $n = {p_1}^{r_1} \ldots {p_m}^{r_m}$, (where ${p_1}^{r_1} \ldots {p_m}^{r_m}$ is a prime factorization of *n*) which generated by *k* elements denoted by $h(n,k) \leq n^{k \log\log n + 2\beta(n)+3}$. Where $\beta(n) = r_1 + \cdots + r_m$. (Mann, 2006). Gerard (2016) used class-equation and classified non-abelian groups of order less than 16. (Gerard, 2016). Gerard approach is independent to Sylow's theorems, but the approach is not applicable on abelian groups and fails to classify the non-abelian groups of order 8, 16, ..., because we can have two or more non-isomorphic non-abelian groups with same class equation. This motivates us to use a maximal order element which exist in any finite group to classify groups of order less than 18.

Preliminaries 2

Definition 2.01: (Roman, 2011)**.** Let G be a group and $H, K \leq G$

i) A collection of one element from each left (right) coset of H in G is called a left (right) transversal for H in G.
ii) H and K are said to be essentially disjoint if $H \cap K = \{e\}$.
iii) H is said to be a complement of K in G if $H \cap K = \{e\}$ and $G = HK$..

Theorem 2.02: (Lagrange's) (Judson and Thomas W., 2010). Let G be a finite group, $H \leq G$ and $a \in G$ such that $a \neq e$, then $|H|$ divides $|G|$ and $o(a)$ divides $|G|$.

Theorem 2.03: (Roman, 2011)**.** Let G be a group and $H, K \leq G$

i) Every element $a \in G$ can be written uniquely as $a = hk$ for some $h \in H$ and $k \in K$ if and only if H and K are essentially disjoint.
ii) $\forall\ a \in G$, $\exists$ a unique $h \in H$ and $k \in K$ such that $a = hk$ if and only if H and K are complements.
iii) K is a complement of H in G if and only if K is a left transversal of H in G.

Now, we give the following facts. Let G be a finite group and p be prime.

1) A groups of order p is unique (up to isomorphism). (Wehrfritz, 1999).
2) All elements of a group G are of order p if and only if $|G| = p^n$, for some positive integer *n*. (Roman, 2011).
3) If $|G| = 2p$, then either $G \cong Z_{2p}$ or $G \cong D_{2p}$. for an odd prime p. (Grabowski, 2008).
4) Let $|G| = p^2$, then either $G \cong Z_{p^2}$ or $G \cong Z_p \times Z_p$. (Grabowski, 2008).
5) Let $|G| = pq$, such that $p < q$ and q does not divide $p - 1$ for some prime q, then
$G \cong Z_{pq}$. (Grabowski, 2008).

That is, we have exactly one group of order 2, 3, 5, 7, 11, 13 and 17 up to isomorphism by fact (**1**)**.** There are two non-isomorphic groups of order 6, 10 and 14, by fact **(3)**. There exist two non-isomorphic groups of order 4 and 9, by fact **(4)**. Group of order 15 is unique up to isomorphism by fact **(5)**. Now we are left with groups of order 8, 12 and 16.

Main work 3

Now we begin the classification by maximal order element.

Groups of order eight 3.01:
If $|G| = 8$, then by Theorem 2.02, for every non-identity element $a \in G, o(a) = 8, 4$ or 2.
[**Case 1**.] If $a \in G$ is of order 8, then a generates G and $G = \langle a: a^8 = e \rangle \cong Z_8$.
[**Case 2**.] Suppose 4 is the highest order of elements of G, let $a \in G$ be of order 4, then $\langle a \rangle$ is of index 2 in G, therefore if $b \in G$ *and* $b \notin \langle a \rangle$, *then* $G = \langle a \rangle \cup \langle a \rangle b$. Now $b^2 \notin \langle a \rangle$ since $b \notin \langle a \rangle$, $b^2 \neq a$ and $b^2 \neq a^3$ since $o(b)$ cannot exceed 4. Therefore $b^2 = e$ or $b^2 = a^2$.

[(**1**)] For $b^2 = e$.
$ba \notin \langle a \rangle$ since $b \notin \langle a \rangle$, $ba \neq b$ and $ba \neq a^2 b$ since o(a) = 4.
Therefore we have $ba = ab$ or $ba = a^3 b = a^{-1} b$.
If $ba = ab$ then $G = \langle a, b: a^4 b^2 = e, ba = ab \rangle \cong Z_4 \times Z_2$.
If $ba = a^{-1} b$, then $G = \langle a, b: a^4 b^2 = e, ba = a^{-1} b \rangle \cong D_8$.

[(**II**)] For $b^2 = a^2$.
$ba \notin \langle a \rangle$ since $b \notin \langle a \rangle$, $ba \neq b$ since $a \neq e$.
If $ba = a^2 b$ then $ba = b^3$ and $b^3 ba = b^3 b^3$, i.e., $a = b^2 = a^2$ which is a contradiction. Therefore $ba = ab$ or $ba = a^{-1} b$.
If $ba = ab$ then

$$\begin{aligned} G &= \langle a, b: a^4 = e, a^2 = b^2, ba = ab \rangle \\ &= \langle a, b, c: a^4 = e, a^2 = b^2, c = ab^{-1}, ba = ab, aca = aac \rangle \\ &= \langle a, c: a^4 = c^2 = e, ca = ac \rangle \cong Z_4 \times Z_2. \end{aligned}$$

If $ba = a^{-1} b$, then $G = \langle a, b: a^4 = e, a^2 = b^2, ba = a^{-1} b \rangle \cong Q_8$

[**Case 3**.] Suppose every non-identity element of G is of order 2, then for non-identity elements $a, b, c \in G$, $G = \langle a, b, c: a^2 = b^2 = c^2 = e, ba = ab, ca = ac, cb = bc \rangle \cong Z_2 \times Z_2 \times Z_2$.
Hence, we get all the five groups of order 8, (Z_8, $Z_4 \times Z_2, Z_2 \times Z_2 \times Z_2, D_8, Q_8$) through maximal order element.

Groups of order twelve 3.02:
If G is of order 12, then by Theorem 2.02, for every non-identity element $a \in G$, we have $o(a) = 12, 6, 4, 3,$ or 2.
[**Case 1**.] If there exists $a \in G$, of order 12, then a generates G, and $G = \langle a: a^{12} = e, \rangle \cong Z_{12}$.
[**Case 2**.] Suppose 6 is the highest order of elements of G, let $a \in G$ be of order 6, then $\langle a \rangle$ is of index two in G, for $b \in G$, $b \notin \langle a \rangle$, $G = \langle a \rangle \cup \langle a \rangle b$.
Now, $b^2 \notin \langle a \rangle$ since $b \notin \langle a \rangle$,
$b^2 \neq a$ and $b^2 \neq a^5$ since order of b cannot exceed 6.
Therefore $b^2 = e, a^2, a^3$ or a^4 but $a^4 = (a^2)^{-1}$. Therefore we consider only $b^2 = a^3, a^2$ and e.

[(**I**)] For $b^2 = a^3$,
$ba \notin \langle a \rangle$ since $b \notin \langle a \rangle$,
$ba \neq b$ since $a \neq e$,
$ba \neq a^2, a^3$ and a^4 since $o(a) = 6$,
$ba \neq ab$ since $o(ab) \leq 6$.
Therefore $ba = a^{-1} b$ and $G = \langle a, b: a^6 = e, a^3 = b^2, ba = a^{-1} b \rangle \cong Dic_3$.

[(**II**)] For $b^2 = a^2$, here b is of order 6.
$ba \notin \langle a \rangle$ since $b \notin \langle a \rangle$,
$ba \neq b$ since $a \neq e$,
$ba \neq a^2, a^3$ and a^4 since $o(a) = 6$.
If $ba = a^{-1} b$, then b has no inverse element in G since for $a^2 b \in G$, $ba^2 b = baab = a^4 b^2 = e$ and $a^2 bb = a^2 b^2 = a^4 \neq e$. Therefore $ba = ab$ and

$$\begin{aligned} G &= \langle a, b: a^6 = e, a^2 = b^2, ba = ab \rangle \\ &= \langle a, b, c: a^6 = e, a^2 = b^2, c = a^{-1} b, ba = ab, aca = aac \rangle \\ &= \langle a, c: a^6 = c^2 = e, ca = ac \rangle \cong Z_6 \times Z_2. \end{aligned}$$

[(**III**)] For $b^2 = e$,
$ba \notin \langle a \rangle$ since $b \notin \langle a \rangle$,
$ba \neq b$ since $a \neq e$,
$ba \neq a^2, a^3$ and a^4 since $o(a) = 6$.
Therefore $ba = ab$ or $ba = a^{-1} b$, if $ba = ab$ then $G = \langle a, b: a^6 = b^2 = e, ba = ab \rangle \cong Z_6 \times Z_2$.
If $ba = a^{-1} b$ then $G = \langle a, b: a^6 = b^2 = e, ba = a^{-1} b \rangle \cong D_{12}$.

[**Case 3**.] Suppose 4 is the highest order of elements of G and let $a \in G$ be of order 4, then $\langle a \rangle$ is of index three in G and for $b \in G$, $b \notin \langle a \rangle$,

[(**I**)] If b is of order 3 then $b^2 = b^{-1} \notin \langle a \rangle$ since there exists no element in $\langle a \rangle$ of order 3 and $G = \langle a \rangle \cup \langle a \rangle b \cup \langle a \rangle b^2$.
But, $ba \notin \langle a \rangle$ since $b \notin \langle a \rangle$,
$ba \neq b$ and $ba \neq a^2 b$ since $o(a) = 4$,
$ba \neq ab$ since $o(a) \leq 4$,
$ba \neq b^2$ since $a \neq b$,
$ba \neq ab^2, a^2b^2$, and a^3b^2 since $o(ab) \leq 4$. Therefore $ba \notin G$. This implies G is not a group.

[(**II**)] Suppose b is of order 4, if $\langle a \rangle$ and $\langle b \rangle$ are essentially disjoint then G is not of order 12. Now, let $x \in \langle a \rangle \cap \langle b \rangle$ be non-identity then $x = a^2 = b^2$ since a^2 and b^2 are of the same order and only non-identity, non-generators of the respective subgroups. Then there exists $c \in G$, of order 2 or 4 such that $c \notin \langle a \rangle$ and $c \notin \langle a \rangle b$, if $a^2 = c^2$ then $G = \langle a \rangle \cup \langle a \rangle b \cup \langle a \rangle c$.
Now, $bc \notin \langle a \rangle$ since $b \notin \langle a \rangle c$,
$bc \notin \langle a \rangle$c since $b \notin \langle a \rangle$,
$bc \neq b$ since $c \neq e$,
$bc \neq ab, a^2b$, and a^3b since $o(bc) \leq 4$. This implies $bc \notin G$ and G is not a group.

[(**III**)] Suppose b is of order 2, this is almost similar to [(**II**)] and by similar method it can be shown that G is not a group. Hence 4 cannot be highest order of G provided G is a group of order 12.
[**Case 4**.] Suppose 3 is the highest order of elements of G and let $a \in G$ be of order 3, then $\langle a \rangle$ is of index 4 in G, for $b \in G$, $b \notin \langle a \rangle$, $\langle a \rangle \cup \langle a \rangle b = \{e, a, a^2, b, ab, a^2b\}$.
But, $ba \notin \langle a \rangle$ since $b \notin \langle a \rangle$, $ba \neq b$ since $a \neq e$ and $ba \neq ab$ as $o(a) \leq 3$, therefore $ba = a^{-1}b$. Now, for $c \in G$, $c \notin \langle a \rangle \cup \langle a \rangle b$, $G = \langle a \rangle \cup \langle a \rangle b \cup \langle a \rangle c \cup \langle a \rangle bc$ and $c^2 = e$ since $o(c) \leq 3$. Now, $ca \notin \langle a \rangle$ since $c \notin \langle a \rangle$,
$ca \notin \langle a \rangle$bc since $o(a) = 3$,
$ca \neq ac, ab,$ and a^2b since $o(ac) \leq 3$,
$ca \neq b$ since $o(a) = 3$. Therefore $ca = a^{-1}c$.
Also $cb \notin \langle a \rangle$ since $c \notin \langle a \rangle b$,
$cb \notin \langle a \rangle b$ since $c \notin \langle a \rangle$,
$cb \neq c$ since $b \neq e$,
$cb \neq ac, a^2c, abc$ and a^2bc since $o(bc) \leq 3$. Therefore $cb = bc$ and

$$G = \langle a, b, c : a^3 = b^2 = c^2 = e, ba = a^{-1}b, ca = a^{-1}c, cb = bc \rangle \cong A_4.$$

[**Case 5**.] Suppose 2 is the highest order of elements of G that is all non-identity elements of G are of order 2, this cannot hold by fact (**2**).
Hence, we get all the five groups of order 12, ($Z_{12}, Z_6 \times Z_2, D_{12}, Dic_3$ and A_4) through maximal order element.

Groups of order sixteen 3.03:

Let G be a group of order 16, by Theorem 2.02, given any element $a \in G$, $a \neq e$, $o(a) = 16, 8, 4$ or 2.
[**Case 1**.] If there exists an element $a \in G$ of order 16, then a generates G and $G = \langle a \rangle \cong Z_{16}$.
[**Case 2**.] Suppose 8 is the highest order of elements of G. Let $a \in G$ be of order 8, $\langle a \rangle$ is of index 2 in G, for $b \in G$, $b \notin \langle a \rangle$, $G = \langle a \rangle \cup \langle a \rangle b$.
Now $b^2 \notin \langle a \rangle b$ since $b \notin \langle a \rangle$,
$b^2 \neq a, a^2, a^3, a^5$ and $a^7 = a^{-1}$ since $o(b) \leq 8$. Therefore, $b^2 = e, a^2, a^4$ or a^6.
But $a^6 = (a^2)^{-1}$ so we consider only $b^2 = e, a^2$ and a^4.
Now, $\langle a \rangle$ is normal in G, this implies $b\langle a \rangle b^{-1} = \langle a \rangle$ therefore, $a = bab^{-1}, ba^3b^{-1}, ba^5b^{-1}$ or ba^3b^{-1} since $o(a) = 8$.

[(**I**)]. For $b^2 = e$.
(**i**) If $bab^{-1} = a$ then $G = \langle a, b, : a^8 = b^2 = e, ba = ab \rangle \cong Z_8 \times Z_2$.
(**ii**) If $ba^3b^{-1} = a$, then $bab^{-1} = a^3$ and $G = \langle a, b, : a^8 = b^2 = e, ba = a^3b \rangle \cong SD_{16}$.
(**iii**) If $ba^5b^{-1} = a$, then $bab^{-1} = a^5$ and $G = \langle a, b, : a^8 = b^2 = e, ba = a^5b \rangle \cong M_{16}$.
(**iv**) If $ba^7b^{-1} = a$, then $bab^{-1} = a^7 = a^{-1}$ and $G = \langle a, b, : a^8 = b^2 = e, ba = a^{-1}b \rangle \cong D_{16}$.

[(**II**)]. For $b^2 = a^2$.
(**i**) If $bab^{-1} = a$, then

$$\begin{aligned} G &= \langle a, b : a^8 = e, a^2 = b^2, ba = ab \rangle \\ &= \langle a, b, c : a^8 = c^2 = e, a^2 = b^2, c = a^3b, ba = ab, (a^5c)a = a(a^5c) \rangle \\ &= \langle a, c : a^8 = c^2 = e, ca = ac \rangle \cong Z_8 \times Z_2. \end{aligned}$$

(**ii**) If $ba^3b^{-1} = a$, then $(ba^3b^{-1})^2 = ba^3b^{-1}ba^3b^{-1} = a^6 \neq a^2$, i.e., $ba^3b^{-1} \neq a$.
(**iii**) If $ba^5b^{-1} = a$, then $bab^{-1} = a^5$ and

$$\begin{aligned} G &= \langle a, b : a^8 = e, a^2 = b^2, ba = a^5b \rangle \\ &= \langle a, b, c : a^8 = e, a^2 = b^2, c = ab, ba = a^5b, (a^{-1}c)a = a^5(a^{-1}c) \rangle \\ &= \langle a, c : a^8 = c^2 = e, ca = a^5c \rangle \cong M_{16}. \end{aligned}$$

(**iv**) If $ba^7b^{-1} = a$, then $(ba^7b^{-1})^2 = ba^7b^{-1}ba^7b^{-1} = a^6 \neq a^2$, i.e., $ba^7b^{-1} \neq a$.

[(**III**)]. For $b^2 = a^4$.
[(**i**)] If $bab^{-1} = a$, then

$G = \langle a, b: a^8 = e, a^4 = b^2, ba = ab\rangle$
$= \langle a, b, c: a^8 = c^2 = e, 4 = b^2, c = a^2b, ba = ab, (a^6c)a = a(a^6c)\rangle$
$= \langle a, c: a^8 = c^2 = e, ca = ac\rangle \cong Z_8 \times Z_2.$

[(**ii**)] If $ba^3b^{-1} = a$, then $bab^{-1} = a^3$ and
$G = \langle a, b: a^8 = e, a^4 = b^2, ba = a^3b\rangle$
$= \langle a, b, c: a^8 = e, a^4 = b^2, c = ab, ba = a^3b, (a^{-1}c)a = a^3(a^{-1}c)\rangle$
$= \langle a, c: a^8 = c^2 = e, ca = a^3c\rangle \cong SD_{16}.$

[(**iii**)] If $ba^5b^{-1} = a$, then $bab^{-1} = a^5$ and
$G = \langle a, b: a^8 = e, a^4 = b^2, ba = a^5b\rangle$
$= \langle a, b, c: a^8 = e, a^4 = b^2, c = a^2b, ba = a^5b, (a^6c)a = a^5(a^6c)\rangle$
$= \langle a, c: a^8 = c^2 = e, ca = a^5c\rangle \cong M_{16}.$

[(**iv**)] If $ba^7b^{-1} = a$, then $bab^{-1} = a^7 = a^{-1}$ and
$G = \langle a, b,: a^8 = e, a^4 = b^2, ba = a^{-1}b\rangle \cong Q_{16}.$

[**Case 3**.] Suppose 4 is the highest order of elements of G, let $a \in G$ be of order 4

[(**I**)]Suppose there exists another element $b \in G$ of order 4, $b \notin \langle a\rangle$, such that $\langle a\rangle$ and $\langle b\rangle$ are essentially disjoint, then $\langle b\rangle$ is a transversal of $\langle a\rangle$ in G and by Theorem 2.08
$G = \langle a\rangle \times \langle b\rangle = \langle a, b: a^4 = b^4 = e, ba = ab\rangle \cong Z_4 \times Z_4$ or
$G = \langle a\rangle \rtimes \langle b\rangle = \langle a, b: a^4 = b^4 = e, ba = ab\rangle \cong Z_4 \rtimes Z_4.$

[(**II**)]If there exists an element $b \in G$ of order 4, $b \neq a$, $\langle a\rangle$ and $\langle b\rangle$ are not essentially disjoint in G and $\langle a\rangle \neq \langle b\rangle$, then there exists non-identity element $x \in \langle a\rangle \cap \langle b\rangle$. Now b and b^3 generates $\langle b\rangle$ also a and b^3 generates $\langle a\rangle$ therefore$x = a^2 = b^2$.

Consider$\langle a\rangle \cup \langle b\rangle = \{a, a, a^2, a^3, b, ab, a^2b, a^3b\}$.

$ba \notin \langle a\rangle$ since $b \notin \langle a\rangle$,
$ba \neq b$ and $ba \neq a^2b$ since a is of order 4.
$ba \neq ab$ since by setting $d = ab^{-1}$, we have $d^2 = ab^{-1}ab^{-1} = e$. Therefore, $ba = a^{-1}b$. But G is of order 16, then there exists $c \in G$, $c \notin \langle a\rangle \cup \langle a\rangle b$ such that $G = \langle a\rangle \cup \langle a\rangle b \cup \langle a\rangle c \cup \langle a\rangle bc$, and $o(c) = 2$ or 4.

[(i)] For $c^2 = e$.
Now, $ca \notin \langle a\rangle$ since $c \notin \langle a\rangle$,
$ca \notin \langle a\rangle b$ since $o(c) = 2$,
$ca \notin \langle a\rangle bc$ since $o(ac) \leq 4$,
$ca \neq c$ since $a \neq e$ and $ca \neq a^2c$ as $o(a) \neq 2$. Therefore, $ca = ac$ or $a^{-1}c$.
Also $cb \notin \langle a\rangle$ since $c \notin \langle a\rangle b$,
$cb \notin \langle a\rangle b$ since $c \notin \langle a\rangle$,
$cb \neq c$ since $b \neq e$ and $cb \neq ac, a^2c, a^3c, abc$ and a^2bc as $o(bc) \leq 4$. Therefore, $cb = bc$ or $b^{-1}c$.
Now, we have $a^4 = c^2 = e, a^2 = b^2, bab^{-1} = a^{-1}, cac^{-1} = a$ or a^{-1}.
For $bab^{-1} = a^{-1},\ cac^{-1} = a^{-1},\ cbc^{-1} = b^{-1}$,
$G = \langle a, b, c: a^4 = c^2 = e, a^2 = b^2, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = b^{-1}\rangle$
$= \langle a, b, c, d: a^4 = c^2 = e, a^2 = b^2, d = a^3c, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = b^{-1}\rangle$
$= \langle a, b, d: a^4 = d^2 = e, a^2 = b^2, bab^{-1} = a^{-1}, dad^{-1} = a^{-1}, db = bd\rangle$
$= \langle a, b, d, f: a^4 = d^2 = e, a^2 = b^2, f = bd, bab^{-1} = a^{-1}, dad^{-1} = a^{-1}, db = bd\rangle$
$= \langle a, d, f: a^4 = d^2 = e, a^2 = f^2, dad^{-1} = a^{-1}, fa = af, df = fd\rangle \cong D_8oZ_4.$
For $bab^{-1} = a^{-1},\ cac^{-1} = a^{-1},\ cb = bc$,
$G = \langle a, b, c: a^4 = c^2 = e, a^2 = b^2, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cb = bc\rangle$
$= \langle a, b, c, d: a^4 = c^2 = e, a^2 = b^2, d = bc, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = b^{-1}\rangle$
$= \langle a, c, d: a^4 = c^2 = e, a^2 = d^2, cac^{-1} = a^{-1}, da = ad, dc = cd\rangle \cong D_8oZ_4.$
For $bab^{-1} = a^{-1},\ ca = ac,\ cbc^{-1} = b^{-1}$,
$G = \langle a, b, c: a^4 = c^2 = e, a^2 = b^2, bab^{-1} = a^{-1}, ca = ac, cbc^{-1} = b^{-1}\rangle$
$= \langle a, b, c, d: a^4 = c^2 = e, a^2 = b^2, d = ac, bab^{-1} = a^{-1}, ca = ac, cbc^{-1} = b^{-1}\rangle$
$= \langle b, c, d: d^4 = c^2 = e, d^2 = b^2, cac^{-1} = a^{-1}, db = bd, dc = cd\rangle \cong D_8oZ_4.$
For $bab^{-1} = a^{-1},\ ca = ac,\ cb = bc$,
$G = \langle a, b, c: a^4 = c^2 = e, a^2 = b^2, bab^{-1} = a^{-1}, ca = ac, cb = bc\rangle \cong Q_8 \times Z_2.$

[(ii)] For $c^4 = e$.
G is of order 16 so this holds only if $a^2 = c^2$.
$cb \notin \langle a\rangle$ since $c \notin \langle a\rangle$b,
$cb \notin \langle a\rangle b$ since $c \notin \langle a\rangle$,
$cb \neq c$ and $cb \neq a^2c$ since $o(b) = 4$,
$cb \neq ac, cb \neq a^3c, cb \neq abc$ and $cb \neq a^3bc$ since $o(bc) \leq 4$. Therefore, $cb = bc$ or $b^{-1}c$ $=a^2bc$
$ca \notin \langle a\rangle$ since $c \notin \langle a\rangle$,
$ca \notin \langle a\rangle b$ since $o(a) = 4$,
$ca \notin \langle a\rangle bc$ since $o(ca) \leq 4$,
$ca \neq c$ and $ca \neq a^2c$ since $o(a) = 4$. Therefore $ca = ac$ or $a^{-1}c$ and we have $bab^{-1} = a^{-1}$, $cac^{-1} = a$ or a^{-1}, $cbc^{-1} = b$ or b^{-1}.
For $bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = b^{-1}$,
$G = \langle a, b, c: a^4 = e, a^2 = b^2 = c^2, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = b^{-1}\rangle$
$= \langle a, b, c, d: a^4 = e, a^2 = b^2 = c^2, d = abc, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cbc^{-1} = b^{-1}\rangle$
$= \langle a, b, d: a^4 = d^2 = e, a^2 = b^2, bab^{-1} = a^{-1}, da = ad, db = bd\rangle \cong Q_8 \times Z_2.$

For $bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cb = bc$,
$$G = \langle a, b, c: a^4 = e, a^2 = b^2 = c^2, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cb = bc\rangle \cong Q_8 \times Z_2.$$
For $bab^{-1} = a^{-1}, ca = ac, cbc^{-1} = b^{-1}$,
$$G = \langle a, b, c: a^4 = e, a^2 = b^2 = c^2, bab^{-1} = a^{-1}, cbc^{-1} = b^{-1}, ca = ac\rangle \cong Q_8 \times Z_2.$$
For $bab^{-1} = a^{-1}, ca = ac, cb = bc$,
$$G = \langle a, b, c: a^4 = e, a^2 = b^2 = c^2, bab^{-1} = a^{-1}, ca = ac, cb = bc\rangle \cong D_8 o Z_4.$$
[(**III**)]Now suppose $b \in G$ is of order 2 and $b \notin \langle a\rangle$, then $bab^{-1} = a$ or a^{-1}.
Also since G is of order 16, there exists $c \in G$, $c \notin \langle a\rangle \cup \langle a\rangle b$ and $o(c) = 2$ or 4.
But we take care by the case of $o(c) = 4$, so we consider $o(c) = 2$,
$$G = \langle a\rangle \cup \langle a\rangle b \cup \langle a\rangle c \cup \langle a\rangle bc.$$
Now, $ca \notin \langle a\rangle$ since $c \notin \langle a\rangle$,
$ca \notin \langle a\rangle b$ since $o(ac) \leq 4$,
$ca \neq c, ca \neq a^2c, ca \neq bc$ and $ca \neq a^2bc$ since $o(a) = 4$.
Therefore, $ca = ac, a^{-1}c, abc$ or $=a^{-1}bc$.
$cb \notin \langle a\rangle$ since $c \notin \langle a\rangle b$,
$cb \notin \langle a\rangle b$ since $c \notin \langle a\rangle$,
$cb \neq c, cb \neq ac$, and $cb \neq a^3c$ since $o(b) = 2$,
$cb \neq a^2c$ since $b \neq a^2$,
$cb \neq abc$ and $cb \neq a^3bc$ since $o(bc) \leq 4$. Therefore, $cb = bc$ or a^2bc.
Now we have $ba = ab$ or $a^{-1}b, cb = bc$ and $ca = ac, a^{-1}c, abc$ or $a^{-1}bc$.
For $ba = a^{-1}b, ca = a^{-1}c, cb = bc$.
$$G = \langle a, b, c: a^4 = b^2 = c^2 = e, bab^{-1} = a^{-1}, cac^{-1} = a^{-1}, cb = bc\rangle \cong Q_8 \times Z_2.$$
For $ba = a^{-1}b, ca = ac, cb = bc$.
$$G = \langle a, b, c: a^4 = b^2 = c^2 = e, bab^{-1} = a^{-1}, ca = ac, cb = bc\rangle \cong D_8 \times Z_2.$$
For $ba = a^{-1}b, ca = abc, cb = bc$.
Then $a = cabc \implies a^2 = cabccabc = cababc = caa^3bbc = e$.
But a is of order 4, therefore this cannot hold.
For $ba = a^{-1}b, ca = a^3bc, cb = bc$.
Now $a = a^3bc \implies a^2 = a^3bca^3bc = e$. But a is of order 4, therefore this cannot hold.
For $ba = ab, ca = ac, cb = bc$.
$$G = \langle a, b, c: a^4 = b^2 = c^2 = e, ba = ab, ca = ac, cb = bc\rangle \cong Z_4 \times Z_2 \times Z_2.$$
For $ba = ab, ca = a^{-1}c, cb = bc$.
$$G = \langle a, b, c: a^4 = b^2 = c^2 = e, ba = ab, ca = a^{-1}c, cb = bc\rangle \cong D_8 \times Z_2.$$
For $ba = ab, ca = abc, cb = bc$.
$$G = \langle a, b, c: a^4 = b^2 = c^2 = e, ba = ab, ca = abc, cb = bc\rangle \cong Small\ gruop(16,3).$$
For $ba = ab, ca = a^3bc, cb = bc$.
$$\begin{aligned} G &= \langle a, b, c: a^4 = b^2 = c^2 = e, ba = ab, ca = a^3bc, cb = bc\rangle \\ &= \langle a, b, c, d: a^4 = b^2 = c^2 = e, d = a^3c, ba = ab, ca = a^3bc, cb = bc\rangle \\ &= \langle a, d: a^4 = d^4 = e, da = a^{-1}d^{-1}\rangle \cong Small\ gruop(16,3). \end{aligned}$$
For $ba = a^{-1}b, ca = a^{-1}c, cb = a^2bc$.
$$\begin{aligned} G &= \langle a, b, c: a^4 = b^2 = c^2 = e, ba = a^{-1}b, ba = a^2bc, ca = a^{-1}c\rangle \\ &= \langle a, b, c, d: a^4 = b^2 = c^2 = e, d = a^3c, ba = a^{-1}b, ba = a^2bc, ca = a^{-1}c\rangle \\ &= \langle a, b, d: a^4 = b^2 = d^2 = e, ba = a^{-1}b, (ad)a = a^{-1}(ad), (ad)b = a^2b(ad)\rangle \\ &= \langle a, b, d: a^4 = b^2 = d^2 = e, ba = a^{-1}b, db = bd, da = a^{-1}d\rangle \\ &= \langle a, b, d, f: a^4 = b^2 = d^2 = e, f = bc, ba = a^{-1}b, db = bd, da = a^{-1}d\rangle \\ &= \langle a, b, f: a^4 = b^2 = f^2 = e, ba = a^{-1}b, fb = bf, fa = af\rangle \cong D_8 \times Z_2. \end{aligned}$$
For $ba = a^{-1}b, ca = ac, cb = a^2bc$.
$$\begin{aligned} G &= \langle a, b, c: a^4 = b^2 = c^2 = e, ba = a^{-1}b, ca = a^2bc, ca = ac\rangle \\ &= \langle a, b, c, d: a^4 = b^2 = c^2 = e, d = a^{-1}c, ba = a^{-1}b, ca = ac, cb = a^2bc\rangle \\ &= \langle a, b, d: a^4 = b^2 = e, a^2 = d^2, ba = a^{-1}b, da = aa, db = bd\rangle \cong D_8 \times Z_2. \end{aligned}$$
For $ba = ab, ca = a^{-1}c, ca = a^2bc$.
$$\begin{aligned} G &= \langle a, b, c: a^4 = b^2 = c^2 = e, ba = ab, ca = a^{-1}c, cb = a^2bc\rangle \\ &= \langle a, b, c, d: a^4 = b^2 = c^2 = e, d = a^{-1}b, ba = ab, ca = a^{-1}c, cb = a^2bc\rangle \\ &= \langle a, c, d: a^4 = c^2 = e, a^2 = d^2, ca = a^{-1}c, da = ad, dc = cd\rangle \cong D_8 o Z_4. \end{aligned}$$
For $ba = ab, ca = ac, cb = a^2bc$.
$$\begin{aligned} G &= \langle a, b, c: a^4 = b^2 = c^2 = e, ba = ab, ca = ac, cb = a^2bc\rangle \\ &= \langle a, b, c, d: a^4 = b^2 = c^2 = e, d = a^{-1}b, ba = ab, ca = ac, cb = a^2bc\rangle \\ &= \langle a, c, d: a^4 = c^2 = e, a^2 = d^2, ca = ac, da = ad, cd = d^{-1}c\rangle \cong D_8 o Z_4. \end{aligned}$$
For $ba = ab, ca = abc, cb = a^2bc$.
$G = \langle a\rangle \cup \langle a\rangle b \cup \langle a\rangle c \cup \langle a\rangle bc$, but $(abc)(a^3c) = abcaa^2 = a^2ca^2c = e$ and $(a^3c)(abc) = a^3abcbc = ba^2bcc = a^2c^2 = a^2$. So, we have some elements in G without inverse in G, that is G is not a group.
For $ba = ab, ca = a^3bc, cb = a^2bc$.
$(a^3bc)(ac) = a^3ba^3bcc = a^2b^2c^2 = a^2$ and $(ac)(a^3bc) = acaa^2bc = a^2bcbc = e$.
Therefore some of elements in G have no inverse in G. Hence G is not a group.
[**Case 4**.] Suppose any non-identity element is of order 2.

Let $a, b, c, d \in G$ be non-identity elements such that $c \notin \langle a \rangle \cup \langle a \rangle b$ and $d \notin \langle a \rangle \cup \langle a \rangle b \cup \langle a \rangle c$. Then, $G = \langle a \rangle \cup \langle a \rangle b \cup \langle a \rangle c \cup \langle a \rangle d$,

$$= \langle a, b, c, d : a^2 = b^2 = c^2 = d^2 = e, ba = ab, ca = ac, da = ad, cb = bc, db = bd, cd = dc, \rangle$$
$$\cong Z_2 \times Z_2 \times Z_2 \times Z_2.$$

Hence, we get all the 14 non-isomorphic groups of order 16 through maximal order element.

[1)] Z_{16}, [2)] $Z_8 \times Z_2$, [3)] M_{16}, [4)] D_{16}, [5)] Q_{16}, [6)] SD_{16}, [7)] $Z_4 \times Z_4$, [8)] $Z_4 \rtimes Z_4$, [9)] $Z_4 \times Z_2 \times Z_2$, [10)] $D_8 \times Z_2$, [11)] $Q_8 \times Z_2$, [12)] $Z_2 \times Z_2 \times Z_2 \times Z_2$, [13)] $D_8 o Z_4$ and [14)] Small group(16,3).

Conclusion 4

Maximal order elements play very important role in classifying finite groups. In this paper, we have presented a method of classifying groups of order ≤ 17 solely by use of maximal order element and without appealing to Sylow theorems..

Acknowledgments 5

The authors are really grateful to referee for giving enough background of the area which made the paper more readable.

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