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A STUDY ON AXIOMS AND MODELS OF ZERMELO-FRAENKELSET THEORY

***Balogun F., Wahab O.A.**

Department of Mathematical Sciences, Federal University Dutsin-Ma

Corresponding Author:

ojowahab@gmail.com

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ABSTRACT

Background: Sets are given axiomatically, thus their existence and basic properties are asserted by appropriate formal axioms. Axiomatic systems address the inconsistencies associated with naïve set theory. The most commonly used system of axioms for sets is the Zermelo-Fraenkel set theory (ZF).
Objectives: In this work, we intend to investigate some set-theoretic results. We begin with a study on the standard axioms of Zermelo-Fraenkel set theory, together with the axiom of choice (ZFC). A comparative analysis of the von Neumann-Bernays-Godel (NBG), which is a conservative extension of ZFC, is presented.
Methods: Axiomatic set theory is developed in the framework of the first-order predicate calculus; this allows a formalization of all mathematical notions and arguments. A model of the Zermelo-Fraenkel set theory that is obtained through an iterative construction that follows the von Neumann hierarchy is presented .
Results: The axioms of Zermelo-Fraenkel set theory and the axiom of choice ZFC were discussed in this work. A comparative analysis of the von Neumann-Bernays-Goedel set theory, which is a conservative extension of ZFC, was presented. The iterative conception of set proposed by von Neumann is a model of set theory and by varying the technique of recursion on the ordinals, different graph models of set theory can be constructed.
Conclusions: These graph models can be employed in establishing independence and consistency results in set theory.

Keywords: Axiomatic system, Formal language, Models, First-order logic and von Neumann hierarchy

INTRODUCTION

Set theory is a branch of mathematics that studies collections of objects. Modern set theory began in the1870s with the works of George Cantor and Richard Dedekind. Researchers like Bertrand Russell later discovered some paradoxes in this early approach to the study of sets. Other set theoretical paradoxes include Burali-Forti paradox, Richard's paradox, Koenig's paradox, Banach-Tarski paradox and paradox of Loewenheim and Skolem (see Jech, 2003 and Wagon, 1985 for details). Russell's paradox became the most famous among the set-theoretical paradoxes. The paradox arises within naïve set theory by considering the set of all sets that are not members of themselves. Thus a set is considered to be a member of itself if it is not a member of itself. The desire for a self-consistent language created a need for an axiomatic system that eliminates the occurrence of paradoxes. Ernst Zermelo proposed the first axiomatic set theory and some contributions were made by Abraham Fraenkel. The resulting axiomatic set theory became known as Zermelo-Fraenkel set theory (**ZF**),or(**ZFC**) when considered with the axiom of choice (Ferreiros, 2007;Enderton, 1977). The Zermelo-Fraenkel set theory with the axiom of choice (**ZFC**) is presently the most widely used system for set theory. Graph models of set theory are constructed using inductive sets. The models, i.e., structures that satisfy a system of axioms under certain interpretation, are tools for establishing independence and consistency results (Paul, 2014). These results aid our understanding of set theory and test the strengths of the axioms that constitute the theory (Gitman et al., 2016; Mathias, 2001).For instance, with the assumption that **ZF** is consistent; the axiom of choice can be shown to be independent of **ZF**. Gitman et al. (2016) proved the essentiality of the power set axiom by showing that the theory **ZFC-,** i.e. the axioms of **ZFC** without the power set axiom, is weaker than expected and inadequate for establishing a number of foundational results. In this paper, we study the standard axioms of Zermelo-Fraenkel set theory and extension of this theory. In section 2, we present preliminaries; notations, symbols and definitions that will be used. In section 3, axioms of the Zermelo-Fraenkel set theory together with the axiom of choice will be studied. A related theory - the von Neumann-Bernays-Goedelset theory – which is based on **ZFC** will be investigated in section 4. We give some concluding remarks in section 5.

1. PRELIMINARIES

The first-order formal language of set theory$\mathcal{L}$ consists of quantified variables defined over non-logical objects. The predicate symbols $=$ (identity) and $\in$ (binary predicate) together with the usual logical symbols, and quantifiers are used. For a function $f: x \to y$, the domain and range of f will be denoted by $dom(f)$ and $ran(f)$, respectively.A formula sayψ follows from Φ if there exists a proof of ψ from a system of axioms ofΦ.A *class* will be a collection of sets defined by a formula whose quantifiers range only over sets. If a structure $\mathcal{T}$ is a model of some axioms of set theory , say extensionality and powerset axioms, we denote this by$\mathcal{T} \vDash Extensionality + powerset$, where$\mathcal{T}$ is a model ofany collection of axioms if the axioms are true in $\mathcal{T}$.

2. THE AXIOMS OF ZFC

There is no universal agreement on the order of the Zermelo-Fraenkel axioms, the exact wording of the axioms, or even how many axioms they are. However, the standard axioms of **ZFC** can be classified into two groups. The first group usually consists of the following axioms: *Extensionality*, *Empty set*, *Pairing*, *Union*, *Powerset* and *Infinity*. These axioms are otherwise known as the basic axioms. The second group contains the complex axioms viz: *Separation*, *Replacement*, *Foundation* and the *axiom of Choice*. These are referred to as *axiom schemas*. The *axiom of foundation* also known as *regularity* was stated by von Neumann as a modification of **ZF**. Formal definitions of these axioms are presented below (see Kunen, 2011; Devlin, 1993 for details).

3.1 Extensionality

$\forall A, B(\forall x(x \in A \leftrightarrow \mathrm{x} \in \mathrm{B}) \to \mathrm{A} = \mathrm{B})$

Any two sets are equal if and only if they contain the same elements. The axiom of extensionality on its own does not guarantee the existence of any set, rather it ensures the uniqueness of a set; a set can be uniquely determined by its elements.

The next axiom guarantees that at least the empty set exists.

3.2 Empty Set

$$\exists A \forall x(x \notin A)$$

There exists a set with no elements. The empty set, $\emptyset$, is unique by extensionality.

3.3 Axiom of Pairing

$$\forall\, x, y \;\exists\, C\; \forall\, z\; (z \in C\; \leftrightarrow z = x \vee z = y)$$

If we're given two sets, a natural desire could be to combine the elements of the two into one set. So for *x* and *y* it is possible to define some whole new set *C* such that *C* = *{x, y}*.

The axiom of pairing implies the existence of singleton sets: the set $\{x\}$ is equal to the unordered pair$\{x, x\}$; equality holds by extensionality. A standard or formal representation of the ordered pair (x, y) is the set$\{\{x\}, \{x, y\}\}$, this can be obtained via repeated applications of the axiom of pairing. Since sets are unordered, we have that ({a, b} = {b, a}), this definition allows us to express ordered pairs as a unique set of a singleton {a} and an unordered pair {a, b}. Using this system we can further define ordered triples: (a, b, c) = ((a, b), c) = {{{a}, {a, b}}, {{{a}, {a, b}}, c}}. We take ordered pair $(a, b) = (c, d)$ if and only if a = c and b = d.

Ordered quadruples $(a, b, c, d) = ((a, b, c), d)$

$$\vdots$$

Ordered n-tuple $(a_1 \ldots \ldots a_{n+1}) = ((a_1 \ldots \ldots a_n), a_{n+1})$

It follows that two ordered n-tuples $(a_1 \ldots . a_n) = (b_1 \ldots \ldots b_n)$ are equal if and only if

$$a_1 = b_1 \ldots \ldots . a_n = b_n$$

A general form of the axiom of pairing is given by:

$$(\forall x_1 \ldots x_n) \exists c \forall w (w \in c \leftrightarrow \bigvee_{1 \le i \le n} w = x_i)$$

3.4 Axiom of Union

The general form of the axiom of union is as follows:

$\forall A \exists B \forall x ((x \in B) \leftrightarrow \exists c ((c \in A) \wedge x \in c)))$

For any collection of A there is a set whose members are those sets belonging to the sets inside A.

The set$B$ is customarily notated as $\bigcup A$.

Stated another way, for every collection C, there exists a set B such that if $x \in X$ for some X in C, then $x \in B$.

Finite sets can easily be constructed by applying the axioms of pairing and union.

3.5 Axiom of Power Set

$$\forall A \exists P \forall x \big(x \in P \leftrightarrow \forall y \in x (y \in A)\big)$$

Intuitively, we can think that for each set, there exists a collection of sets that contain among its elements all the subsets of the given set. If we assume for a moment that there exists some set P for which all the sets are subsets of E, then we write $P = \{X : X \subset E\}$.

As a consequence, it is perfectly possible that P could contain other elements than the ones that are in X. An easy fix would be to apply *Extensionality* which would imply that the set is unique. Which give us P(∅) = {∅}.

Using the Powerset axiom, we can define other basic notions of set theory.

The product of X and Y is the set of all pairs (x, y) such that $x \in X \text{ and } y \in Y$:

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

The notation {(x, y) : . . .} in above is justified because

$\{(x, y) : \varphi(x, y)\} = \{u : \exists x \exists y \big(u = (x, y) \text{and } \varphi(x, y)\big)\}$.

The product $X \times Y$ is a set because $X \times Y \subset PP(X \cup Y)$.

We can also define $X \times Y \times Z = (X \times Y) \times Z$ and in general

$X_1 \times \ldots \times X_{n+1} = (X_1 \times \ldots \times X_n) \times X_{n+1}$.

Thus

$$(X_1 \times \ldots \times X_n) = \{(x_1, \ldots, x_n) : x_1 \in X_1 \wedge \ldots \wedge x_n \in X_n\}.$$

Also let $X^n = \underbrace{X \times . . . \times X}_{n \text{ times}}$.

An n-ary relation R is set of n-tuples. R is a relation on X if $R \subset X^n$. It is customary to write $R(x_1, . . ., x_n)$ instead of $(x_1, . . ., x_n) \in R$, and in case that R is binary, then we also use $x\ R\ y$ for (x, y) $\in R$.

If R is a binary relation, then the domain of R is the set $dom(R) = \{u : \exists v (u, v) \in R\}$ and the range of R is the set$ran(R) = \{v : \exists u (u, v) \in R\}$.

3.6 Axiom of Infinity

An infinite set exists.

$\exists \boldsymbol{x}(\emptyset \in \boldsymbol{x} \wedge \forall \boldsymbol{y}(\boldsymbol{y} \in \boldsymbol{x} \rightarrow \cup \{\boldsymbol{y}, \{\boldsymbol{y}\}\} \in \boldsymbol{x})$.

We may think of this as follows. Let us define the union of x and y, $(x \cup y)$, as the union of the pair set of x and y, i. e ., as $\cup \{x, y\}$. Then the axiom of infinity asserts that there is a set x which contains $\emptyset$ as a member and which is such that whenever a set y is a member of x, then $y \cup \{y\}$ is a member of x. Consequently, this axiom guarantees the existence of a set of the following form: $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \left\{\emptyset, \{\emptyset,\}\{\emptyset, \{\emptyset\}\}\right\}, \ldots\}$

An infinite set of the indicated above contains a copy of the natural numbers, modeled as follows: first 0 corresponds to the empty set, then 1 corresponds to $\{\phi\}$, then 2 correspond to$\{\phi, \{\phi\}\}$ and so on.

3.7 The Axiom (Scheme) of Separation

If φ is a property (with parameter p) then for any X and p there exists a set $Y = \{ u \in X : \varphi(u, p)\}$ that contains all those $u \in X$ that have the property φ

$$\forall X \forall p \exists Y \ \forall u(u \ \in Y \equiv (u \in X \wedge \varphi(u,p)\ ))$$

Let $P(x)$ be a property of x. For any set A, there exists aset$B$ such that $x \in B$ if and only if $x \in A$ and $P(x)$ holds. In other words, if given a formula φ and a set w there exists a set v which contains members of w that satisfy the formula φ.

3.8 The Axiom (Scheme) of Replacement

If F is a function, then for any X there exists a set $Y = F[X] = \{\ F(x) \colon x \in X\ \}$

$$\forall x \forall y \forall z[\varphi(x, y, p) \wedge \varphi(x, z, p) \rightarrow y = z] \rightarrow \forall x \exists Y \forall y[y \in Y \equiv (\exists x \in X)\ \varphi(x, y, p))]$$

Let $P(x, y)$ be a property such that for every x there is a unique y for which $p(x, y)$ holds. For every A there exists B such that for every $x \ \in A$ there is $y \ \in B$ for which $P(x, y)$ holds. This axiom aims to correct some of the paradoxes that arise out of the use of the axiom schema of comprehension. The major difference between the two is that the property $P(x, y)$ depends both on x as well as the unique y for while $P(x)$ depends on x only.

3.9 Axiom of Foundation (or Regularity)

Every nonempty set has an element that is disjoint from the set.

$$\forall\ S\ [(S\ \neq\ \emptyset) \rightarrow (\ \exists\ x((x \ \epsilon\ S) \wedge (S\ \cap x = \ \emptyset))]$$

In contrast to most of the other axioms, the axiom does not guarantee the existence of any sets.

3.10 Axiom of Choice

Every family of nonempty sets has a choice function.

$$\forall\ x\ \ \in a \ \exists\ A\ (x, y) \rightarrow \exists\ y \ \forall\ x \in a\ A\big(x, y(x)\big).$$

Given any infinite collection of nonempty sets, it is possible to choose (simultaneously) one element from each set. More precisely if f is a function whose domain is a nonempty set A and whose co-domain is a set B whose elements are nonempty sets, then there is a choice function g with the property that $g(x)\ \in f(x)$ for each x in A. In other words, if A is a set the elements of which are nonempty sets, then there exists a function f with domain A such that, for member B of A,$f(B) \in B$.

Some well-known results that are equivalent to the axiom of choice include, the *Well-ordering Principle* and *Zorn's Lemma*.

3. EXTENSIONS OF THE ZERMELO-FRAENKEL THEORY

In this section, an extension of **ZFC** set theory is studied. In particular, the von Neumann-Bernays-Goedel set theory (**NBG)** and a model will be presented.

4.1 von Neumann-Bernays-Goedel Set Theory

The von Neumann-Bernays-Goedel set theory (**NBG)** can be viewed as a conservative extension of the Zermelo-Fraenkel set theory. Unlike **ZFC**, classes and sets make up the objects of **NBG** (Bernays, 1991).Intuitively, all sets are classes, but not all classes are sets. The theory **NBG** is constructed by extending the language of **ZF** set theory to classes. This theory is closely related to **ZFC** and any statement about sets is provable in **NBG** if and only if it is provable in Zermelo-Fraenkel set theory. The theories **ZFC** and **NBG** are equiconsistent. We present a comparative analysis of **ZFC** and **NBG** set theories.

1. In contrast to **ZFC**, which is a study of a single kind of objects called sets, the theory **NBG** has among its objects, *proper classes*. These are different from sets because they do not belong to other classes. Thus in **NBG** we have

   $$\boldsymbol{Set}(x) \leftrightarrow \exists y(x \in y)$$

2. An equivalence of the set existence axioms of ZFC, with the exception of the axioms of powerset and infinity, is the *limitation of size principle* given by von Neumann:

   $$\neg\boldsymbol{Set}(x) \leftrightarrow |x| = |V|$$

   Where V is the set theoretic universe.
3. The Zermelo-Fraenkel set theory is not finitely axiomatized. The axiom of replacement, for example, is an axiom schema consisting of an infinite family of axioms, since replacement is true for any set-theoretic formula $A(u, v)$.Whereas **NBG** has only finitely many axioms; this was the main motivation in its construction.

4.1.2 Axiomatization of NBG

The **NBG** theory can be axiomatized using a *two-sorted* approach proposed by Bernays, or via the *Goedel class construction functions*. The major distinction between the two approaches is in the way statements are written. In order to eliminate sorts, Goedel uses primitive predicates for classes and sets. Nevertheless, all statements that are provable in the former approach are also provable in the latter. The latter allows the use of statements such as;

$\exists x\phi(x)$ Instead of $\exists x(\exists C(x \in C) \wedge \phi(x))$ and

$\forall x\phi(x)$ Instead of $\forall x(\exists C(x \in C) \rightarrow \phi(x))$

With the introduction of classes to the language of **ZFC**, we have the following:

1. The axiom scheme of *class comprehension* was added.

Axiom (scheme) of Class Comprehension

For every formula $\phi(x_1, \dots, x_n)$ that quantifies only over sets, there exists a class A consisting of the $n - tuples$ satisfying the formula, $\forall x_1, \dots, \forall x_n[(x_1, \dots, x_n) \in A \leftrightarrow \phi(x_1, \dots, x_n)]$. This axiom is used in its restricted form to avoid the paradoxes encountered in naïve set theory.

2. The axiom of extensionality of **ZFC** is generalized to accommodate classes.

Generalized Axiom of Extensionality

If two classes have the same elements, then they are identical.

$$\forall A \forall B[\forall x(x \in A \leftrightarrow x \in B) \rightarrow A = B]$$

3. The axiom (scheme) of replacement is replaced by a single axiom that uses a class.

To make the theory finitely axiomatized, the axiom schema of class comprehension is replaced with finitely many class existence axioms, these are employed in the proof of the *Class Existence Theorem* which is a basic theorem of **NBG**.

Theorem 4.1 (Class Existence Theorem)

Let $\phi(x_1, \dots, x_n, Y_1, \dots, Y_m)$ be a formula that quantifies over sets and contains no free variables other than $x_1, \dots, x_n, Y_1, \dots, Y_m$. Then for all $Y_1, \dots, Y_m$, there exists a unique class $\boldsymbol{A}$of n-tupples such that:

$$\forall x_1, \dots, \forall x_n[(x_1, \dots, x_n) \in \boldsymbol{A} \leftrightarrow \phi(x_1, \dots, x_n, Y_1, \dots, Y_m)]$$

The class $\boldsymbol{A}$ is denoted by $\{(x_1, \dots, x_n): \phi(x_1, \dots, x_n, Y_1, \dots, Y_m)\}$

Axiom of Global Choice

The **NBG** version of the axiom of choice is known as *axiom of global choice*. This axiom is a stronger form of the axiom of choice and it is implied by the axiom of limitation of size.

$$\exists G[G \text{ is a function } \wedge \forall x \left(x \neq \emptyset \to \exists y(y \in x \wedge (x, y) \in G)\right)]$$

I.e. there exists a function that chooses an element from every nonempty set.

The axiom of global choice implies ZFC's axiom of choice.

Other axioms that were introduced or modified to handle classes include:

The Axiom of Specification

Let $\phi\ (A_{1,}A_{2,} . . ., A_n,\ x)$ be a propositional function such that $A_{1,}A_{2,} . . ., A_n$ are a finite number of free variables whose domain ranges over all classes, and$x$ a free variable whose domain ranges over all sets. Then, the axiom of specification gives that $A_{1,}A_{2,} . . ., A_n$: $\exists\ B\ \forall x\ (x \in B) \leftrightarrow \phi(A_{1,}A_{2,} . . ., A_n,\ x)$ where each of B range over arbitrary classes.

The Axiom of Foundation

For any non- empty class, there is an element of the class that shares no element with the class

$$\forall\ S : \sim\left(S = Z : \forall y : \left(\sim(y\ \in Z)\right)\right) \to \exists\ x \in S : \sim(\exists w : w \in S \wedge w \in x)$$

Membership Axiom

There exists a class E containing the ordered pairs whose first component is a member of the second component.

$$\exists\ E\ \forall x\ \forall y\ [(x, y) \in E\ \leftrightarrow x\ \in y]$$

Intersection (conjunction)

For any two classes $A$and$B$, there is a class$C$ consisting precisely of thesets that belong to both A and B.

$$\forall\ A\ \forall\ B \exists C \forall x\ [\ x\ \in C\ \leftrightarrow (x \in A \wedge x \in B)]$$

Complement (Negation)

For any class A there is a class B consisting precisely of the sets not belonging to A

$$\forall\ A\ \exists B \forall x\ [\ x \in B\ \leftrightarrow \sim(x \in A)]$$

Circular Permutation

For any class A there is a class B whose 3-tuples are obtained by applying the circular permutation $(y, z, x) \to (x, y, z)$ to the 3-tuples of A.

$$\forall A \exists B \forall x \forall y \forall z\ [(x, y, z) \in B\ \leftrightarrow (y, z, x) \in A]$$

Transposition

For any class$A$ there is a class B whose 3-tuples are obtained by transposing the last two components of the 3-tuples of A.

$$\forall \boldsymbol{A} \exists \boldsymbol{B} \forall \boldsymbol{x} \forall \boldsymbol{y} \forall \boldsymbol{z}\ [(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{B}\ \leftrightarrow (\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}) \in \boldsymbol{A}]$$

4.2 von Neumann Hierarchy

Models of set theory can be obtained through an iterative construction that follows the von Neumann hierarchy. The structures used are graphs of the form $G = (V, E)$ where V is a set, containing nodes and $E \subseteq V \times V$ is a binary edge relation.

In the von Neumann hierarchy, the zero-th stage corresponds to the empty set. At any successor stage, the powerset of the previous stage is taken. A limit step that is nonzero contains collections of previously existing sets; this corresponds to taking the union of all previous stages. With this iterative conception of sets, the $\omega - th$ step for an ordinal ω in the von Neumann hierarchy is a model of ZFC.

For αan ordinal, V_α can be defined as follows:

$$V_0 = \emptyset$$

For an ordinal say β, a successor ordinal $\beta + 1$ is given by

$V_{\beta+1} = \mathcal{P}(V_\beta)$, where $\mathcal{P}$ is the powerset

For any limit ordinal α, we have

$V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ for $\alpha \neq 0$

For each ordinal α, the graph $(V_\alpha, \in)$ models a set theory. With this construction, the axioms of **ZFC** without *infinity* are true in V_α, thus $(V_\omega, \in)$ is a model of finite set theory.

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