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AN M-STAGE IMPLICIT RUNGE-KUTTA METHOD FOR THE NUMERICAL SOLUTION OF AN INDEX-3 DIFFERENTIAL ALGEBRAIC EQUATIONS

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ABSTRACT

In this paper, numerical solution of Differential-Algebraic Equations with index-3 systems is considered using implicit two stage second order Runge-Kutta method. This method preserved the stability property of the numerical scheme of the differentiation-algebraic equation. The derivation of the index-3 system method using the M-stage implicit Runge-Kutta method is attempted and presented. The analysis show clearly that the spectral radius of the iteration matrix is less than one. By contraction mapping theorem, we conclude that it also converges to a solution that satisfies the Lipchitz condition. Solutions obtained and numerical error estimated, have been favorably compared with some of the existing methods and those obtained by exact solutions.

Keywords: Differential -Algebraic Equations, Index-3, M- Stage Implicit Runge-Kutta, Hessenberg form of DAEs

INTRODUCTION

In this paper the authors focus on the derivation and the numerical solution of Differential-Algebraic Equations with index-3 systems “using a derived implicit” two stage second order Runge-Kutta method. The Differential Algebraic Equations (DAEs) are general form of Differential Equations for vector value function in an independent variable.

$$F(\dot{x}(t), x(t), t) = 0 \tag{1}$$

Where $x: [a, b] \rightarrow R^n$ is a vector of dependent variable $x(t) = (x_i(t), \ldots, x_n(t))$

Differential-Algebraic Equations (DAEs) is one of the important areas in mathematical field, that result from modeling of certain physical, economical and engineering problems, including real-time simulation of mechanical (multi-body) system, power system, circuit analysis, optimal control and computer-aided design (Brenan et al, 1989).

Earlier works by Petzold (1986), Ascher and Petzold (1995), Yelogu and Celik (2008), and Fatokun (2011) reported the development and adaption of Differential Algebraic Equation with index-1 and index-2 to solve to problems using the approximation methods. The aim of this present paper is to derive and apply the implicit Runge-kutta method to nonlinear semi-explicit index-3 system and the sufficient condition which will ensure the accuracy of the method, which is a follow up of Ascher and Petzold (1995) and Fatokun (2011). The important aspect of the derivation of nonlinear semi-explicit index-3 system is that of order, stability and convergence which may be used as basis for the procedure.

Definition 1: Differential Index

For general DAEs system, the index along a solution $x(t)$ is the minimum number of differentiation steps required to transform a differential-algebraic equation into an ordinary differential equation for the variable. The concept of index is very paramount because, it enables us to quantify the level of difficulty that is involved in solving a given DAES.

Definition 2: Hessenberg forms of DAEs

Many of the higher index problems encountered in practice can be expressed as a combination of more restrictive structures of ODEs coupled with constraints. In such systems, the algebraic and differential variables are explicitly identified for higher index DAEs as well, and the algebraic variables may all be eliminated by subsequent differentiation. When this occurs, we call such *Hessenberg forms of DAEs*. These are categorized as follows:

Hessenberg index-1

This is a semi explicit index -1 system of the form

$$x' = f(t, x, z) \quad (2a)$$

$$0 = g(t, x, z) \quad (2b)$$

The Jacobian matrix function $\frac{\partial g}{\partial z}$ is assumed to be nonsingular for all t. they are closely related to implicit ODEs. The index-1 property require that it is solvable in other words, the differentiation of index-1 is by differentiation of the algebraic equation for which result in an implicit ODEs system.

Hessenberg index-2

This is the system of the form

$$x' = f(t, x, z) \quad (3a)$$

$$0 = g(t, x) \quad (3b)$$

where the product of the Jacobian is non-singular for all t. This system is pure index -2 DAEs, and all algebraic variables play the role of index- 2 variables.

Hessenberg index-3

This is the system of the form

$$x' = f(t, x, y, z) \quad (4a)$$

$$y' = g(t, x, y) \quad (4b)$$

$$0 = h(t, y), \quad (4c)$$

where the product of the three matrix functions $\frac{\partial h}{\partial y}\frac{\partial g}{\partial x}\frac{\partial f}{\partial z}$ is nonsingular.

Definition 3: An M-stage Implicit Runge-Kutta Method

An M-stage Runge-Kutta method (IRK) applied to a DAE (1) is given by

$$F(y_{n-1} + h\sum_{j=1}^{M} a_{ij}Y_i^{'}, Y_j^{'}, t_{n-1} + c_{ih}) = 0,\ i = 1,2,\dots,M,\ y_n = y_{n-1} + h\sum_{i=1}^{M} b_i Y_i^{'} \quad (5)$$

Where $h = t_n - t_{n-1}$ The intermediateY's are given by $y_i = y_{n-1} + h\sum_{j=1}^{M} a_{ij}Y_j^{'}$ Petzold(1984)

2.0 DERIVATION OF THE METHODS

Given a non-linear semi-explicit index-3 system of the form

$$x^{'} + g_1(x,y,z,t) = 0,\ g_2(x,y,t) = 0,\ g_3(x,t) = 0 \quad (6)$$

To solve equation (6) numerically by M-stage implicit Runge-Kutta method, we consider the M –Stage implicit Runge-Kutta applied to (6) thus yielding

$$X_i' + g_1\left[x_{n-1} + h\sum_{j=1}^{M} a_{ij}X_j',\ y_{n-1} + h\sum_{j=1}^{M} a_{ij}Y_j',\ z_{n-1} + h\sum_{j=1}^{M} a_{ij}Z_j', t_i\right]$$

$$Y_i^{'} + g_2\left[x_{n-1} + h\sum_{j=1}^{M} a_{ij}X_j^{'},\ y_{n-1} + h\sum_{j=1}^{M} a_{ij}Y_j^{'},\ t_i\right] = 0$$

$$g_3\left[x_{n-1} + h\sum_{j=1}^{M} a_{ij}X_j^{'},\ t_i\right] = 0, \quad i = 1,2,\dots,M \quad (7)$$

By substituting the value of x in terms of i, where the intermediate stage value for t_i is define as

$$X_i = x_{n-1} + h\sum_{j=1}^{M} a_{ij}X_j^{'},\ Y_i = y_{n-1} + h\sum_{j=1}^{M} a_{ij}Y_j^{'},\ and\ Z_i = z_{n-1} + h\sum_{j=1}^{M} a_{ij}Z_j^{'}, t_i$$

The true solution of (6) with respect to satisfies

$$x(t_i) = x(t_{n-1}) + h\sum_{i=1}^{M} b_j x'(t_{n-1} + c_i h) - \delta_{M+1}^{x(n)}$$

$$y(t_i) = y(t_{n-1}) + h\sum_{i=1}^{M} b_j y'(t_{n-1} + c_i h) - \delta_{M+1}^{y(n)}$$

$$z(t_i) = z(t_{n-1}) + h\sum_{i=1}^{M} b_j z^{'}(t_{n-1} + c_i h) - \delta_{M+1}^{z(n)} \quad (8)$$

where $x(t_i) = x(t_{n-1}) + h\sum_{j=1}^{M} a_{ij}x^{'}(t_{n-1} + c_j h) - \delta_i^{x(n)}$

$$y(t_i) = y(t_{n-1}) + h\sum_{j=1}^{M} a_{ij}y^{'}(t_{n-1} + c_j h) - \delta_i^{y(n)}$$

$$z(t_i) = z(t_{n-1}) + h\sum_{j=1}^{M} a_{ij}z^{'}(t_{n-1} + c_j h) - \delta_i^{z(n)} \quad i = 1,2,3,\dots,M$$

Let$G_{31}(t_i) = \frac{\partial g_3}{\partial x}, G_{21}(t_i) = \frac{\partial g_2}{\partial x}, G_{22}(t_i) = \frac{\partial g_2}{\partial x}, G_{11}(t_i) = \frac{\partial g_1}{\partial x}, G_{12}(t_i) = \frac{\partial g_1}{\partial y}, G_{13}(t_i) = \frac{\partial g_1}{\partial z}$

where the partial derivatives are evaluated along the true solutions by subtracting (8) from (7), and substituting the values of$t_i = t_{n-1} + c_i h$, then we obtain

$$e_n^x = e_{n-1}^x + h\sum_{i=1}^{M} b_i E_i^{x'} + \delta_{M+1}^{x(n)} \tag{9}$$

Also for the value of y, by Subtracting (8) from (7), and substituting the values where $t_i = t_{n-1} + c_i h$ then,

$$e_n^y = e_{n-1}^y + h\sum_{i=1}^{M} b_i E_i^{y'} + \delta_{M+1}^{y(n)} \tag{10}$$

Similarly, for the value of z, by substituting for y and then

$$e_n^z = e_{n-1}^z + h\sum_{i=1}^{M} b_i E_i^{z'} + \delta_{M+1}^{z(n)} \tag{11}$$

$$\left[X_i' + g_1\left(x_{n-1} + h\sum_{j=1}^{M} a_{ij}X_j', y_{n-1} + h\sum_{j=1}^{M} a_{ij}Y_j', z_{n-1} + h\sum_{j=1}^{M} a_{ij}Z_j', t_i\right)\right] - \left[x'(t_i) + g_1(x(t_i), y(t_i), t_i)\right] = 0$$

$$X_i' - x'(t_i) + g_1[X_i - x_i(t)] + g_1[Y_i - y_i(t)] + g_1[Z_i - z_i(t)] = 0$$

$$E_i^{x'} + G_{11}(t_i)E_i^x + G_{12}(t_i)E_i^y + G_{13}(t_i)E_i^z = \eta_i^x \tag{12}$$

Similarly, by evaluating and substitute as done for (12) we have

$$E_i^{y'} + G_{21}(t_i)E_i^x + G_{22}(t_i)E_i^y = \eta_i^y \tag{13}$$

By evaluating and substituting this implies that we thus,

$$G_{31}(t_i)E_i^x = \eta_i^z \tag{14}$$

Bringing equation (9), (10), (11), (12), (13), and (14) together we have

$$E_i^{x'} + G_{11}(t_i)E_i^x + G_{12}(t_i)E_i^y + G_{13}(t_i)E_i^z = \eta_i^x \tag{15a}$$

$$E_i^{y'} + G_{21}(t_i)E_i^x + G_{22}(t_i)E_i^y = \eta_i^y \tag{15b}$$

$$G_{31}(t_i)E_i^x = \eta_i^z \tag{15c}$$

$$e_n^x = e_{n-1}^x + h\sum_{i=1}^{M} b_i E_i^{x'} + \delta_{M+1}^{x(n)},\ e_n^y = e_{n-1}^y + h\sum_{i=1}^{M} b_i E_i^{y'} + \delta_{M+1}^{y(n)}$$

$$and\ e_n^z = e_{n-1}^z + h\sum_{i=1}^{M} b_i E_i^{z'} + \delta_{M+1}^{z(n)}$$

Where $E_i^x = e_{n-1}^x + h\sum_{i=1}^{M} a_{ij}E_j^{x'} + \delta_n^{x(n)}$, $E_i^y = e_{n-1}^y + h\sum_{i=1}^{M} a_{ij}E_j^{y'} + \delta_n^{y(n)}$ and

$$E_i^z = e_{n-1}^z + h\sum_{i=1}^{M} a_{ij}E_j^{z'} + \delta_n^{z(n)} \tag{16}$$

where terms are higher order terms by multiplying the first equation in 15 by

$$E_i^z = -M_i E_i^{x'} - M_i G_{11}(t_i)E_i^x - M_i G_{12}(t_i)E_i^y + M_i\eta_i^x \tag{17}$$

Suppose that the product of (15a) and substituting equation (17) for then we have

$$(I-H_i)E_i^{x'}+N_iE_i^x+D_iE_i^y+(I-H_i)M_iG_{11}(t_i)E_i^x-G_{13}(t_i)M_iG_{12}(t_i)E_i^y+ G_{13}(t_i)M_i\eta_i^x=\eta_i^{\sim x} \quad (18)$$

By applying the definition of (17), we have

$$(I-H_i)E_i^{x'}+N_iE_i^x+D_iE_i^y+[I-H_i]\left[-E_i^{x'}-G_{11}(t_i)E_i^x-G_{12}(t_i)E_i^y+\eta_i^x\right]=\eta_i^{\sim x} \quad (19)$$

$$E_i^{x'}+G_{11}(t_i)E_i^x+G_{12}(t_i)E_i^y=\eta_i^x \quad (20)$$

Thus (19) becomes

$$(I-H_i)E_i^{x'}+N_iE_i^x+D_iE_i^y+[I-H_i]\left[-(E_i^{x'}+G_{11}(t_i)E_i^x+G_{12}(t_i)E_i^y)+\eta_i^x\right]=\eta_i^{\sim x} \quad (21)$$

Substituting (20) in (21) and evaluating we have

$$(I-H_i)E_i^{x'}+N_iE_i^x+D_iE_i^y=\eta_i^{\sim x} \quad (22)$$

where $N_i=(I-H_i)G_{11}(t_i),\ D_i=(I-H_i)G_{12}(t_i),\ \eta_i^{\sim x}=(I-H_i)\eta_i^x$

Similarly, multiply the second equation (15) by and solving for

$$E_i^x=-Q_iE_i^y-Q_iG_{22}(t_i)E_i^y+Q_i\eta_i^y \quad (23)$$

by multiplying (15b) and substitute (23) for also by applying the definition, we obtain

$$(I-Q_i)E_i^{y'}+[I-Q_i]\left[-E_i^y-G_{22}(t_i)E_i^y+\eta_i^x\right]+(I-Q)G_{22}(t_i)E_i^y=\eta_i^{\sim y} \quad (24)$$

Thus, we can rewrite (15b) as

$$E_i^{y'}+G_{22}(t_i)E_i^y=\eta_i^y \quad (25)$$

Substituting (25) into (24) we have

$$(I-Q_i)E_i^{y'}+\rho_iE_i^y=\eta_i^{\sim y} \quad (26)$$

By multiplying (15c) we obtain

$$H_iE_i^y=\eta_i^{\sim z} \quad (27)$$

Thus equation (15a) - (15c) can now be written as:

$$(I-H_i)E_i^x+N_iE_i^x+\ D_iE_i^y=\eta_i^{\sim x},\ (I-Q_i)E_i^{y'}+\rho_iE_i^y=\eta_i^{\sim y},\ E_i^x=\eta_i^{\sim z} \quad (28)$$

by substituting the value and from the first equation in (28), we obtain

$$D_iE_i^y=\eta_i^{\sim x}-(I-H_i)E_i^{x'}-N_iE_i^x \quad (29)$$

From the second equation in (28) we obtain

$$\rho_i E_i^{y} = \eta_i^{\sim y} - (I - Q_i),\ E_i^{y'} \tag{30}$$

Subtracting (29) from (16) we have

$$\left.\begin{array}{l} (I - Q_i)E_i^{y'} + \rho_i E_i^{y} = \eta_i^{\sim y}, \\ (I - H_i)E_i^{x'} + N_i E_i^{x} = \eta_i^{\sim x}, \\ H_i E_i^{x} = \eta_i^{\sim z},\ i = 1,2,\dots,M \end{array}\right\} \tag{31}$$

Multiplying the third equation in (31) by we have

$$N_i H_i E_i^{x} = N_i \eta_i^{\sim z} \tag{32}$$

Adding the second equation in (31) with (32) we obtain

$$(I - H_i)E_i^{x'} + N_i[(I - H_i)E_i^{x} + \ H_i E_i^{x}] = \eta_i^{\sim x} \tag{33}$$

Subtracting (32) from (33) and substituting the definition of giving in equation (16), and by evaluating we obtain

$$E_i^{\sim x'} + N_i(I - H_i)e_{n-1}^{x} + h\sum_{j=1}^{M} a_{ij}E_j^{\sim x'} + h\sum_{j=1}^{M} a_{ij}\left(H_j - H_i\right)E_j^{\sim x'} + \delta'^{x(n)}_{n} = -N_i\eta_i^{\sim z} + \eta_i^{\sim x} \tag{34}$$

From the third equation in (31) we can obtain the expression for

$$e_{n-1}^{x} + h\sum_{j=1}^{M\Sigma} a_{ij}\tilde{E}_j^{x'} + h\sum_{j=1}^{M\Sigma_i^{\sim z}} a_{ij}(H_j - H_i)E_j^{x'} + \delta'^{x(n)}_{n} \tag{35}$$

By expanding (34) and (35), and substituting the definition, we have

$$\begin{gathered} E_i^{\sim x'} + N_i((I - H_i)e_{n-1}^{x} + \delta_n^{x(n)}) + \\ h\sum_{j=1}^{M} a_{ij}N_iE_j^{\sim x'} + h\sum_{j=1}^{M} a_{ij}N_i(H_j - H_i)(E_j^{\sim x'} + E_j^{\sim\sim x'}) = -N_i\eta_i^{\sim z} + \eta_i^{\sim x} \end{gathered} \tag{36}$$

$$h\sum_{j=1}^{M} a_{ij}E_j^{\sim x'} + H_i(e_{n-1}^{x} + \delta_n^{x(n)}) + h\sum_{j=1}^{M} a_{ij}(H_i - H_j)(E_j^{\sim x'} + E_j^{\sim\sim x'}) = \eta_i^{\sim z} \quad i = 1,2,\dots,M$$

Let, where are evaluated along the true solution at time and Let be substituted

Thus Equation (36) can be written in matrix form as

$$\begin{pmatrix} T_1 & h^2T_2 \\ h^2T_3 & hT_4 \end{pmatrix}\begin{pmatrix} E^{\sim x'} \\ E^{\sim\sim x'} \end{pmatrix} = \begin{pmatrix} S_1 & 0 \\ 0 & S_4 \end{pmatrix}\begin{pmatrix} e_{n-1}^{x} + \delta^{x(n)} \\ e_{n-1}^{x} + \delta^{x(n)} \end{pmatrix} + \begin{pmatrix} \eta^{\tau\eta^{\sim\sim x}} \\ \eta^{\sim y} \end{pmatrix} \tag{37}$$

Equation (37) can be rewritten as

$$T_1 = \underset{\wedge}{T_1} + 0(h^2), T_4 = \underset{\wedge}{T_4} + 0(h) \text{ and } \quad S_1 = \hat{S}_1 + 0(h), S_4 = \hat{S}_4 + 0(h) \quad (38)$$

where is the dimension of (8)

Supposing T_n is the left hand matrix in (37), since the matrix A of coefficients of the method is invertible, is also invertible. Thus,

$$T_n^{-1} = \begin{pmatrix} \hat{T}_1^{\ -1} + & 0(h) \\ 0(h) & \frac{\hat{T}^{-1}{}_4}{h+0(1)} \end{pmatrix}$$

Solving and substituting in (37), we have

$$\begin{pmatrix} \tilde{E}^{E^{\sim x'}} \\ \tilde{\tilde{E}}^{E^{\sim\sim x'}} \end{pmatrix} = -\begin{pmatrix} \hat{T}_1^{\ -1}\hat{S}_1 + 0(h) & 0(h) \\ 0(h) & \frac{\hat{T}_4^{\ -1}\ \hat{S}_4}{h+0(1)} \end{pmatrix}\begin{pmatrix} e_{n-1}^x + \delta^{x(n)} \\ e_{n-1}^x + \delta^{x(n)} \end{pmatrix} + \begin{pmatrix} \eta^{\sim\sim x} + 0(h)\tilde{\eta}^{\eta^{\sim z}} + 0(h)\eta^{\tau\eta^{\sim\sim x}} \\ \frac{\hat{T}_4^{\ -1}\ \eta^{\sim z}}{h} + 0(h)\eta^{\tau\eta^{\sim\sim x}} + 0(1)\eta^{\tau\eta^{\sim z}} \end{pmatrix} \quad (39)$$

Recall that. from (39)

$$E^{x'} = -(\tfrac{1}{h})(\hat{T}_1^{-1}\hat{S}_4(e_{n-1}^x + \delta^{x(n)}) + (\tfrac{1}{h})\hat{T}_4^{-1}\eta^{\sim z} + \eta^{\sim\sim x} + 0(1)(e_{n-1}^x + \delta^{x(n)}) + 0(1)\eta^{\tau\eta^{\sim z}} + 0(h)\eta^{\sim\sim x}) \quad (40)$$

Using equation (40) above and the forth equation (15) we have

$$e_n^x = e_{n-1}^x + hb^T E^{x'} + \delta_{M+1}^{x(n)} \quad (41)$$

Where $b^T = (b_1 I_d, b_2 I_d, \dots, b_M I_d) = b^T \times I_d$ by substituting (39) into the above expression, we have

$$e_n^x = e_{n-1}^x - (b^T \times I_d)(A^{-1} \times I_d)(I_M \times H)(e_{n-1}^x + \delta^{x(n)}) + \delta_{M+1}^{x(n)} + (b^T \times I_d)(A^{-1} \times I_d)\eta^{\sim z} + 0(h\delta^{x(n)} + 0(he_{n-1}^x)) + 0(he_{n-1}^x) + 0(h\eta^{\sim y}) + 0(h\eta^{\sim\sim x}) \quad (42)$$

By strict stability condition, where $I_d = (1,1,\dots,1)^T$

Then from (41) we have

$$e_n^x = (I - (1-\gamma)H_{n-1} + 0(h))e_{n-1}^x + ((b^T A^{-1}) \times I_d)\eta^{\sim z} - H_{n-1}((b^T A^{-1}) \times I_d)\delta^{x(n)} + \delta_{M+1}^{x(n)} + 0(h\delta^{x(n)}) + 0(h\eta^{\sim\sim x}) + 0(h\eta^{\sim y}) \quad (43)$$

By expanding (42) we obtain

$$e_n^{\sim(\sim x)} = \gamma(I + 0(h))e_{n-1}^{\sim(\sim x)} - H_{n-1}((b^T A^{-1}) \times I_d)\delta^{x(n)} - \delta_{M+1}^{x(n)} + 0(h\delta^{x(n)}) + 0(h\delta_{M+1}^{x(n)}) + 0(\eta^{\sim z}) + 0(h\eta^{\sim x}) \quad (44)$$

By definition of algebraic order, we obtain

$$H((b^T A^{-1}) \times I_d)\delta^{x(n)} - \delta_{M+1}^{x(n)} = 0(h^{k_{a,1}+1})$$

Multiplying (42) by definition, we obtain

$$e_n^{\sim x} = (I + 0(h))e_{n-1}^{\sim x} + 0(h\delta^{x(n)}) + 0(\delta_{M+1}^{x(n)}) + 0(\eta^{\sim y}) + 0(h\eta^{\sim x}) \quad (45)$$

Where for

Suppose that | | and ‖ ‖ are well defined, we can now rewrite (44) and (45) as

$$e_n^{\sim\sim x} = \gamma(I + 0(h))e_{n-1}^{\sim x} + 0(h^{k_{a,1}+1}) + 0(h^{k_I+2}) + 0(\varepsilon_2) + 0(h\varepsilon_1) \quad (46)$$

$$e_n^{\sim x} = \gamma(I + 0(h))e_{n-1}^{\sim x} + 0(h^{k_I+2}) + 0(h^{k_d+1}) + 0(h\varepsilon_2) + 0(h\varepsilon_1) \quad (47)$$

Supposing $\|e_n^x\| = 0(h^{k_G}) + 0(\varepsilon_2) + 0(\varepsilon_1) + 0(I - H)e_0^x$ (48)

Substituting for$\|e_n^x\|$ in the expansion for (15) and substituting also by simplifying we obtain

$$e_n^z = \rho e_{n-1}^z - b^T(A^{-1} \times I_d)(I_M \times M + 0(h))(E^{x'} + (I_M \times G_{11})E^x) + 0(\eta^x) + 0(h^{k_{a,1}}1) \quad (49)$$

and are evaluated along the true solution. Also, substitute from (38) and simplify, and solving the recurrence in (45) and simplifying we have

$$\|e_i^z\| = 0(h^{k_I}) + 0(\frac{\varepsilon_2}{h}) + 0(\varepsilon_1) + 0(I - H)e_0^x \quad (50)$$

From the above, we have the solution to be

$$\varepsilon_1 = k_1(h^{k_G} + h^{k_I}\varepsilon_1 + \varepsilon_2 + \frac{\varepsilon_1\varepsilon_2}{h} + h^{k_I-1}\varepsilon_2 + \frac{\varepsilon_2^2}{h^2}) \quad (51)$$

$$\varepsilon_2 = k_2(h^{k_I+1} + h^{k_G}\varepsilon_1 + \varepsilon_1\varepsilon_2 + \frac{\varepsilon_1\varepsilon_2}{h} + h^{k_G}\varepsilon_2 + \varepsilon_2^2)$$

Assuming and letting the inertial values. Solving (49) above, we observe that the spectral radius is less than one.

By contraction mapping theorem, we can conclude that the spectral radius converges to a solution that satisfies the above initial condition.

From the first equation in (31),$(I - Q_i)E_i^{y'} + (\rho_i - D_i)E_i^y = \tilde{\eta}^y$

Then,$(I - Q_i)E_i^{y'} + \rho_i E_i^y = \tilde{\eta}^y - D_i E_i^y = 0$ this implies that or.

From the second equation in (28), $\rho_i E_i^y = \tilde{\eta}^y - (I - Q_i)E_i^{y'}$. Substituting this in the above, we have $(I - Q_i)E_i^{y'} + \tilde{\eta}^y - (I - Q_i)E_i^{y'} = E_i^y = 0$. Similarly, we observe that the spectral radius of the iteration matrix is less than one. By contraction mapping theorem, we conclude that it also converges to a solution that satisfies the Lipschitz condition. Hence it is stable

3.0 THE CONVERGENCE/STABILITY ANALYSIS OF THE METHOD

3.1 Strangeness-free DAE problem

Hypothesis 3.1. *There exist integ*ers $\mu, a,$ *and* d *such that the set*

$\mathbb{L}_\mu = \{(t, x, \dot{x}, \ldots, x^{(\mu+1)}) \in \mathbb{R}^{(\mu+2)n+1} | F_\mu(t, x, \dot{x}, \ldots, x^{(\mu+1)}) = 0\}$ associated with F is nonempty and such that for every $(t_0, x_0, \dot{x}_0, \ldots, x_0^{(\mu+1)}) \in \mathbb{L}_\mu$, there exists a (sufficiently small) neighborhood in which the following properties hold:

1. We have rank $M_\mu(t, x, \dot{x}, \ldots, x^{(\mu+1)}) = (\mu+1)n - a$ on $\mathbb{L}_\mu$ such that there exists a smooth matrix function Z_2 of size $(\mu+1)n \times a$ and a pointwise maximal rank, satisfying $Z_2^T M_\mu = 0$on $\mathbb{L}_\mu$.

2. We have rank $\hat{A}_2(t, x, \dot{x}, \ldots, x^{(\mu+1)}) = a$ where $\hat{A}_2 = Z_2^T N_\mu [I_n 0 \ldots 0]^T$ such that there exists a smooth matrix function T_2 of size $n \times d, d = n - a$, and a pointwise maximal rank, satisfying $\hat{A}_2 T_2 = 0$on $\mathbb{L}_\mu$.

3. We have rank $F_{\dot{x}}(t, x, \dot{x}) T_2(t, x, \dot{x}, \ldots, x^{(\mu+1)}) = d$ such that there exists a smooth matrix function Z_1 of size $n \times d$ and a pointwise maximal rank, satisfying $\hat{E}_1 T_2 = d$, where $\hat{E}_1 = Z_1^T F_{\dot{x}}$.

Let $E \in C^\ell(\mathbb{D}, \mathbb{R}^{m,n})$, $\ell \in \mathbb{N}_0 \cup \{\infty\}$, with rank $E(x) = r$ for all

$x \in \mathbb{M} \subseteq \mathbb{D}, \mathbb{D} \subseteq \mathbb{R}^k$ open. For every $\hat{x} \in \mathbb{M}$ there exists a sufficiently small neighborhood $\mathbb{V} \subseteq \mathbb{D}$ of $\hat{x}$ and matrix functions $T \in C^\ell(\mathbb{V}, \mathbb{R}^{n,n-r})$, $Z \in C^\ell(\mathbb{V}, \mathbb{R}^{n,n-r})$, with pointwise orthonormal columns such that

$$ET = 0, \quad Z^T E = 0 \tag{52}$$

on $\mathbb{M}$.

Proof. For $\hat{x} \in \mathbb{M}$, using the singular value decomposition, there exist orthogonal matrices $\dot{U} \in \mathbb{R}^{m,m}, \dot{V} \in \mathbb{R}^{n,n}$ with

$$\hat{U}^T E(\hat{x}) \hat{V} = \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \tag{53}$$

and $\hat{\Sigma} \in \mathbb{R}^{r,r}$ nonsingular.

Splitting $\hat{V} = [\hat{T}' \hat{T}]$ according to the above block structure, we consider the linear system of equations

$$\begin{bmatrix} \hat{Z}' E(x) \\ \hat{T}^T \end{bmatrix} T = \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}.$$

Since

$$\begin{bmatrix} \hat{Z}'E(x) \\ \hat{T}^T \end{bmatrix} [\hat{T}'\hat{T}] = \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & I_{n-r} \end{bmatrix}, \tag{54}$$

Which obviously has full column rank. Moreover, by construction, $[\hat{T}'\hat{T}(x)]$ is nonsingular. By the definition of T, it follows that

$$\hat{U}^T E(\hat{x})[\hat{T}'\hat{T}(x)] = \begin{bmatrix} \hat{Z}'^T E(x)\hat{T}' & 0 \\ \hat{Z}^T E(x)\hat{T}' & \hat{Z}^T E(x)T(x) \end{bmatrix} \tag{55}$$

Such that

rank $E(x) =$ rank $\hat{Z}'^T E(x)\hat{T}'$ +rank $\hat{Z}^T E(x)T(x) = r$ +rank $\hat{Z}^T E(x)T(x)$.

If $\hat{x} \in \mathbb{M}$, then we have rank $E(x) = r$ implying that $\hat{Z}^T E(x)T(x) = 0$.

Together with $\hat{Z}'^T E(x)T(x) = 0$, this gives $E(x)T(x) = 0$. Orthonormality of the columns of $T(x)$ can be obtained by the smooth Gram-Schmidt orthonormalization process.

The corresponding result for Z follows by considering the pointwise transposeE^T.

Given a differential-algebraic equation $F(t, x, \dot{x}) = 0$, the smallest value of such that satisfies Hypothesis 3.1 is called the *strangeness* index of $F(t, x, \dot{x}) = 0$. If $\mu = 0$, then the differential-algebraic equation is called strangeness-free.

3.2 Stability concepts for ODEs

We briefly recall classical stability concepts for ordinary differential equations

$$\dot{x} = f(t, x), \quad t \in \mathbb{I}. \tag{56}$$

We include proofs when we need the notation and parts of them when we discuss similar results for DAEs.

Theorem 1: The trivial solution of the linear homogenous ODE (54)

1. is stable if and only if there exists a constant $L > 0$ with $\|\Phi(t, t_0)\| \leq L$ on $\mathbb{I}$;
2. is asymptotically stable if and only if $\|\Phi(t, t_0)\| \to 0$ for $t \to \infty$;
3. is exponentially stable if there exists $L > 0$, and $\gamma > 0$ such that $\|\Phi(t, t_0)\| \leq Le^{-\gamma(t-t_0)}$ on $\mathbb{I}$.

In the general nonlinear case, we can only expect sufficient conditions that guarantee the specific stability properties. The classical result is given in the so-called Lyapunov stability theorems.

Definition 2: Let $\mathbb{U}$ be an (open) neighborhood of an equilibrium solution x^* of the ODE (56). A function $V \in C^1(\mathbb{I} \times \mathbb{U}, \mathbb{R}_0^+)$ is called Lyapunov function associated with x^* if

1. $V(t, x^*) = 0$ for all $t \in \mathbb{I}$,
2. $\dot{V}(t, x) \leq 0$ for all $(t, x) \in \mathbb{I} \times \mathbb{U}$, where $\dot{V}(t, x) = V_x(t, x) f(t, x) + V_t(t, x)$,
3. There exists a continuous function $(W: \mathbb{U} \rightarrow \mathbb{R}_0^+)$ with $W(x) > 0$ for all $x \in \mathbb{U} \backslash \{x^*\}$ and $V(t, x) \geq W(x)$ for all $(t, x) \in \mathbb{I} \times \mathbb{D}$.

4.0 NUMERICAL EXPERIMENTS AND RESULTS

In this section, the newly derived method is applied to some classical problems in literature to prove the efficiency and accuracy of the new scheme in comparison with some known methods in the literature. All the computations in this section were performed using MATLAB 2017Ra.

Problem 1

Consider the following index-3 differential–algebraic equations with hessenbergindex-3 of the form

$$x_2' + x_1 - 1 = 0, \quad x x_2' + x_3' + 2x_2 = 0, \quad x x_2 + x_3 + e^x = 0 \tag{57}$$

with initial conditions given as$x_1(0) = 0, \quad x_2(0) = -1, \quad x_3(0) = 1$

The exact solution is given as$x_1(x) = e^x - 1, \quad x_2(x) = 2x - e^x, \quad x_3(x) = (1 + x)e^x - 2x^2$

Where represents the differential variables and represent the algebraic variable.

Using the transformation with

$$x = 1, \; x_3' + x_1 = 1, \quad x_2' + x_3' + 2x_2 = 2, \; x_2 + x_3 = e^x \tag{58}$$

After three times differentiation, we obtain

$$x_1'(x) = e^x, \quad x_2'(x) = 2 - e^x, \quad x_3'(x) = x x_1 - 2x_2 + x \tag{59}$$

From the equation in equation (51), $x(e^x - 1) - 2(2x - e^x) + x$. This implies that

$x_3'(x) = 3e^x - 4x$.

Thus, (51) becomes $x_1'(x) = e^x, \quad x_2'(x) = 2 - e^x, \quad x_3'(x) = 3e^x - 4x$ (60)

Solving using implicit two-stage second order Runge-Kutta method

Table 1: Numerical Solutions of x_1 for Problem One

X	h	Exact solution	Ralston's Method	Error	RungKutta Method	Error	Variational iteration Method
0.0	0	0	0	0	0	0	
1.0	0.1	1.718281828	0.105192276	0.9387805	0.101265756	0.941065688	0.105170918
2.0	0.2	6.389056099	0.221577899	0.965319149	0.205127109	0.967893988	0.221402758
3.0	0.3	19.08553692	0.5234	0.972576092	0.311682622	0.83669171	0.349858808
4.0	0.4	53.59815003	0.493295682	0.990273338	0.421034183	0.992144613	0.491824698
5.0	0.5	147.4131591	0.651663805	0.987841673	0.533287113	0.996382364	0.648721271
6.0	0.6	402.4287935	0.827324874	0.9979417	0.648550272	0.998388409	0.822118800
7.0	0.7	1095.633158	1.022214129	0.9990701	0.766936175	0.999300006	1.013752707
8.0	0.8	2980.957987	1.23846336	0.999584541	0.888561103	0.999701921	1.225540928
9.0	0.9	8102.083928	1.478419785	0.999757589	1.01354222	0.999874903	1.459603111
10.0	1.0	22025.46579	1.744666678	0.9999208	1.14201270	0.999948	1.71828183

Solution Graph for Problem One

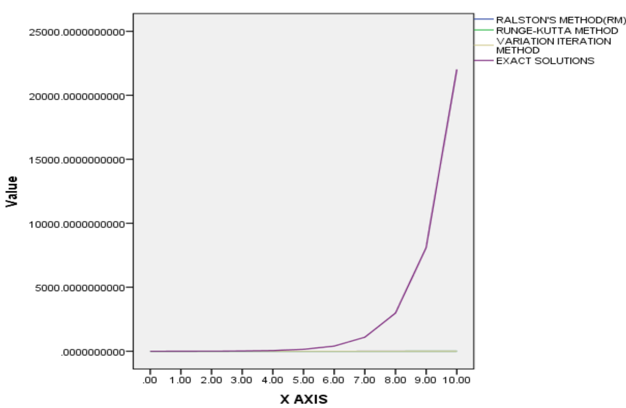

Figure 1: Solution curve showing comparison between Ralston's method (RM), Runge-Kutta Method(RKM), Variational Iteration Method(VIM) and Exact Solution(ES)

Applying implicit two stage second order Runge- Kutta method to solve for in (58) for the solution of we obtain the following results as presented on Table 2

Table 2: Numerical Solutions of x_2 for Problem One

x	H	Exact Solution	Ralston's Method	Error	RungeKutta Method	Error	Variational iteration Method
0.0	0.0	-1	0	1	0	1	-1
1.0	0.1	-0.718281828	0.094807723	1.1319922373	0.098734243	1.137458918	-0.905170918
2.0	0.2	-3.389056099	0.178422101	1.052646547	0.19437289	1.057353105	-0.821402758
3.0	0.3	-14.08553692	0.249535458	1.017715722	0.288317377	1.020469037	-0.749858808
4.0	0.4	-46.59815003	0.306704318	1.006581899	0.378965816	1.008132637	-0.691824698
5.0	0.5	-138.4131591	0.48336195	1.003492168	0.466712886	1.003371882	-0.648721271
6.0	0.6	-391.4287935	0.372675125	1.00952089	0.551449727	1.001408812	-0.622118800
7.0	0.7	-1082.63358	0.37778587	1.000348951	0.633063824	1.000584744	-0.613752707
8.0	0.8	-2964.957987	0.361536639	1.000121937	0.711438896	1.000239949	-0.625540928
9.0	0.9	-8085.083928	0.321580214	1.000039775	0.786454777	1.000097272	-0.659603111
10.0	1.0	-22006.40579	0.255333322	1.000011603	1.142012708	1.000051894	-0.718281828

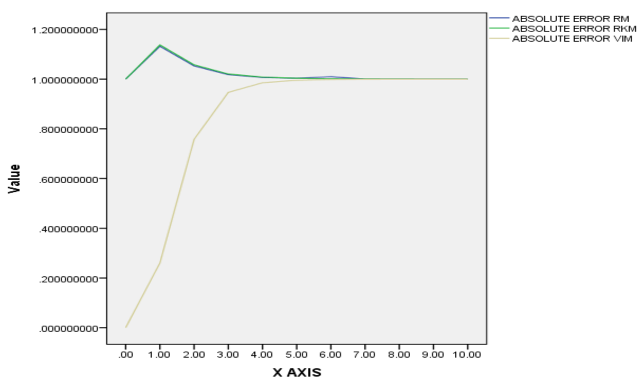

Figure 2: Solution curve showing comparison between Ralston's Method(RM), Runge-Kutta Method(RKM), Variational Iteration Method(VIM) and Exact Solution (Es)

Table 3: Numerical Solutions of x_3 for Problem One

x	h	Exact Solution	Ralston's Method	Error	RungeKutta Method	Error	Variational iteration Method
0.0	0.0	1	-1	2	-1	2	1
1.0	0.1	3.436563657	-0.503905772	1.146630711	-0.49033837	1.142682755	1.19568010
2.0	0.2	14.1671683	-0.035458138	0.997497161	0.006386139	0.999549229	1.385683310
3.0	0.3	62.34214769	0.393783764	0.993683506	0.498984887	0.991996026	1.574816450
4.0	0.4	240.9907502	0.824420019	0.996579038	0.98466946	0.995914077	1.768554577
5.0	0.5	840.4789546	1.219201149	0.998549397	1.4563556096	0.998267231	1.97381906
6.0	0.6	2752.001554	1.593067437	0.999421124	1.9357764936	0.999296597	2.195390080
7.0	0.7	8675.065267	1.948154688	0.99977543	2.401420155	0.999723181	2.44337962
8.0	0.8	26700.62188	2.286815627	0.999914353	2.860650087	0.999892862	2.725973670
9.0	0.9	80868.83928	2.611640734	0.999967705	3.31358735	0.999959025	3.053245911
10.0	1.0	242091.1237	2.925481008	0.999987915	3.76036899	0.999984467	3.436563656

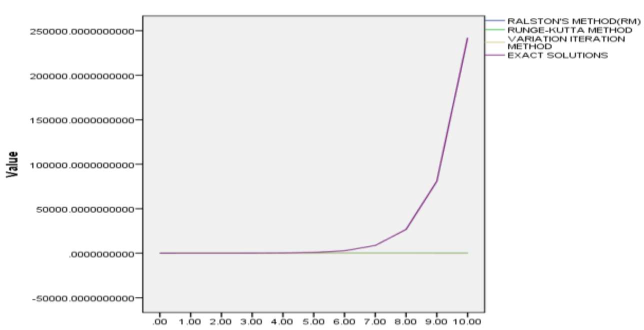

Figure 3: Solution curve showing comparison between Ralston's Method(RM), Runge-Kutta Method(RKM), Variational Iteration Method(VIM) and Exact Solution (ES)

Problem 2

Consider the problem below

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1'\\ x_2'\\ x_3'\end{pmatrix}+\begin{pmatrix}1 & 1 & x\\ e^x & x+1 & 0\\ 0 & x^2 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}2x\\ x^2+x+2\\ x^3\end{pmatrix} \qquad (61)$$

with initial conditions as $\begin{pmatrix}x_1(0)\\ x_2(0)\\ x_3(0)\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$

The exact solutions are$x_1(x) = e^{-x}1,\ x_2(x) = x,\ x_3(x) = 1$

From equation (i) above, we obtain $x_1' + x_1 + x_2 + xx_3 = 2x$

$$x_2' + e^x + (x+1)x_2 = 2 + x + x^2,\ x^2x_2 = x_3 \qquad (62)$$

From first equation in (ii) above, we obtain

$$x_1' + e^x + x + x = 2x,\ x_1 + e^x + 2x = 2x,\ x_1' = -e^x \qquad (63)$$

From the second equation above, we obtain

$$x_2' + e^x(x+1)x_2 = x + x^2 + 2,\ x_2' + e^x + x + x^2 + 2,\ x_2' = 2 - e^x \qquad (64)$$

From the third equation , we obtain

$$x^2x_2 = x_{3,}\ x_3' = 2x^2 + x^2 = 3x^2 \qquad (65)$$

Table 4: Numerical Solutions of x_1 for Problem Two

x	h	Exact Solution	Ralston's Method	Error	RungeKutta Method	Error	Variational iteration Method
0.0	0.0	1	0	1	0	1	1
1.0	0.1	0.367879441	-0.095182899	4.864974117	-0.987654956	1.372477695	0.904837418
2.0	0.2	0.135335283	-0.18142773	1.734411037	-0.975614712	1.13871796	0.818730753
3.0	0.3	0.049787068	-0.259703243	1.191707533	-0.289161523	1.172`177361	0.740818221
4.0	0.4	0.018315638	-0.330884858	1.055353509	-0.380967483	1.0748076644	0.670320046
5.0	0.5	0.006737946	-0.395763092	1.017025201	-0.470624225	1.01431704	0.606530659
6.0	0.6	0.002478752	-0.45505126	1.13730047	-0.558212392	1.004440518	0.548811636
7.0	0.7	0.000911881	-0.509392503	1.001790134	-0.643809957	1.001416382	0.499658530
8.0	0.8	0.000335462	-0.559366205	1.000599718	-0.727492301	1.000461121	0.499328964
9.0	0.9	0.000123409	-0.605493852	1.000203815	-0.809332298	1.000152482	0.406569659
10.0	1.0	0.0000045399	-0.648244368	1.000700338	-0.889400391	1.0000051045	0.367879441

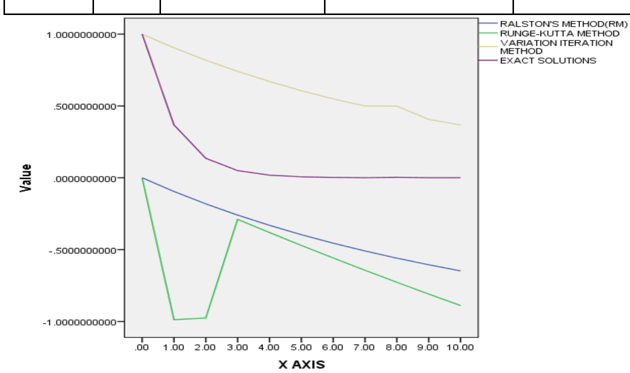

Figure 4: Solution curve showing comparison between Ralston's Method(RM), Runge-Kutta Method(RKM), Variational Iteration Method(VIM) and Exact Solution (ES)

Table 5: Numerical Solutions of x_2 for Problem Two

X	h	Exact Solution	Ralston's Method	Error	RungeKutta Method	Error
0.0	0.0	0	1	1	1	0
1.0	0.1	1.0	0.914057745	0.085942255	0.924731135	0.075268865
2.0	0.2	2.0	0.797688823	0.601155885	0.842406505	0.578796747
3.0	0.3	3.0	0.6473386	0.784220466	0.75275879	0.749080403
4.0	0.4	4.0	0.459083312	0.885229172	0.655510429	0.836122392
5.0	0.5	5.0	0.228594121	0.954281175	0.55037533	0.889924934
6.0	0.6	6.0	-0.048902171	1.008150362	0.437057578	0.92715707
7.0	0.7	7.0	-0.378666091	1.054095156	0.31525329	0.954963815
8.0	0.8	8.0	-0.766492446	1.09581156	0.184640499	0.976999937
9.0	0.9	9.0	-1.218761637	1.13541796	0.04489844	0.9950111284
10.0	1.0	10.0	0.840381816	0.915961818	0.104312392	1.010431239

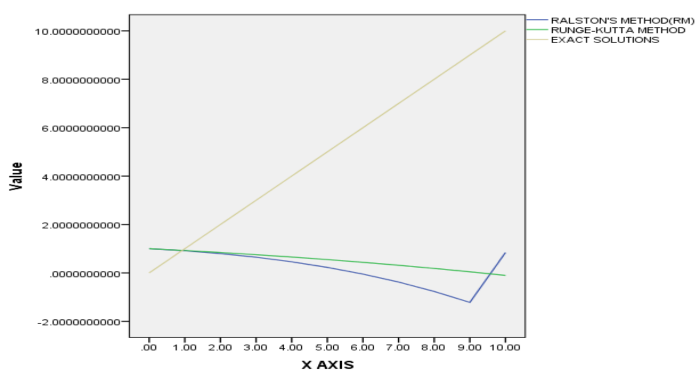

Figure 5: Solution curve showing comparison between Ralston's Method (RM), Runge-Kutta Method (RKM) and Exact Solution (ES)

Table 6: Numerical Solutions of x_3 for Problem Two

X	h	Exact Solution	Ralston's Method	Error	RungeKutta Method	Error
0.0	0.0	0	0	0	0	0
1.0	0.1	1.0	0.001125	0.998875	0.00009375	0.99990625
2.0	0.2	1.0	0.009	0.991	0.00075	0.0.9925
3.0	0.3	1.0	0.030375	0.969625	0.00253125	0.99746875
4.0	0.4	1.0	0.072	0.928	0.024	0.976
5.0	0.5	1.0	0.140625	0.85937	0.01171875	0.98828125
6.0	0.6	1.0	0.243	0.757	0.02025	0.97975
7.0	0.7	1.0	0.385875	0.614142	0.03215625	0.0.96784375
8.0	0.8	1.0	0.576	0.424	0.048	0.952
9.0	0.9	1.0	0.820125	0.1798875	0.06834375	0.0.93165625
10.0	1.0	1.0	1.125	-0.125	0.09375	0.90625

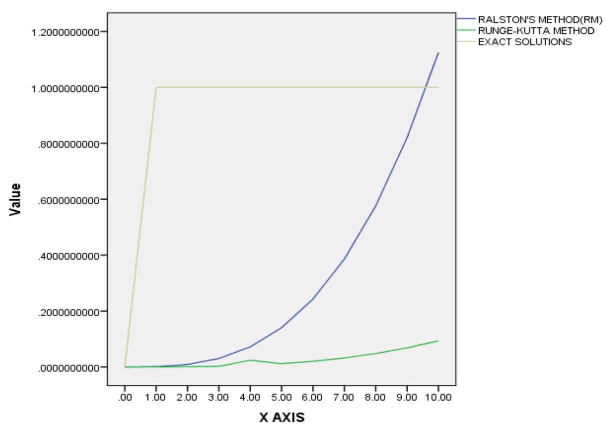

Figure 6: Solution curve showing comparison between Ralston's Method(RM), Runge-Kutta Method(RKM) and Exact Solution (ES)

CONCLUSION

The numerical results presented, show the implementation of the method when applied to the systems of higher index differential algebraic equations. It is obvious that the methods compete favorably with the existing method for the class of problems considered and also, yields desire accuracy

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