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A Generalization of Dilworth's Decomposition Theorem for Partially Ordered Multisets

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ABSTRACT

Dilworth's decomposition theorem characterizes the size of the largest antichain of any finite partially ordered set in terms of a partition of the order into a minimum number of chains. In this study, Dilworth's theorem is generalized using multisets i.e., mathematical entities that admit repetition. The theorem is formulated analogously using a partially ordered multiset (or pomset), with combinatorial parameters defined in the multiset setting. Basic conditions are established for constructing antichains $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of an ordered multiset such that no two elements are comparable or equal. These conditions are adopted in proving Dilworth's theorem in the multiset setting, showing that the smallest number of chains in a partition of a pomset coincides with the size of the largest subpomset containing incomparable elements. Lastly, an algorithm for recursively constructing antichains of an ordered multiset is formulated. Python programming language is used in implementing this algorithm on an ordered multiset structure. It is decidable and runs in quadratic time complexity $O(n^2)$. For any finite pomset, the number of antichains constructed via this algorithm is the same as the size of a maximum multiset chain in the pomset.

Keywords: Ordered-multiset, multiset-chain (antichain), maximum-multiset chain (antichain), maximal-multiset chain (antichain), partially ordered multiset, Dilworth's theorem.

1. INTRODUCTION

The size of the largest chain (antichain) in any partially ordered set (or poset) P is the *height* (*width*) of P (see Schroeder, 2003 for details on posets). One of the goals of combinatorial research is to characterize combinatorial parameters. The height and width of a poset are important combinatorial concepts and, not surprisingly, a great deal of effort has gone into characterizing these parameters (Knauer et al.,

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2018; Joret et al., 2016; Streib and Trotter, 2014; Dilworth, 1950). Dilworth proved that the minimum number of chains into which a finite poset P can be partitioned is the same as the width of P. The size of the longest chain was characterized in Mirsky (1971). In the case of infinite posets, if there exists a partition of the order into finitely many chains, or if a finite upper bound exists for the size of an antichain, then the width of the poset is the same as the minimum number of chains in a partition of the order. The height of a partially ordered structure has been used in proposing interesting characterizations for its *dimension* i.e. the minimum number of linear orders needed to characterize it (Joret et al., 2016; Streib and Trotter, 2014; Kelly, 1981). For instance, Joret et al., (2016) proved that the dimension of a poset is bounded in terms of its height and tree-width of its cover graph. Ordered set theory has applications in different spheres of computer science, economics, and biology. In mathematics, ordered sets abound in fields like algebra, graph theory, fixed point theory, and category theory (Caspard et al., 2012). If the requirement for a set to contain *distinct elements* is relaxed, we would have a multiset (Yager, 1986). These are models of entities that admit repetitions. Multisets are defined on the basis of classical set theory; usually in terms of first-order predicate calculus with equality and the usual logical symbols. Thus, multiset theory generalizes the well-known Zermelo-Fraenkel set theory (Felisiak et al., 2020; Dang, 2014; Blizard, 1988 are good expositions on foundational works on multisets). Though a relatively new field of study, the theory of multisets is well-established and has various applications (Felisiak et al., 2020; Jurgensen, 2020; Singh et al., 2007; Knuth, 1981). In particular, ordered multisets have applications in computer science, biology, and linguistics (Paun and Rozenberg, 2002; Basten, 1997; Dershowitz and Manna, 1979). In Dershowitz and Manna (1979), ordered multisets are used to prove that certain types of computer programs terminate. Basten (1997) defined a partially ordered multiset (or pomset) as a generalization of a string in which the total order has been relaxed to a partial order. Based on the LR-parsing technique, the author developed the fundamentals of a parsing theory for pomsets called PLR parsing. In the light of diverse applications of ordered multisets, it becomes imperative to generalize results on ordered sets and establish new results that are peculiar to multisets (Balogun et al., 2020,2022; Girish and Sunil, 2009; Rensink, 1996; Pratt, 1986). In this paper, Dilworth's decomposition theorem for ordered sets is generalized using ordered multisets. We define basic combinatorial parameters and establish conditions under which Dilworth's theorem holds in the multiset setting. The rest of the work is organized as follows: for convenience, we present basic terminologies to be used in this study in section 2. The concept of multiset partition is discussed in section 3. In section 4, an analogous form of Dilworth's decomposition theorem is presented and a proof of the theorem is provided using ordered multisets. Lastly, we present an algorithm to show that the conditions in the dual of

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dual of Dilworth's result also hold for ordered multisets.

2. Preliminaries

Basic terminologies used will be presented here (see Balogun et al., 2021; Singh et al., 2007; Blizard, 1988 for details).

2.1 Multisets: Unlike the classical set, a multiset admits repetition thus in multiset theory, the sets $A = \{x_1, x_2, x_1, x_3, x_3, x_1\}$ and $B = \{x_1, x_2, x_3\}$ do not coincide. The number of times an object occurs is the *multiplicity* of the object. The multiplicity of an object takes value from any of the sets of *natural numbers*, *integers*, or *real numbers* (Felisiak et al., 2020; Blizard, 1988, 1989). The *root* (or support) set of a multiset M is usually denoted by M^*, Blizard (1988) established that $M^* \subseteq M \wedge Set(M^*) \wedge \forall x(x \in M \rightarrow x \in M^*)$ through the *separation schema* of set theory. Blizard's result showed that the *root* set M^* of a multiset M is the set containing the distinct elements of M; we will refer to these distinct elements as *objects*. In the example above, B is the root set for the multiset A. The multiset theory **MST** developed in Blizard (1988) is adopted for this work. As in **MST**, multisets used in this study are modelled by integer-valued functions and the multiplicity of an object is assumed to be a (finite) positive integer. A multiset M is finite if its root set is finite and each object has a finite multiplicity, it is infinite otherwise; this study focuses on finite multisets. The cardinality of M is the sum of the multiplicities of all objects in M. If we denote the class of all finite multisets defined on a set S by $M(S)$, then two multisets M and N in $M(S)$ are related by inclusion as follows: $M \subseteq N$ if $M(x) \leq N(x)$ for all $x \in S$, and $M \subset N$ if $M(x) < N(x)$ for at least one $x \in S$, where $M(x)$ and $N(x)$ represent the multiplicities of x in M and N, respectively. For convenience, we will denote an arbitrary *point* in a multiset M, i.e. an object together with its multiplicity, by m_i/x_i; this will represent the atomic formula $x_i \in^{m_i} M$. Thus the multiset A above would be written as $[3/x_1, 1/x_2, 2/x_3]$. A new instance of the relation $\in$ in the three place predicate symbol $x \in^z y$ was proposed in Singh and Singh (2007). The symbol $\in_+$ (which they call *dressed epsilon*) was introduced, this indicates the minimum value that the multiplicity of an object can assume. For instance, $x \in_+^z y$ implies x belongs to y at least z times. The predicate $\in_+$ is useful for modelling problems that do not require the exact value of the multiplicity of an object. Given any two multisets M and N, the additive union (also

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called multiset union) of M and N, denoted by $M \uplus N$, for each object x_i is the multiset defined by

$$Multiplicity_{(M \uplus N)}(x_i) = (multiplicity\ of\ x_i\ in\ M) + (multiplicity\ of\ x_i\ in\ N) = m_i + n_i$$

For instance, given multisets A and B as above we have

$A \uplus B = [4/x_1, 2/x_2, 3/x_3]$. The additive union of two or more multisets is assumed in sections 3 and 4 to obtain the number of elements in a partition of a given multiset. For more on multiset operations see Singh et al., (2007).

2.2 Ordered Multisets: An ordered multiset is a multiset with a reflexive, antisymmetric and transitive multiset relation defined on it (details on multiset relations is presented in Girish and Sunil, 2009). Throughout this work, we will assume that the multiset M is defined on a partially ordered set $P = (S, \preccurlyeq)$. Also, $\mathcal{M} = (M, \preccurlyeq\leq)$ will be a partially ordered multiset (or pomset), where M is a finite multiset and $\preccurlyeq\leq$ is a partial multiset order on M induced by the ordering $\preccurlyeq$.

Definition 1.3: Let $m_i/x_i, m_j/x_j$ be any two points of M. Then $m_i/x_i \preccurlyeq\leq m_j/x_j$ in $\mathcal{M}$ if and only if $x_i \preccurlyeq x_j$ in P.

Remark 1.4: The two points $m_i/x_i, m_j/x_j$ are comparable if $(m_i/x_i \preccurlyeq\leq m_j/x_j) \vee (m_j/x_j \preccurlyeq\leq m_i/x_i)$, they are incomparable otherwise.

Let N be a submultiset of M. Then a subpomset $\mathcal{C} = (N, \preccurlyeq\leq)$ of $\mathcal{M}$ is a multiset chain if for any two points n_i/x_i and n_j/x_j in N, either $n_i/x_i \preccurlyeq\leq n_j/x_j$ or $n_j/x_j \preccurlyeq\leq n_i/x_i$, where $n_i \leq m_i$ for all i. A subpomset $\mathcal{A}$ is an antichain if it contains only incomparable pairs; we will write $(l_i/x_i)||(l_j/x_j)$ whenever l_i/x_i and l_j/x_j are incomparable in $\mathcal{M}$. A point m_i/x_i is maximal in $\mathcal{M}$ if it is not covered by any other point in the pomset i.e., if $\nexists(m_j/x_j \in M)$ such that $m_i/x_i << m_j/x_j$. Similarly, m_i/x_i is minimal if $\nexists(m_j/x_j \in M)$ such that $m_j/x_j << m_i/x_i$ (where $<<$ is the strict ordering on M). A subpomset of $\mathcal{M}$ is maximal if it is not strictly contained in any other subpomset, it is maximum if it contains the most number of elements. The height (width) of a pomset is the cardinality of a maximum multiset chain (antichain), we will denote this by $\hbar$ and w, respectively.

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3.1. MULTISET PARTITION

In this section, the notion of a multiset partition is presented following Jouannaud & Lescanne (1982). The adopted multiset partition is particularly useful for constructing subpomsets of $\mathcal{M}$ such that no two elements are equal or comparable.

Definition 3.1: Let M be a multiset then, $\{M_i | i = 1, \dots, p\}$ is a partition of M if and only if $M = \sum_{i=1}^{p} M_i$.

Definition 3.2: If M is a multiset defined over a poset $(S, \preccurlyeq)$ and $\mathcal{M} = (M, \preccurlyeq\leq)$ is the pomset induced by $\preccurlyeq$. Then $\bar{M} = \{M_i | i = 1 \dots p\}$ is a partition of M if it satisfies the following conditions:

$M_i(x_i) \leq 1$, for each i, where $M_i(x_i)$ represents the multiplicity of x_i in M_i.

$x_i \in M_i$ and $x_j \in M_i \Rightarrow x_i || x_j$,

$\forall i \in [2, \dots, p],\ x_i \in M_i \Rightarrow (\exists x_j \in M_{i-1}) x_j \succcurlyeq x_i$.

Remark 3.3: Each multiset M_i in the partition $\bar{M}$ will consist of incomparable elements. Unlike in classical set theory, the intersection of any two elements of the partition need not be empty since the structure admits repetition. By condition i of definition 2.2, $Set(M_i)$ holds $\forall i \in \{1, \dots, p\}$, we have

$M_i = \emptyset \vee \forall x \forall n (x \in^n M_i \to n = 1)$.

We need the following result from Balogun et al., (2021). The lemma would be used in proving the generalized Dilworth's decomposition theorem, hence a detailed proof is provided here.

Lemma 3.4: Suppose a finite pomset $\mathcal{M}$ is partitioned into multiset chains $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$. If $\mathcal{A}$ is any antichain in $\mathcal{M}$, then at most one element of $\mathcal{A}$ occurs in each multiset chain in the partition, thus $n \geq |\mathcal{A}|$.

Proof: Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a finite pomset induced by the poset $(S, \preccurlyeq)$. Suppose $\{\mathcal{C}_k | k = 1, \dots, n\}$ is a partition of $\mathcal{M}$, where each $\mathcal{C}_k$ is a maximal multiset chain. If $\mathcal{A}$ is an antichain in $\mathcal{M}$, we need to show that the intersection of $\mathcal{A}$ and $\mathcal{C}_k$ has at most one element, for any k.

Since a multiset admits repetition, antichains of $\mathcal{M}$ will be constructed via definition 2.2 so that

$\forall x_i \forall x_j \in \mathcal{A} \sim [(x_i \prec x_j) \vee (x_i \succ x_j) \vee (x_i = x_j)]$

i.e., such that no two elements are comparable or equal in $\mathcal{A}$. We then show that

$|\mathcal{A} \cap \mathcal{C}_k| \leq 1$ for all k.

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Let $\mathcal{C}_k$ be a maximal multiset chain in $\mathcal{M}$ for all $k = 1,2,\ldots,n$. Then the multiset, say $\boldsymbol{A}$, consisting of all maximal points from each $\mathcal{C}_k$ will contain only incomparable points. Applying the construction in definition 2.2 to $\boldsymbol{A}$ gives a partition of the form $\bar{\boldsymbol{A}} = \{\boldsymbol{A}_i | i = 1 \ldots p\}$, for an integer p such that $\forall x_i \forall x_j \in \boldsymbol{A}_i, (x_i || x_j \wedge Set(\boldsymbol{A}_i))$ i.e. $\forall x_i \forall x_j \in \boldsymbol{A}_i (x_i || x_j \wedge \forall i (\boldsymbol{A}_i(x_i) \leq 1))$. If $\mathcal{C}_k$ and $\boldsymbol{A}_i$ are disjoint, the result is straight forward. Suppose $|\boldsymbol{A}_i \cap \mathcal{C}_k| \neq \emptyset$, we need to show that $|\boldsymbol{A}_i \cap \mathcal{C}_k| = 1$. Suppose $|\boldsymbol{A}_i \cap \mathcal{C}_k| > 1$. Then $\exists x_1, \ldots, x_n$ in $|\boldsymbol{A}_i \cap \mathcal{C}_k|$ with $n \geq 2$. This implies $x_1, \ldots, x_n \in \boldsymbol{A}_i$ and $x_1, \ldots, x_n \in \mathcal{C}_k$. This is a contradiction since all elements in $\mathcal{C}_k$ are comparable and no two elements in $\boldsymbol{A}_i$ are comparable. Hence for any antichain in $\bar{\boldsymbol{A}}$, there is at most one element in the intersection. Thus $|\mathcal{A} \cap \mathcal{C}_k| \leq 1$.

4. DILWORTH'S DECOMPOSITION THEOREM FOR PARTIALLY ORDERED MULTISETS

In this section, we present an analogous form of Dilworth's decomposition theorem in the multiset setting and prove that the conditions of Dilworth's theorem hold for ordered multisets. Lastly, an algorithm that establishes the conditions in Dilworth's theorem and its dual is constructed and implemented on an ordered multiset structure.

Theorem 4.1 (Generalized Dilworth's decomposition theorem): Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a finite pomset defined over a poset $(S, \preccurlyeq)$. If w is the width of the pomset $\mathcal{M}$, then $\mathcal{M}$ has a partition into w-many multiset chains.

Proof: Firstly, antichains of the pomset $\mathcal{M}$ are constructed following definition 2.2 with respect to the multiset order $\preccurlyeq\leq$. If $\mathcal{M}$ is empty then the statement holds trivially. Also, the statement is true if $w = 1$, thus the statement holds if $\mathcal{M}$ is a trivial pomset with a single element x_i or point $m_i x_i$.

Assume the statement is true for pomsets $\mathcal{N}_1, \ldots, \mathcal{N}_k$ with $|\mathcal{N}_i| < \mathcal{M}$ for all i. If $\mathcal{A}$ is a one-point antichain in $\mathcal{M}$, we have $w = 1$. Without loss of generality, we can have $|\mathcal{M}| = |\mathcal{N}_k| + 1$. Assume that $\mathcal{A}$ has more than one point and let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be maximal multiset chains in $\mathcal{M}$. By lemma 2.4, we know that $|\mathcal{A} \cap \mathcal{C}_i| \leq 1$ for all $i \in \{1, .., n\}$ with $n \geq |\mathcal{A}|$. For some $i \in \{1, .., n\}$, consider

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the subpomset $\mathcal{J} = \mathcal{M} \backslash \mathcal{C}_i$ and width($\mathcal{J}$) = $w - |\mathcal{A}|$. Since | $\mathcal{J}| < |\mathcal{M}|$, therefore $\mathcal{J}$ can be partitioned into $w - |\mathcal{A}|$ multiset chains via the inductive hypothesis. The desired partition of $\mathcal{M}$ can be obtained by adding the multiset chain $\mathcal{C}_i$. Thus the pomset $\mathcal{M}$ can be partitioned into at most w multiset chains. It remains to show that there are exactly w (not fewer) multiset chains in the smallest partition of $\mathcal{M}$. Suppose the pomset $\mathcal{M}$ can be partitioned into a minimum of $\mathcal{C}_1, \dots, \mathcal{C}_n$ multiset chains with $n < w$. Consider a maximum antichain, say $\mathcal{A}$, in $\mathcal{M}$ i.e. $|\mathcal{A}| = w$. Since $\mathcal{A}$ belongs to the set-based partition, every element of $\mathcal{A}$ must belong to a different multiset chain $\mathcal{C}_i$. Thus $n < w$ gives a contradiction. Therefore $n = w$.

4.2. Dual of Dilworth's Decomposition Theorem

The following is the dual of Dilworth's result:

A *poset of height* h *can be partitioned into* h *antichains*. The proof of this is outlined in Mirsky (1971).

Analogously, we have: *The height* h *of a pomset* $\mathcal{M}$ *coincides with the minimum number of antichains in a partition of* $\mathcal{M}$.

By recursively removing the maximal elements (based on the defined ordering) from a pomset $\mathcal{M}$ and applying set-based partitioning (definition 2.2), we would have antichains $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ of $\mathcal{M}$ whose additive union $\uplus$ will be $\mathcal{M}$.

Antichains of $\mathcal{M}$ can be constructed via the following steps:

Step 1: Choose $\mathcal{A}_1$ to be the set of all maximal elements in $\mathcal{M}$

Step 2: Choose $\mathcal{A}_2$ to be the set of all maximal elements after obtaining the set $\mathcal{A}_1$

Step 3: Choose $\mathcal{A}_{i+1}$ to be the set of maximal elements after obtaining the sets $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_i$

Consequently, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ obtained will be antichains. Also $\mathcal{M} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \dots \uplus \mathcal{A}_n$. The number of antichains thus constructed will be the same as the cardinality of a multiset chain with the greatest size possible in $\mathcal{M}$.

Example 3.2: Consider the multiset $M =$ [8, 2, 2, 6, 7, 4, 8, 3, 5, 6, 7, 2, 3, 5, 6, 3, 18, 24, 12, 18, 18, 12, 10, 10, 10, 9, 9, 12] with the following ordering defined on it:

$x \prec y$ if and only if x and y are both even (odd) and $x < y$ (i.e. the natural ordering),

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clearly $\prec$ is a partial order.

Based on the definition of the multiset order (definition 1.3), $\mathcal{M} = (M, \prec\prec)$ is a pomset induced by $\prec$, where $\prec\prec$ is irreflexive and transitive. Hence, $m_i/x_i \prec\prec m_j/x_j$ if and only if x_i and x_j are both even (odd) and $x_i < x_j$.

5. ALGORITHM

An algorithm for obtaining multiset chains and antichains of a pomset is constructed and implemented using example 3.2. The pseudocode for obtaining the desired output based on the input $M =$ [8, 2, 2, 6, 7, 4, 8, 3, 5, 6, 7, 2, 3, 5, 6, 3, 18, 24, 12, 18, 18, 12, 10, 10, 10, 9, 9, 12] and the defined ordering is presented below.

Pseudocode

INPUT – [8, 2, 2, 6, 7, 4, 8, 3, 5, 6, 7, 2, 3, 5, 6, 3, 18, 24, 12, 18, 18, 12, 10, 10, 10, 9, 9, 12]

```
from collections import Counter
# Read a list of numbers from the user
input_str = input("Enter a list of numbers separated by spaces: ")
# Split the input string into a list of strings
input_list = input_str.split()
# Define a function to duplicate elements based on their values in the dictionary
def duplicate_elements(input_list, dictionary):
+    duplicated_list = []
for element in input_list:
if element in dictionary:
duplicated_list.extend([element] * dictionary[element])
else:
duplicated_list.append(element)
return duplicated_list
def remove_duplicates(input_list):
result_list = []
for item in input_list:
if item not in unique_elements:
result_list.append(item)
unique_elements.add(item)
return result_list
```


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```
else:
duplicated_list.append(element)
return duplicated_list
def remove_duplicates(input_list):
unique_elements = set()
result_list = []
for item in input_list:
if item not in unique_elements:
result_list.append(item)
unique_elements.add(item)
return result_list
# Convert the list of strings to a list of integers (or floats if needed)
try:
numbers = [int(x) for x in input_list]  # Convert to integers
print('########################################')
print(numbers)
value_counts = Counter(numbers)

# Sort the list in-place
numbers.sort()
a_lists = {}
c_lists = {}
chain = []
chain1 = []
chain2 = []
evenNum = [x for x in numbers if x % 2 == 0]
oddNum = [x for x in numbers if x % 2 != 0]
even=remove_duplicates(evenNum)
odd=remove_duplicates(oddNum)
chain.append(even)
chain.append(odd)
index = 0
```


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```
while index<len(chain):
chain_name = f"c_{index}"
c_lists[chain_name] = chain[index]
index+=1
print('#########################################')
print(even)
print('#########################################')
print(odd)
print('#########################################')
print(c_lists)
# Apply the function to each inner list using a list comprehension
my_dicts=dict(value_counts)
duplicated_lists = {key: duplicate_elements(inner_list, my_dicts) for key, inner_list in c_lists.items()}
print('#########################################')
#print(duplicated_lists)
total_elements = sum(len(i_list) for i_list in duplicated_lists.values())
i=0
while i <len(duplicated_lists) or total_elements!=0:
a_name = f"a_{i}"
a = []
for key, in_list in duplicated_lists.items():
if in_list:
max_value = max(in_list)
a.append(max_value)
duplicated_lists[key].remove(max_value)
total_elements = sum(len(i_list) for i_list in duplicated_lists.values())
a_lists[a_name] = a
i+=1
print('')
print('#########################################')
print(a_lists)
except ValueError:
numbers = "contains an invalid number"
```


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6. DISCUSSIONS

Python programming language was used to implement the algorithm. The first step creates two arrays from the input:

Root set $M^* = [2,3,4,5,6,7,8,9,10,12,18,24]$, and

Multiplicity $= [3,3,1,2,3,2,2,2,3,3,3,1]$

The array M^* is then sorted based on the ordering $<<$. The multiplicity of the object x_i in M^* is also picked from the multiplicity array at each stage, and the following multiset chains are produced:

$\mathcal{C}_1 = [3/2, 1/4, 1/6, 2/8, 3/10, 3/12, 2/18, 1/24]$ i.e.

$3/2 << 1/4 << 1/6 << 2/8 << 3/10 << 3/12 << 2/18 << 1/24$.

and

$\mathcal{C}_2 = [3/3, 2/5, 2/7, 2/9]$

i.e $3/3 << 2/5 << 2/7 << 2/9$

The multiset chain $\mathcal{C}_1 = [3/2, 1/4, 1/6, 2/8, 3/10, 3/12, 2/18, 1/24]$ is a maximum multiset chain with $|\mathcal{C}_1| = 16$, thus the height $\hbar$ of the pomset $\mathcal{M}$ is 16.

The next stage of the algorithm sorts the elements into a minimum number of set-based antichains by picking the maximal element from each multiset chain.

$\mathcal{A}_1 = [24,9]$ $\mathcal{A}_7 = [10,3]$ $\mathcal{A}_{13} = [4]$

$\mathcal{A}_2 = [18,9]$ $\mathcal{A}_8 = [10,3]$ $\mathcal{A}_{14} = [2]$

$\mathcal{A}_3 = [18,7]$ $\mathcal{A}_9 = [10,3]$ $\mathcal{A}_{15} = [2]$

$\mathcal{A}_4 = [12,7]$ $\mathcal{A}_{10} = [8]$ $\mathcal{A}_{16} = [2]$

$\mathcal{A}_5 = [12,5]$ $\mathcal{A}_{11} = [8]$

$\mathcal{A}_6 = [12,5]$ $\mathcal{A}_{12} = [6]$

The minimum number of antichains in a partition of the pomset is 16. This coincides with the height $\hbar$ of the pomset. Thus the output establishes the conditions in the dual of Dilworth's decomposition theorem. Also, the width w of the pomset based on the output of the above algorithm is 2 (i.e. the size of a maximum antichain), this coincides with the minimum number of multiset chains in a chain decomposition of the pomset, thus establishing the conditions in Dilworth's decomposition theorem. The outputs are a consequence of the set-based partitioning method adopted.

7. CONCLUSION

Dilworth's decomposition theorem was generalized using a partially ordered multiset structure. An algorithm that establishes the conditions in Dilworth's theorem and its dual was constructed and implemented on an ordered multiset. The algorithm is decidable with a time complexity of $O(n^2)$. It is efficient for solving problems with large inputs. The approach proposed in this study

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can be employed in modeling application problems involving ranking and decision-making in a process that admits repetition.

References

Anusuya Ilamathi, V. S., & Vimala, J. (2018). Multi-criteria decision making on lattice ordered multisets. In: Thampi, S., Mitra, S., Mukhopadhyay, J., Li,KC., James, A., Berreti, S. (eds) Intelligent Systems Technologies and Applications. ISTA 2017. Advances in Intelligent Systems and Computing, vol 683. Springer, Cham. https://doi.org/10.1007/978-3-319-68385-0_34

Balogun, F., Singh, D. and Aliyu, S. (2022). Multiset linear extensions with a heuristic algorithm. *Annals of Fuzzy Mathematics and Informatics* **24**(2), 129-136.

Balogun, F., Singh, D. and Tella, Y. (2021). Maximal and maximum antichains of ordered multisets. *Annals of Fuzzy Mathematics and Informatics* **21**(1), 105-112.

Balogun, F., Singh, D. and Tella, Y. (2020). Realizers of partially ordered multisets. *Theory and Applications of Mathematics and Computer Science*, **10**(2), 1-6.

Basten, T. (1997). Parsing partially ordered multisets. *International Journal of Foundations of Computer science*, **8**(4), 379-407.

Blizard, W. (1988). Multiset theory. *Notre Dame Journal of Formal Logic*, **30**(1), 36-66.

Blizard, W. (1989). Real-valued multisets and fuzzy sets. *Fuzzy Sets and Systems*, **33**(1), 77-97.

Caspard, N., Leclerc, B. and Monjardet, B. (2012). *Finite ordered sets: concepts, results and uses*. Cambridge University Press.

Dang, H. (2014). A single-sorted theory of multisets. *Notre Dame Journal of Formal Logic*, **55**(3), 299-332.

Dershowitz, N. and Manna, Z. (1979). Proving termination with multiset orderings. *Communications of the Journal of Association for Computing Machinery*, **22**, 465-476.

Dilworth, R. P. (1950). A decomposition theorem for partially ordered sets. *Annals of Mathematics*, **51**(1), 161-166.

Felisiak, P. A., Qin, K. and Li, G. (2020). Generalized multiset theory. *Fuzzy Sets and Systems*, **380**, 104-130.

Girish, K. P. and Sunil, J. J. (2009). General relations between partially ordered multisets and their chains and antichains. *Mathematical Communications*, **14**(2), 193-205.

Joret, G., Micek, P., Milans, K., Trotter, W. T., Walczak, B. and Wang, R. (2016). Tree-width and dimension. *Combinatorica*, **36**, 431-450.

Jouannaud, J-P., and Lescanne, P. (1982). On multiset orderings. *Information Processing Letters*, **15**(2), 57-63.

Jürgensen, H. (2020). Multisets, heaps, bags, families: What is a multiset? *Mathematical Structures in Computer Science,* **30**(2), 139-158.

Knauer, K., Micek, P., & Ueckerdt, T. (2018). The queue-number of posets of bounded width or height. In: Biedl, T., Kerren, A. (eds) Graph

https://dx.doi.org/10.4314/aujst.v5i1.6

Knauer, K., Micek, P., Ueckerdt, T. (2018). The queue-number of posets of bounded width or height. In: Biedl, T., Kerren, A. (eds) Graph Drawing and Network.

Mirsky, L. (1971). A dual of Dilworth's decomposition theorem. *American Mathematical Monthly,* **78**(8), 876-877.

Paun, G., and Rozenberg, G. (2002). A guide to membrane computing. *Theoretical Computer Science*, **287**, 73-100.

Pratt, V. R. (1986). Modelling concurrency with partial orders. *International Journal of Parallel Programming*, **15**, 33-71.

Rensink, A. (1996). Algebra and theory of order-deterministic pomsets. *Notre Dame Journal of Formal Logic*, **37**(2), 283-320

Schroeder, B. (2003), *Ordered sets*. Boston: Birkhaeuser.

Singh, D., Ibrahim, A. M., Yohanna, T. and Singh, J. N. (2007). An overview of the applications of multisets. *Novi Sad Journal of Mathematics*, **37**(2), 73-92.

Singh, D. and Singh, J. N. (2007). A note on the definition of multisubset. Association of Symbolic Logic. *In Annual Conference, USA*.

Streib, N. and Trotter, W. T. (2014). Dimension and height for posets with planar cover graphs. *European Journal of Combinatorics* **32**, 474-489.

Yager, R. R. (1986). On the theory of bags, *International Journal of General Systems*, 13; 23-37

https://dx.doi.org/10.4314/aujst.v5i1.6