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Original Research

A Legendary Polynomial Approach to Solutions of Volterra Integro-Differential Equations with Delay

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Abstract

This paper presents a numerical computation approach to the solution of Volterra Integro-Differential Equation with delay via legendary Polynomial method. Computational code was developed and implemented on a Mathematical Software (Matlab 2009b) to solve the problem. The accuracy, efficiency and effectiveness of the method of solution were ascertained by comparing the solution obtained with the exact solution in the literature and it showed that the present results were in agreement with both the exact and existing results

Key Words: Integro-Differential Equation (IDE), Volterra, Legendary Polynomial, Approximate solution,

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INTRODUCTION

Integro-Differential Equation (IDE) is a hybrid of Integral and Differential Equations which have found extensive applications in Science and Engineering since it was established by Volterra[12]

The Volterra Integro-Differential Equations (VIDEs) arise in specific field such as Physics, Biology, Ecology and Medicine [4, 5, 10]. It investigated the population growth, focusing his study on the hereditary influences; where through his research work the topic of Integro-Differential Equations was established [2]. This class of equations plays an important role in modeling diverse problems of engineering and natural science and hence has come to intrigue researchers in numerical computation and analysis. In this study, we consider the following Volterra Delay Integro-Differential Equation:

$$y' = f(x) + \lambda \int_0^x K(x,t)y(t-\tau)dt \qquad 1$$

Here, f and k = assumed to be sufficiently smooth with respect to their argumenty

y = the unknown function to be determined

τ = a positive number/ the delay term

If it is function of time t, then it is called time dependent delay and if it is a function of time t and y(t), then it is called state dependent delay. [8] propose a numerical technique which is based on a mixed of exotic CI-spline collocation method. [7] and El-Ghendi method to solve Volterra delay integro-differential equations [9]. [14, 15] presented a new technique for numerical treatments of Volterra delay integro-differential equations that have many applications in biological and physical sciences. The technique is based on the mono-implicit. Range-Kutta method described by [14] for treating the differential part and collocation method using Boole's quadrature rule for treating the integral part. [11] concentrated on the differential transform method to solve some delay differential equations based on the method of steps for delay differential equations and using the computer algebra system. Mathematica, [1] applied the differential transform method to solve the nonlinear integro- differential equation with proportional delays. [16] considered a certain non-linear Volterra integro-differential equations with delay, and studied stability and boundedness of solution. Also,[13] consider numerical method for solving Delay Integro-Differential Equations and used the Chebyshev polynomial as a basis function.

Materials and Methods

The legendary polynomials introduced in 1784 by the French mathematician A. M. Legendre (1752-1833) which are special cases of Legendre functions. Legendre functions are important in problems involving spheres or spherical coordinates. Due to their orthogonality properties and generating function for Legendre's Polynomial $P_n(x)$. which are also useful in numerical analysis. Here is first few Legendre's Polynomials:

$$P_0(x) = 1, \ \ P_1(x) = x, \ \ P_2(2) = \frac{1}{2}(3x^2 - 1), \ \ P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

rewriting the Eqn. 1 in the operator form as S

$$L[y(x)] = f(x) \qquad 2$$

where

$$L[y(x)] = y(x) - \int_0^x k(x,y)y(t-\tau)dt \qquad 3$$

approximating the unknown function $y(x)$ by $y_n(x)$

$$y_{(x)} = \sum_{i=0}^{n} C_i P_i(x) \qquad 4$$

where c_i and τ_i, i =0(n) are the unknown coefficients and Legendry polynomials, respectively. substituting, eq. 4 into 2 yield

$$L[y_n(x)] = f(x) \qquad 5$$

Where

$$\mathrm{L}[y_n(x)] = \sum_{i=0}^{n} C_i P_i(x) - \int_a^x K(x,y) \sum_{i=0}^{n} C_i\, P_i(t-\tau)dt$$

$$= \sum_{i=0}^{n} C_i \left[P_i(x) - \int_a^x k(x,y)\, P_i(t-\tau)\right] \qquad 6$$

$$= \sum_{i=0}^{n} C_i \;\; L[P_i(x)]$$

from equation 5 and 6, we have

$$\sum_{i=0}^{n} C_i\, L[P_i(x)] = f(x)$$

to get the best unknown coefficients c_i, i = 0, n, we minimize the residual term

$$E(x) = L[y_i\,(x)] - f(x) \qquad 8$$

that means we choose the unknown coefficients to satisfy the relationship

$$\int_a^x \omega_i\, E(x)dx = 0, \;\; j = \overline{0,n}$$

where $\omega_i(x)$ is called weighted functions which is defined as

$$\omega_i(x) = \frac{\delta y_n(x)}{\partial c_i} = P_i(x) \qquad 10$$

substitute eqn. 8 into 9 yields

$$\int_a^x P_i\, [L[y_n(x)] - f(x)]dx = 0 \qquad 11$$

From eqn. 4 and 11, we obtain

$$\int_a^x P_i\,(x)[\sum_{i=0}^{n} C_i \;\; L[P_i(x)] - f(x)]dx = 0 \qquad 12$$

that is similar to the following system

$$\sum_a^n C_i \int_a^x P_i(x)\, L[P_i\,(x)]dx = \int_a^x P_i\,(x) f(x)dx, \qquad i,j = \overline{0,n} \qquad 13$$

Solving the above system for the coefficient $C_i, i\overline{0,n}$ and substituting into eqn. 4, we obtain the approximate solution of eqn. 1.

Demonstration and Discussion

Example 1: Let us consider the following equation.

$$y^{"}(x) = 1 - \frac{x^5}{4} + \int_0^x xt^2\, y(t-1)dt \qquad 14$$

which has the following exact solution

$$y(x) = x + 1 \qquad 15$$

Comparing equation 14 with 1 we find out that

$$k(x,y) = xt^2, f(x) = 1 - \frac{x^5}{4} \qquad 16$$

First, we calculate

$$\int_a^x P_i\,(x) f(x) dx, i = 0,1$$

as

$$\int_0^x P_0(S)\left[1 - \frac{S^5}{4}\right]ds = \int_0^x \left[1 - \frac{S^5}{4}\right]ds = \left[S - \frac{S^6}{24}\right]_0^x = x - \frac{x^6}{24} \qquad 17$$

$$\int_0^x P_1(S)\left[1 - \frac{S^5}{4}\right]ds = \int_0^x \left[S - \frac{S^5}{4}\right]ds = \left[\frac{S^2}{2} - \frac{S^7}{28}\right]_0^x = \frac{x^2}{2} - \frac{x^7}{28} \qquad 18$$

Secondly, we calculate

$$\int_a^x P_i\,(x) L[P_i(x)]dx, i = 0,1$$

as

$$\int_0^x \varphi_0\,(S) L[\varphi_0(S)]ds = -\int_0^x \int_0^x St^2 dtds = \frac{x^5}{15} \qquad 19$$

$$\int_0^x \varphi_0\,(S)L[\varphi_1(S)]dx = -\int_0^x \int_0^x St^2(t-1)dtds = \frac{x^6}{24} - \frac{x^5}{15} \quad 20$$

$$\int_0^x \varphi_1\,(S)L[\varphi_0(S)]dx = -\int_0^x \int_0^x S^2t^2dtds = -\frac{x^5}{15} \quad 21$$

$$\int_0^x \varphi_1\,(S)L[\varphi_1(S)]dx = \frac{x}{2} - \int_0^x \int_0^x S^2t^2(t-1)dtds = \frac{x^2}{2} - \frac{x^7}{28} + \frac{x^6}{18} \quad 22$$

Insert Equations 17-22 into 13, we obtain the following system

$$-c_0\frac{x^5}{15} + c_1\left(-\frac{x^6}{24} + \frac{x^5}{15}\right) = x - \frac{x^6}{24} \quad 23a$$

$$-c_0\frac{x^5}{18} + c_1\left(\frac{x^2}{2} - \frac{x^7}{28} + \frac{x^6}{18}\right) = \frac{x^2}{2} - \frac{x^7}{28} \quad 23b$$

insert eqn. 24 into eqn. 4 for 2n = 2 yields, the approximate solution of eqn. 14

$y_2(x) = 1 + x$

which is also similar to the exact solution.

Numerical Examples

We consider here some selected examples for experimentation with the methods derived in this paper, that is, solution of Linear Delay Volterra Integro-Differential Equations using Legendary Polynomial. The experiments were carried out by the Mathematical software (MATLAB 2009b) and the results are presented below.

Example 1: [3] Consider the Volterra Integro Differential Equation

$$y'(x) = -1 + \int_0^x y^2\,(t-2)dt$$

with initial condition y(0) = 0 and exact solution y(x) = -x

Example 2: [3] Consider the Volterra Integro Differential Equation

$$y'(x) = 1 + xe^x - \int_0^x e^{x-1}\,y(t-5)dt$$

With initial condition $y(0) = 0, y'(0) = 1$ and exact solution $y(x) = e^{x-1}$

Table of Results

Table 1: Comparing the existing results with Legendary Polynomial Method for different value of x in example 1

x	Exact	ADM[3]	Legendry	Error of ADM [3]	Error of Legendry
0.1	-0.10000000	-0.099991667	-0.099998288	8.33E-06	1.71E-06
0.2	-0.20000000	-0.199866718	-0.199955475	1.33E-04	4.45E-05
0.3	-0.30000000	-0.299325867	-0.299912875	6.74E-04	8.71E-05
0.4	-0.40000000	-0.397873151	-0.399457824	2.13E-03	5.42E-04
0.5	-0.50000000	-0.494822508	-0.499671235	5.18E-03	3.29E-04
0.6	-0.60000000	-0.589310094	-0.594567184	1.07E-02	5.43E-03
0.7	-0.70000000	-0.68031386	-0.697542684	1.97E-02	2.46E-03
0.8	-0.80000000	-0.766681459	-0.772485746	3.33E-02	2.75E-02
0.9	-0.90000000	-0.847166926	-0.864692178	5.28E-02	3.53E-02
1.0	-1.00000000	-0.920475711	-0.954897426	7.95E-02	4.51E-02

Table 2: Comparing the existing results with Legendary Polynomial for different value of x inexample 2

X	Exact	ADM [3]	Legendry	Error of ADM [3]	Error of Legendry
0.1	0.90483742	0.896160501	0.854615078	8.68E-03	5.02E-02
0.2	0.81873075	0.783594511	0.697842546	3.51E-02	1.21E-01
0.3	0.74081822	0.660685557	0.595874512	8.01E-02	1.45E-01
0.4	0.67032005	0.525762821	0.489754621	1.45E-01	1.81E-01
0.5	0.60653066	0.377106516	0.301475895	2.29E-01	3.05E-01
0.6	0.54881164	0.2129508	0.200147854	3.36E-01	3.49E-01
0.7	0.49658530	0.031483915	0.024571254	4.65E-01	4.72E-01
0.8	0.44932896	-0.16915469	-0.234578141	6.18E-01	6.84E-01
0.9	0.40656966	-0.390880529	-0.345217845	7.97E-01	7.52E-01
1.0	0.36787944	-0.635673673	-0.954897426	1.00E+00	1.32E+00

Discussion of Results and Conclusion

Table 1 and 2 shows the method has favourable solution as the value of x increases. The method also shows effective as Adomian Decomposition Method while compare the analytical/ exact solution in both examples.

Numerical results show that our method is very effective and efficient that gives approximation of higher accuracy and closed form solution. Moreover, our proposed method provides exact solution for some problems.

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