Numerical Solutions of Potential Flow Equations using Finite Differences

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ABSTRACT

This article delves into the numerical solutions of potential flow equations using finite differences, aiming to enhance our understanding of fluid dynamics. The general objective is to obtain numerical solutions to potential flow equations using finite differences, with specific objectives including the investigation of potential flow equations, the solutions of associated PDE and the analysis of the stability of employed numerical schemes. The study employs a combination of numerical methods to achieve its objectives; MATLAB is utilized as a computational tool, while the Gauss-Seidel and Jacobi's iterative methods are implemented for solving PDEs. Central differences are employed for discretization. The study yields valuable insights into the behaviour of potential flow systems. The significance of this research lies in its contribution to advancing our comprehension of fluid dynamics with potential applications. Generally, this work provides a foundation for further exploration and application of numerical methods in the study of potential flow.

Mathematics Subject Classification: Primary 20K30; Secondary 16P10.
Keywords: Potential flow equations, computational tool, fluid dynamics

1 Introduction

Fluid dynamics plays a pivotal role in understanding the behavior of fluids in various engineering applications, ranging from aerodynamics and hydrwodynamics to heat transfer and chemical processing. Potential flow, a fundamental concept in fluid dynamics, simplifies the study of fluid motion by assuming irrotational and incompressible flow. While potential flow equations provide an elegant mathematical framework, their analytical solutions are often limited to simple geometries and boundary conditions.Potential flow describes the velocity field as the gradient of a scalar function: the velocity potential. As a result, a potential flow is characterized by an irrotational velocity field, which is a valid approximation for several applications. The irrotationality of a potential flow is due to the curl of the gradient of a scalar always being equal to zero.

The complexity of real-world fluid dynamics problems necessitates the use of numerical methods to obtain solutions. Numerical techniques, such as finite differences, have proven to be valuable tools for solving partial differential equations, including those governing potential flow. Finite differences discretize the spatial domain, allowing for the transformation of continuous equations into algebraic systems that can be solved numerically on a computer. Numerical methods often operate iteratively, refining solutions through successive approximations. This iterative nature enables us to converge towards accurate solutions, providing a valuable mechanism for improving the precision of results and gaining deeper insights into the behavior of models. This research proposal aims to explore and develop an efficient numerical solution for potential flow equations using the finite differences method.

The accurate prediction of fluid behavior is crucial for the design and optimization of engineering systems. Traditional analytical methods often fall short when addressing the intricacies of complex geometries and boundary conditions. Numerical methods offer a viable alternative, providing the flexibility needed to tackle real-world scenarios. This research is motivated by the need for advanced numerical techniques that enhance the accuracy and efficiency of potential flow simulations, enabling more realistic predictions of fluid behavior.

The proposed research on the numerical solution of potential flow equations using finite differences is motivated by several compelling justifications, each emphasizing the broader impact and significance of the study; Analytical solutions for potential flow equations are limited to simple geometries and boundary conditions. As many real-world fluid dynamics scenarios involve complex structures and diverse boundary conditions, the application of analytical methods becomes impractical. The proposed numerical solution using finite differences serves as a bridge, offering a practical means to tackle the complexities inherent in potential flow problems. While numerical methods have been widely employed in fluid dynamics, there is a continuous need for advancements to improve accuracy, stability, and computational efficiency. The study contributes to the ongoing evolution of numerical techniques by exploring the application of finite differences to potential flow equations, potentially offering new insights and methods that can be extended to other fluid dynamics problems.
The theoretical contribution of numerical solutions of potential flow equations using finite differences lies in advancing computational fluid dynamics (CFD) techniques. It enhances our ability to model and analyze fluid behaviour, offering more accurate predictions for various applications such as aerodynamics, hydrodynamics, and heat transfer. This approach contributes to the understanding of potential flow phenomena through the application of numerical methods, enabling simulations that can guide design

processes and optimize engineering solutions. Numerical solutions of potential flow equations using finite differences hold methodological significance in fluid dynamics. This approach allows for the approximation of continuous mathematical models, providing insights into complex fluid behaviour. Finite differences discretize the domain, enabling the analysis of fluid flow over intricate geometries that may lack analytical solutions. Numerical solutions enhance our ability to study turbulence, vorticity, and lift generation, providing valuable information for optimizing designs and predicting fluid dynamics outcomes. This methodology bridges the gap between theoretical formulations and practical applications, fostering advancements in fluid mechanics and engineering simulations.

Employing finite differences for solving potential flow equations contributes academically by advancing numerical methods by introducing or improving finite difference methods for solving potential flow equations that expands the repertoire of numerical techniques available for fluid dynamics simulations.Interest and passion for understanding the intricacies of fluid behaviour and the desire to contribute to this field motivated our pursuit. curiosity and the drive to explore the uncharted territories of numerical methods in solving potential flow equations inspired us.

2 Methods

2.1 Investigation of Potential Flow Equations

To develop a clear and detailed formulation of potential flow equations, considering various boundary conditions and geometries; Governing equations for potential flow will be extracted from the literature review. Different formulations based on variations in assumptions and boundary conditions will be examined. Mathematical derivations and physical interpretations associated with potential flow equations will be considered. To identify critical factors influencing potential flow behaviour and understand their implications for numerical solution; Sensitivity analyses will be conducted on key parameters such as velocity, pressure, and geometric features. The impact of variations in boundary conditions on potential flow solutions will be evaluated. The influence of different assumptions (e.g., incompressibility, irrotationality) on the behaviour of potential flows will be investigated. To explore mathematical properties and theoretical considerations associated with potential flow equations, the mathematical structure of potential flow equations, including linearity, superposition, and conservation laws, will be examined. The existence and uniqueness of solutions under various conditions will be investigated.

2.2 Solution of the PDEs Associated with Potential Flow

To apply finite difference methods to discretize the spatial domain of the potential flow equations; Appropriate finite difference schemes, such as central differences, will be chosen to approximate spatial derivatives within the discretized domain. We will use Gauss-Seidel and Jacobi's iterative methods to improve results obtained. To implement time integration schemes to advance the solution in time: Implicit or explicit methods will be utilized based on stability and efficiency considerations. To develop a numerical algorithm to solve the discretized potential flow equations: The algorithm will be implemented

in a programming environment, utilizing suitable numerical libraries and tools. To validate the numerical solution against known analytical solutions or benchmark cases; The correctness of the numerical implementation will be verified through convergence studies and comparison with established results.

2.3 Analysis of Stability of Numerical Schemes

To conduct a von Neumann stability analysis to assess the stability properties of the finite difference schemes; amplification factors and stability regions will be analyzed to identify conditions under which the numerical solution remains stable. To investigate the influence of varying time steps on the stability of the numerical solution; Optimal time step sizes will be identified to ensure both stability and computational efficiency. To perform parametric studies to explore the effects of mesh size, boundary conditions, and other numerical parameters on stability; the robustness of the numerical method across a range of scenarios will be evaluated. To document the findings of the stability analysis in a clear and concise manner; insights into the limitations and constraints associated with the numerical method's stability will be provided.

By implementing these methods, a systematic investigation into potential flow equations using finite differences will be conducted, aligning with the research objectives outlined in the proposal.

3 Literature Review

Fluid dynamics, particularly potential flow, has been a subject of significant interest in the scientific community due to its broad applications in engineering and physics. The quest for accurate and efficient solutions to potential flow equations has driven researchers to explore various numerical methods. This literature review provides an overview of existing literature, highlighting key contributions and gaps in the field of numerical solutions for potential flow equations, with a focus on the application of finite differences. The governing equations for potential flow describe irrotational and incompressible fluid motion, simplifying the complexities of real-world fluid dynamics. These equations, derived from the conservation of mass and the Bernoulli equation, form a set of partial differential equations (PDEs). Analytical solutions to these equations are limited to idealized scenarios, prompting the need for numerical methods to address real-world complexities.

[10] provides an extensive overview of finite difference methods applied to potential flow equations. They explore the historical evolution, challenges, and recent advancements, offering a comprehensive understanding of the strengths and limitations of finite differences in modelling potential flows. [5] focuses on recent developments in numerical techniques for potential flow simulations, with an emphasis on finite differences. They discuss innovations in spatial discretization, mesh refinement, and parallel computing, providing insights into how these advancements contribute to the accuracy and efficiency of numerical solutions. [4] conducted a comparative analysis of various finite difference schemes used in solving potential flow equations. They evaluated the accuracy, stability, and computational efficiency of different schemes, offering a critical assessment of their applicability to real-world fluid dynamics

problems. [9] focused on adaptive finite difference methods, this explores strategies for dynamically refining the computational mesh based on solution characteristics. They discussed challenges in adapting finite differences to handle complex geometries and varying boundary conditions in potential flow simulations. [3] specifically addressed the challenges and opportunities in applying finite differences to unsteady potential flow scenarios. They discussed issues related to stability and accuracy, proposing methodologies to extend the applicability of finite difference methods to dynamic fluid flow problems.

Numerical methods have become indispensable tools for solving potential flow equations, offering the flexibility to tackle a wide range of scenarios. Finite difference methods, in particular, have gained prominence due to their simplicity and computational efficiency. The discretization of the spatial domain allows for the transformation of PDEs into algebraic systems, facilitating the implementation of numerical algorithms on modern computing platforms. Finite difference methods involve discretizing the spatial domain into a grid and approximating derivatives using finite differences. The accuracy and stability of these methods depend on the choice of discretization schemes. Classic schemes such as forward, backward, and central differences have been extensively employed, with modifications and higher-order schemes developed to enhance accuracy.

While numerical methods offer practical solutions, challenges persist. Stability issues, numerical dissipation, and dispersion effects are among the concerns that researchers have grappled with. The literature highlights the importance of carefully selecting numerical parameters and discretization schemes to mitigate these challenges. Several studies have undertaken comparative analyses of different numerical methods applied to potential flow equations. These comparisons often assess accuracy, computational efficiency, and applicability to diverse scenarios. While finite difference methods have shown promise, some studies explore alternative numerical techniques, such as finite element and spectral methods, providing insights into their comparative advantages and limitations. Recent literature reflects a growing interest in hybrid and adaptive numerical methods for potential flow. Hybrid approaches combine the strengths of different numerical techniques to overcome individual limitations. Adaptive methods dynamically refine the grid based on solution characteristics, optimizing computational resources while maintaining accuracy.

Despite the progress made, certain gaps in the literature persist. The application of finite differences to three-dimensional potential flow problems, consideration of unsteady flows, and the exploration of parallel computing for scalability are areas where further research is warranted. Additionally, emerging trends, such as machine learning coupled with numerical methods, present exciting possibilities for future investigations.

The literature reviewed underscores the evolution from analytical to numerical methods in solving potential flow equations, with a specific emphasis on the role of finite differences. While finite difference methods have shown promise, challenges persist, and ongoing research aims to address these challenges and explore innovative approaches. This literature review sets the stage for the proposed research, emphasizing the need for a robust numerical solution for potential flow equations using finite differences and identifying areas for further exploration and refinement.

4 Potential Flow Equation

Potential flow is a simplified model of fluid flow that is based on the assumption that the flow is irrotational (i.e., the vorticity is zero). In potential flow theory, the velocity field can be derived from a scalar potential function called the velocity potential.

The potential flow equations are typically derived from the conservation of mass and conservation of momentum principles. The key equations in potential flow theory include:

1. **Laplace's Equation**:
$$\nabla^2 \Phi = 0 \tag{4.1}$$
Where Φ is the velocity potential and ∇^2 is the Laplacian operator. This equation represents the irrotationality condition of the flow field.

2. **Velocity Field**:
$$u = \frac{\partial \Phi}{\partial x} \tag{4.2}$$
$$v = \frac{\partial \Phi}{\partial y} \tag{4.3}$$
$$w = \frac{\partial \Phi}{\partial z} \tag{4.4}$$
Where u, v, and w are the components of the velocity vector and Φ is the velocity potential. These equations indicate that the velocity field can be derived from the gradient of the velocity potential.

3. **Bernoulli's Equation**:
$$P + \frac{1}{2}\rho |V|^2 + \rho g Z = \text{constant} \tag{4.5}$$
Where P is the pressure, ρ is the fluid density, V is the fluid velocity vector, g is the acceleration due to gravity, and Z is the elevation. This equation represents the conservation of energy along a streamline in a potential flow field.

4. **Stream Function**:
$$\psi = -\Phi \tag{4.6}$$
Where ψ is the stream function. The stream function is often used in 2D potential flow problems to visualize streamlines and streamline patterns.

These equations are the foundation of potential flow theory and are commonly used in aerodynamics, hydrodynamics, and other fields of fluid mechanics to analyze and predict fluid flow behavior in a simplified manner.

5 Discretization of the laplace Equation

To discretize the Laplace equation in 2D within the interval $[0, 1]$, we use a finite difference method. Let's assume we have a grid with N_x points in the x direction and N_y points in the y direction, and the interval $[0, 1]$ is divided into $N_x - 1$ intervals of size Δx and $N_y - 1$ intervals of size Δy.

The grid points are located at:

$$x_i = i \cdot \Delta x, \quad i = 0, 1, 2, \ldots, N_x$$

$$y_j = j \cdot \Delta y, \quad j = 0, 1, 2, \ldots, N_y$$

Now, let $u_{i,j}$ represent the value of u at the grid point (x_i, y_j). The Laplace equation can be discretized as:

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)$$

This is the finite difference equation for the Laplace equation at each interior grid point (i, j).

At the boundaries (where $i = 0, N_x$ or $j = 0, N_y$), boundary conditions need to be applied appropriately. Common boundary conditions include Dirichlet, Neumann, or mixed boundary conditions.

Once the discretization is done, you can solve the resulting system of equations iteratively using methods such as Jacobi, Gauss-Seidel, until convergence is achieved.

6 Finite Differential Approximation of the Laplace equation

6.1 standard 4-point formula

The standard 4-point formula for solving the Laplace equation in a rectangular domain is given by:

$$u_{i,j} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right)$$

This formula represents an averaging of the values of the neighboring points to calculate the value of the point at location (i, j) in the computational grid.

7 Improvement of Results

7.1 Jacobi's Method

Jacobi's method is an iterative numerical technique commonly used to solve linear systems of equations. In the context of improving results obtained using the standard 4-point formula for Laplace's equation, Jacobi's method can be applied to iteratively refine the solution until convergence is achieved. Let's see how Jacobi's method can be used to enhance the results obtained from the standard 4-point formula for the Laplace equation in 2D space.

Jacobi's Method: Jacobi's method is an iterative algorithm that updates the values of the solution vector from the previous iteration. The method involves several steps:

1. **Initialization:** - Start with an initial guess for the solution vector u^0 at all grid points.
2. **Iteration:** - For each grid point (i, j), update the solution using the discretized equation. - For the Laplace equation, you will apply the discretized 4-point formula at each grid point.

3. **Update Rules:** - The update rule for Jacobi's method is:

$$u^{(k+1)}(i,j) = \frac{1}{4}\left(u^{(k)}(i-1,j) + u^{(k)}(i+1,j) + u^{(k)}(i,j-1) + u^{(k)}(i,j+1)\right)$$

4. **Convergence:** - Repeat the iteration process until the solution converges within a specified tolerance.

Enhancing Results with Jacobi's Method:

- By applying Jacobi's method iteratively to the solutions obtained from the standard 4-point formula, you can refine the solutions to better approximate the true solution of the Laplace equation.
- Jacobi's method allows for improvement by considering the neighboring grid points and updating the solution iteratively.

7.2 Gauss-Seidel Method

The Gauss-Seidel method is an iterative algorithm that updates the solution vector by taking into account the most recent values of the solution obtained in the same iteration. The method involves the following steps:

1. **Initialization:** - Start with an initial guess for the solution vector u^0 at all grid points.
2. **Iteration:** - For each grid point (i,j), update the solution using the most recent solution values in that iteration. - For the Laplace equation, you will apply the discretized 4-point formula using the most recent values from the iteration.
3. **Update Rules:** - The update rule for the Gauss-Seidel method updates the solution simultaneously:

$$u^{(k+1)}(i,j) = \frac{1}{4}\left(u^{(k+1)}(i-1,j) + u^{(k)}(i+1,j) + u^{(k+1)}(i,j-1) + u^{(k)}(i,j+1)\right)$$

4. **Convergence:** - Repeat the iteration process until the solution converges within a specified tolerance, typically checking the change in the solution values.

Advantages of Gauss-Seidel Method:

- **Faster Convergence:** The Gauss-Seidel method often converges faster than the Jacobi method due to using updated values within the same iteration.
- **Efficiency:** Since newer values are used immediately, the method converges in fewer iterations.

Enhancing Results with Gauss-Seidel Method:

- By applying the Gauss-Seidel method iteratively to the results obtained from the 4-point formula, you can refine the solutions more effectively, reducing error and increasing accuracy.

8 Stability analysis using Von Neumann method

Suppose we use Jacobi iteration for solving a system of equations generated. The recurrence relation for the laplace equation is then given by:

$$u_0^{(k+1)}(i,j) = \frac{1}{4}\left[u_0^{(k)}(i+1,j) + u_0^{(k)}(i-1,j) + u_0^{(k)}(i,j+1) + u_0^{(k)}(i,j-1)\right] \tag{8.1}$$

Let us define the error at the kth iteration as follows:

$$e_{(i,j)}^{(k)} = u_{(i,j)}^{\prime(k)}(Exact) - u_{(i,j)}^{\prime(k)}(Approximate) \tag{8.2}$$

Since the exact solution satisfies the relation (5.1), so is the error. Hence,

$$e_{i,j}^{(k+1)} = \frac{1}{4}\left[e_{i+1,j}^{(k)} + e_{i-1,j}^{(k)} + e_{i,j+1}^{(k)} + e_{i,j-1}^{(k)}\right] \tag{8.3}$$

The error term is represented in the form

$$e_{l,m}^{(k)} = A_{pq}\sin\left(\frac{p\pi l}{m}\right)\sin\left(\frac{q\pi m}{m}\right), \quad 1 \le p, q \le m-1 \tag{8.4}$$

A_{pq} is an arbitrary constant. Following from (5.4), we have the following expressions:

$$\begin{aligned} e_{i+1,j}^{(k)} &= Apq\sin\left(\frac{p\pi(i+1)}{m}\right)\sin\left(\frac{q\pi j}{m}\right) \\ &= Apq\sin\left(\frac{q\pi j}{m}\right)\left[\sin\left(\frac{p\pi i}{m}\right)\cos\left(\frac{p\pi}{m}\right) + \cos\left(\frac{p\pi i}{m}\right)\sin\left(\frac{p\pi}{m}\right)\right] \end{aligned} \tag{8.5}$$

$$\begin{aligned} e_{i-1,j}^{(k)} &= Apq\sin\left(\frac{p\pi(i-1)}{m}\right)\sin\left(\frac{q\pi j}{m}\right) \\ &= Apq\sin\left(\frac{q\pi j}{m}\right)\left[\sin\left(\frac{p\pi i}{m}\right)\cos\left(\frac{p\pi}{m}\right) - \cos\left(\frac{p\pi i}{m}\right)\sin\left(\frac{p\pi}{m}\right)\right] \end{aligned} \tag{8.6}$$

$$e_{i+1,j}^{(k)} + e_{i-1,j}^{(k)} = 2Apq\sin\left(\frac{q\pi i}{m}\right)\sin\left(\frac{p\pi i}{m}\right)\cos\left(\frac{p\pi}{m}\right) = 2\cos\left(\frac{p\pi}{m}\right)Apq\sin\left(\frac{p\pi i}{m}\right)\sin\left(\frac{q\pi j}{m}\right) = 2\cos\left(\frac{p\pi}{m}\right)e_{i,j}^{(k)} \tag{8.7}$$

$$\begin{aligned} e_{i+1,j}^{(k)} + e_{i-1,j}^{(k)} &= 2Apq\cos\left(\frac{q\pi}{m}\right)\sin\left(\frac{p\pi i}{m}\right)\sin\left(\frac{q\pi j}{m}\right) \\ &= 2\cos\left(\frac{q\pi}{m}\right)e_{i,j}^{(k)} \end{aligned} \tag{8.8}$$

Substituting equations (5.7) and (5.8) into equation (5.3) yields:

$$e_{i,j}^{(k+1)} = \xi e_{i,j}^{(k)} \quad \text{where} \quad \xi = \frac{1}{4}[2\cos\left(\frac{p\pi}{m}\right) + 2\cos\left(\frac{q\pi}{m}\right)i] \tag{8.9}$$

For stability;

$$\left|\frac{e_{i,j}^{(k+1)}}{e_{i,j}^{(k)}}\right| = |\xi| < 1 \tag{8.10}$$

This implies that

$$\left|\frac{1}{4}[2\cos\left(\frac{p\pi}{m}\right) + 2\cos\left(\frac{q\pi}{m}\right) i]\right| < 1 \tag{8.11}$$

Let $p = q = 1$. Therefore,

$$-1 < \frac{4}{4 - \cos\left(\pi/m\right)} < 1 \tag{8.12}$$

For stability to be achieved, the above condition must be satisfied.

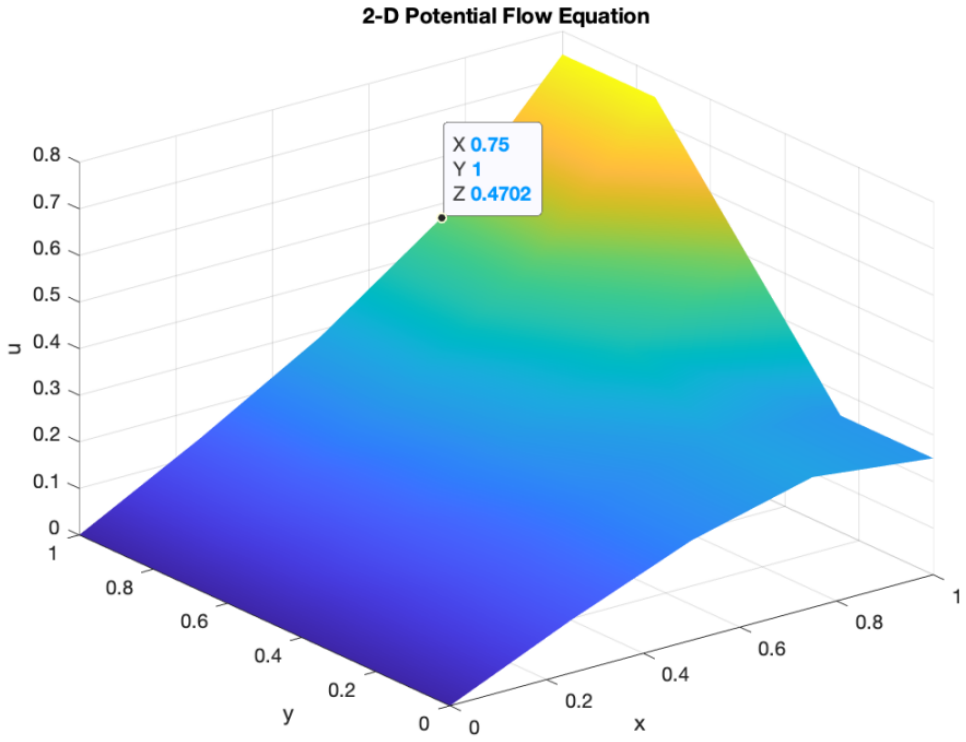

Figure 1: This plot shows 3D surface plot showing the solution to the 2-D Laplace's equation using the Finite Difference Method with the specified parameters as follows: Number of steps in space (x): $nx = 5$, Number of steps in space (y): $ny = 5$, Number of iterations: $niter = 1000$, Width of space step (x): $dx = 0.25$, Width of space step (y): $dy = 0.25$, Range of x: $x = [0, 0.25, 0.5, 0.75, 1]$, Range of y: $y = [0, 0.25, 0.5, 0.75, 1]$

9 Results

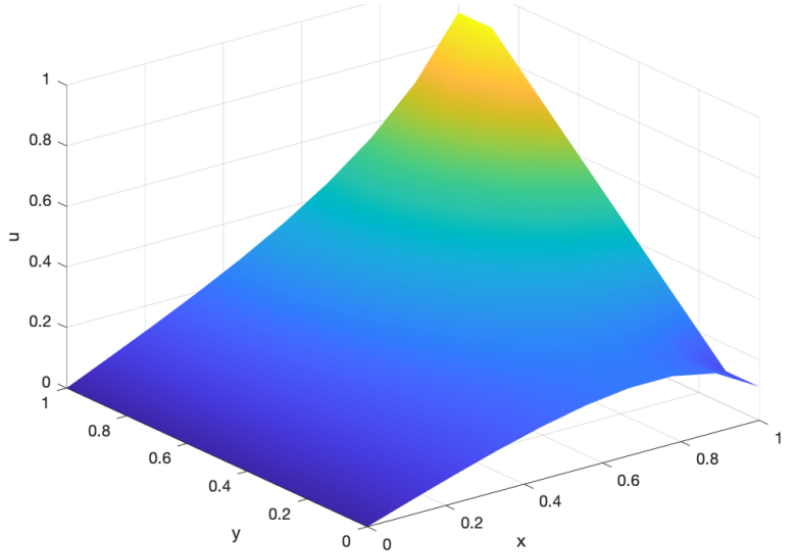

Figure 2: This plot shows 3D surface plot showing the solution to the 2-D Laplace's equation using the Finite Difference Method with the specified parameters as follows: Number of steps in space (x): $nx = 20$, Number of steps in space (y): $ny = 20$, Number of iterations: $niter = 1000$, Width of space step (x): $dx = 0.0526$, Width of space step (y): $dy = 0.0526$, Range of x: $x = [0, 0.0526, 0.1053, \ldots, 1]$, Range of y: $y = [0, 0.0526, 0.1053, \ldots, 1]$

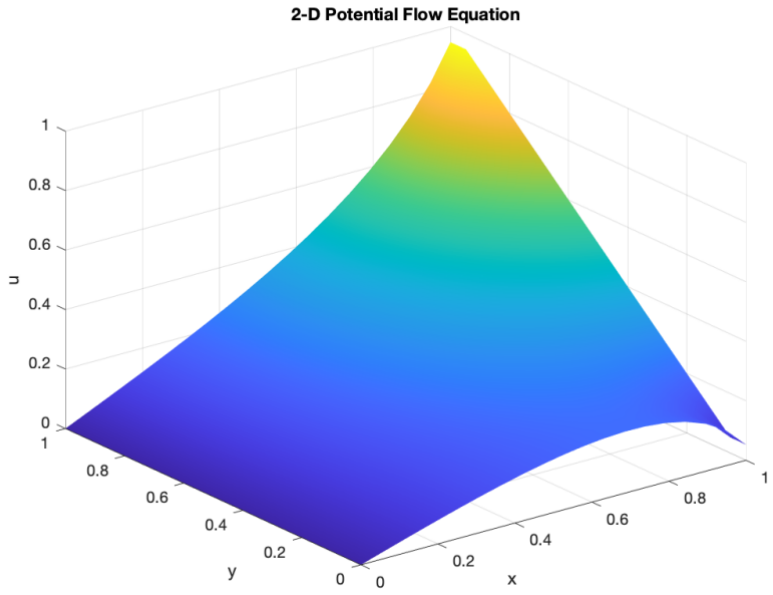

Figure 3: This plot shows 3D surface plot showing the solution to the 2-D Laplace's equation using the Finite Difference Method with the specified parameters as follows: Number of steps in space (x): $nx = 100$, Number of steps in space (y): $ny = 100$, Number of iterations: $niter = 1000$, Width of space step (x): $dx = 0.01$, Width of space step (y): $dy = 0.01$, Range of x: $x = [0, 0.01, 0.02, \ldots, 1]$, Range of y: $y = [0, 0.01, 0.02, \ldots, 1]$

10 Conclusions

Here are the general observations based on the results for different values of $nx = ny = (5, 10, 20, 50, 100)$:

Grid resolution increases with nx and ny: As the number of steps in space nx and ny increases, the grid resolution becomes finer. This means that the interval is discretized into smaller increments, allowing for a more detailed representation of the solution.

Decreasing step size with increasing nx and ny: With larger values of nx and ny, the width of the space step dx and dy decreases. This indicates that the distance between adjacent grid points becomes smaller, resulting in a more precise representation of the spatial domain.

Increasing number of grid points: As nx and ny increase, the number of grid points within the spatial domain also increases. This leads to a denser distribution of grid points, allowing for a more accurate approximation of the solution.

Consistency of range in x and y: Despite varying values of nx and ny, the range in x and y remains consistent from 0 to 1. This ensures that the spatial domain is uniformly discretized across different resolutions.

Stability: Stability analysis of the laplace's equation using Von Neumann method shows that stability is guaranteed when the condition below is satisfied:

$$-1 < \frac{4}{4 - \cos(\pi/m)} < 1 \tag{10.1}$$

11 Recommendations

Optimizing Grid Resolution: Consider the required level of accuracy for the specific problem. Choose nx and ny values that balance accuracy and computational efficiency. Higher values of nx and ny may be needed for problems requiring high accuracy.

Efficient Resource Allocation: Be mindful of computational resources and time constraints. Avoid excessively large values of nx and ny that may lead to unnecessary computational cost. Conduct sensitivity analyses to identify optimal values.

Consider Problem Characteristics: Tailor the choice of nx and ny to the specific characteristics of the problem. Problems with intricate features may require higher grid resolutions for accurate representation.

Verification and Validation: Verify the accuracy of the numerical solution against analytical solutions or experimental data. Perform validation studies to ensure adequate capture of the problem physics by the chosen grid resolution.

Iterative Refinement: Start with coarse grid resolutions and gradually increase nx and ny as needed. Assess the impact on solution accuracy and computational cost at each step.

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