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Original Research

Energy of the Zero-Divisor Graph of the Integers Modulo n ($\mathbb{Z}_n$)

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Abstract

Adding the moduli (absolute values) of the eigenvalues of a matrix generated from a graph gives the energy of the graph. Three different types of energies are computed in this paper; the adjacency energy, Seidel energy and the maximum degree energy. The graph under consideration is the zero-divisor graph of the integers modulo *n* ($\mathbb{Z}_n$), where we considered seven rings of integers modulo *n*, namely, $\mathbb{Z}_{6,}\mathbb{Z}_{8,}\mathbb{Z}_{9,}\mathbb{Z}_{10,}\mathbb{Z}_{12,}\mathbb{Z}_{14}$ *and* $\mathbb{Z}_{15}$. The matrices of the graphs are first generated after which the energies are then computed using the eigenvalues of the respective matrices.

Key Words: Adjacency energy, energy of graph, maximum degree energy, ring of integers modulo *n*, Seidel energy.

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1. INTRODUCTION

Recently, there is a growing interest in studying the energy of graphs. Ivan Gutman first defined the energy of simple graphs in 1978 [1]. He viewed it as adding the modulus of all eigenvalues of a matrix generated from the graph.

There is a lot of work on energy of graphs; Gutman in 2001 worked on energy of graphs and established a connection between energy and the total electron energy of a class of organic molecules [2]. In 2009 Adiga and Smitha introduced the concepts of maximum degree matrix and maximum degree energy of a graph [3], Haemer's work in 2012 was on seidel switching and seidel energy of graphs [4]. As for Meenakshi and Lavanya they computed about 12 different types of energies for a complete graph K_4 in 2014 [5]. Some works were also done on the computation of energies of graphs of finite non-abelian groups.

Here we considered the energy of graphs of rings (abelian groups), i.e the energy of zero-divisor graph of integers modulo n.

A graph *G*=(*V*,*E*) is an ordered pair which consists of vertices (*V*) and edges (*E*) [6]. Anderson and Livingston in 1999 introduced the zero-divisor graph of commutative rings [7]. The graph has zero-divisors as sets of vertices. The zero-divisor graph of integers modulo *n* is a simple and undirected graph which is not complete [8].

The paper is structured in such a way that after the introduction we have the preliminaries where previous results needed for this work are given. Then we looked at the matrices of the zero-divisor graphs of the rings and finally the energies of the graphs are computed.

2. PRELIMINARIES

In this section we look at some known results which are relevant to this paper.

Definition 2.1 Zero-Divisor of A Ring [9]

A non-zero element x of a ring R is called a zero-divisor of the ring if there exist a non-zero element $y \in R$ such that $xy = 0$.

Definition 2.2 Zero-Divisor Graph [7]

Zero divisor graph of a commutative ring R is a graph whose vertex set is the set of nonzero zero-divisors of R and two vertices $x \text{ and } y$ are adjacent iff $xy = 0$.

Definition 2.3 Zero-Divisor Graph of Integers Modulo *n* [8]

This is a graph (G^D) whose vertex set is the set of nonzero zero-divisors of integers modulo n where two vertices $z_1$and z_2 are adjacent iff $z_1 z_2$ = 0. The graphs are simple, undirected and not complete.

Definition 2.4 Adjacency Matrix [6]

The adjacency matrix of a graph G with n vertices is an $n \times n$ matrix with entries a_{ij}; where for $i \neq j$ the entry is 1 if the vertices *i* and *j* are adjacent and 0 if the vertices *i* and *j* are not adjacent. For *i*=*j* the entry is also 0.

Definition 2.5 Seidel Matrix [4]

This is an adjacency matrix with entries s_{ij}; where for $i \neq j$ the entry is -1 if the vertices *i* and *j* are adjacent and 1 if they are not adjacent. For *i*=*j* the entry is 0.

Definition 2.6 Maximum Degree Matrix [3]

Maximum degree matrix is an adjacent matrix with entries m_{ij}; where for $i \neq j$ the entry is the maximum degree between the degrees of vertices *i* and *j* if they are adjacent, otherwise the entry is 0.

Definition 2.7 Energy of A Graph [1]

The energy of a graph is obtained by adding the modulus of eigenvalues of the adjacency matrix of the graph, and this is given by;

$$E(G) = \sum_{I=1}^{n} |\gamma_i|$$

Where γ_i are the respective eigenvalues of the adjacency matrix, with $i = 1,2,3\ldots,n$.

3. Matrices of the Zero-Divisor Graphs

Here we get the adjacency matrices, seidel matrices and
maximum degree matrices of the zero-divisor graph of the first seven rings of integers modulo n which are not integral domains; i.e $\mathbb{Z}_6$, $\mathbb{Z}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_{10}$, $\mathbb{Z}_{12}$ $\mathbb{Z}_{14}$ and $\mathbb{Z}_{15}$.
Note that by definition a zero divisor is a non-zero element. Table 1 gives the set of zero-divisors for the seven rings:

Table 1. The zero-divisors of the rings.

Modulus (n)	Ring ($\mathbb{Z}_n$)	Zero-Divisors
6	$\mathbb{Z}_6$	{2,3}
8	$\mathbb{Z}_8$	{2,4,6}
9	$\mathbb{Z}_9$	{3,6}
10	$\mathbb{Z}_{10}$	{2,4,5,6,8}
12	$\mathbb{Z}_{12}$	{2,3,4,6,8,9,10}
14	$\mathbb{Z}_{14}$	{2,4,6,7,8,10,12}
15	$\mathbb{Z}_{15}$	{3,5,6,9,10,12}

The following Lemmas show the adjacency matrices of graphs;

Lemma 3.1 For the ring $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$ the adjacency matrix of its zero-divisor graph $G^{\mathbb{Z}_6}$ is

$$A(G^{\mathbb{Z}_6}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Proof Given that $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$, Using Table 1 and Definition 2.3 the zero-divisor graph of $\mathbb{Z}_6$ is

Figure 1. Zero-Divisor graph of $\mathbb{Z}_6$.
By Definition 2.4 the adjacent vertices have entries 1 while the non adjacent vertices have 0. Hence, the result follows.

Lemma 3.2 The ring $\mathbb{Z}_8 = \{0,1,2,3,4,5,6,7\}$ has an adjacency matrix for its zero-divisor graph as;

$$A(G^{\mathbb{Z}_8}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Proof
From table 1 and definition 2.3, the ring $\mathbb{Z}_8 = \{0,1,2,3,4,5,6,7\}$ has the zero-divisor graph

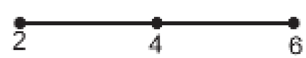

Figure 2. Zero-Divisor graph of $\mathbb{Z}_8$.
The result therefore follows using Definition 2.4.

Lemma 3.3
The ring $\mathbb{Z}_9 = \{0,1,2,3,4,5,6,7,8\}$ has an adjacency matrix for its zero-divisor graph as;

$$A(G^{\mathbb{Z}_9}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Proof
The ring $\mathbb{Z}_9 = \{0,1,2,3,4,5,6,7,8\}$ has the set of zero-divisors as {3,6,}(Table 1), hence its zero-divisor graph according to Definition 2.3 is

Figure 3. Zero-Divisor graph of $\mathbb{Z}_{9.}$

Applying Definition 2.4 yields the result.

Lemma 3.4 For the ring $\mathbb{Z}_{10} = \{0,1,2,3,4,5,6,7,8,9\}$ the adjacency matrix of its zero-divisor graph $G^{\mathbb{Z}_{10}}$ is

$$A(G^{\mathbb{Z}_{10}}) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Proof

Using Definition 2.3 and Table 1, the ring $\mathbb{Z}_{10}$ has its zero-divisor graph as

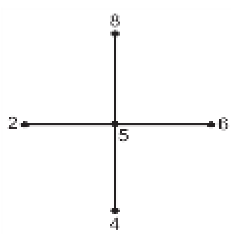

Figure 4. Zero-Divisor graph of $\mathbb{Z}_{10.}$

Applying Definition 2.4 completes the proof.

Lemma 3.5 The ring $\mathbb{Z}_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}$ has an adjacency matrix for its zero-divisor graph as;

$$A(G^{\mathbb{Z}_{12}}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Proof

From Table 1 and Definition 2.3, the zero-divisor graph of $\mathbb{Z}_{12}$ is

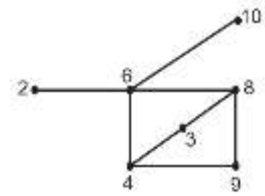

Figure 5. Zero-Divisor graph of $\mathbb{Z}_{12.}$

The result follows using Definition 2.4.

Lemma 3.6 The ring $\mathbb{Z}_{14} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13\}$ has an adjacency matrix for its zero-divisor graph as;

$$A(G^{\mathbb{Z}_{14}}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Proof

By using Table 1 and Definition 2.3 the zero-divisor graph of $\mathbb{Z}_{14}$ is

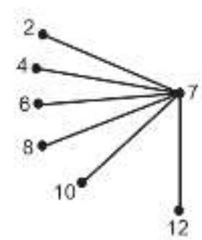

Figure 6. Zero-Divisor graph of $\mathbb{Z}_{14.}$

Definition 2.4 applies to give the result.

Lemma 3.7 The ring $\mathbb{Z}_{15} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$ has an adjacency matrix for its zero-divisor graph as;

$$A(G^{\mathbb{Z}_{15}}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Proof

In view of Definition 2.3 and Table 1, the ring $\mathbb{Z}_{15}$ has the zero-divisor graph

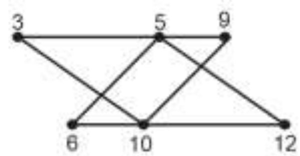

Figure 7. Zero-Divisor graph of $\mathbb{Z}_{15.}$

Using Definition 2.4 gives the result.

Lemmas 3.8 – 3.14 show the Seidel matrices of the seven rings

Lemma 3.8

The Seidel matrix for the zero-divisor graph of the ring $\mathbb{Z}_6$ is given by

$$S(G^{\mathbb{Z}_6}) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

Proof

Using Figure 1 and Definition 2.5 the result follows.

Lemma 3.9 The Seidel matrix for the zero-divisor graph of the ring $\mathbb{Z}_8$ is given by

$$S(G^{\mathbb{Z}_8}) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

Proof

Applying Definition 2.5 to Figure 2 yields the result.

Lemma 3.10

The Seidel matrix for the zero-divisor graph of the ring $\mathbb{Z}_9$ is given by

$$S(G^{\mathbb{Z}_9}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Proof

The result follows when Definition 2.5 is applied to Figure 3.

Lemma 3.11 The Seidel matrix for the zero-divisor graph of the ring $\mathbb{Z}_{10}$ is given by

$$S(G^{\mathbb{Z}_{10}}) = \begin{bmatrix} 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 \\ -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 \end{bmatrix}$$

Proof

The result follows by applying Definition 2.5 to Figure 4.

Lemma 3.12 The Seidel matrix for the zero-divisor graph of the ring $\mathbb{Z}_{12}$ is given by

$$S(G^{\mathbb{Z}_{12}}) = \begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

Proof

Using Definition 2.5 applied to Figure 5 gives the result.

Lemma 3.13 The Seidel matrix for the zero-divisor graph of the ring $\mathbb{Z}_{14}$ is given by

$$S(G^{\mathbb{Z}_{14}}) = \begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

Proof

The Seidel matrix is obtained by using Definition 2.5 on Figure 6.

Lemma 3.14 The Seidel matrix for the zero-divisor graph of the ring $\mathbb{Z}_{15}$ is given by

$$S(G^{\mathbb{Z}_{15}}) = \begin{bmatrix} 0 & -1 & 1 & 1 & -1 & 1 \\ -1 & 0 & -1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ -1 & 1 & -1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

Proof

The result follows by applying Definition 2.5 to Figure 7.

The maximum degree matrix for each of the seven graphs is shown in Lemmas 3.15-3.21

Lemma 3.15 The maximum degree matrix for the zero-divisor graph of the ring $\mathbb{Z}_6$ is given by

$$M(G^{\mathbb{Z}_6}) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Proof

Denote by δ(x) the degree of vertex x and Δ(*x*,*y*) the maximum degree between the degrees of vertices *x* and *y*. Then, from Figure 1, we have

δ (2) = 1; δ(3) = 2; δ(4) = 1

and applying Definition 2.6 gives

Δ(2,2) = Δ(3,3) = Δ(4,4) = 0;

Δ(2,3) = Δ(3,2) =2;

Δ(3,4) = Δ(4,3) =2

as the entries of the maximum degree matrix for the zero-divisor graph of $\mathbb{Z}_6$, hence the result.

Lemma 3.16 The maximum degree matrix for the zero-divisor graph of the ring $\mathbb{Z}_8$ is given by

$$M(G^{\mathbb{Z}_8}) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Proof

From figure 2, note here that

δ (2) = 1; δ(4) = 2; δ(6) = 1

and similarly, applying Definition 2.6 yields

Δ(2,2) = Δ(4,4) = Δ(6,6)= 0;

Δ(2,6) = Δ(6,2) =2;

Δ(4,6) = Δ(6,4) =2

as the required result.

Lemma 3.17 The maximum degree matrix for the zero-divisor graph of the ring $\mathbb{Z}_9$ is given by

$$M(G^{\mathbb{Z}_9}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Proof

Applying Figure 3 and Definition 2.6, we have

. δ(3) = δ(6) = 1 and

Δ(3,3) = Δ(6,6) = 0;

Δ(3,6) = Δ(6,3) =1 proving the result.

Lemma 3.18 The maximum degree matrix for the zero-divisor graph of the ring $\mathbb{Z}_{10}$ is given by

$$M(G^{\mathbb{Z}_{10}}) = \begin{bmatrix} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 4 & 4 & 0 & 4 & 4 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{bmatrix}$$

Proof

Using Figure 4 and Definition 2.6,
δ(2) = δ(4) = δ(6) = δ(8) = 1; δ(5) = 4
while
Δ(2,5) = Δ(5,2) = Δ(4,5) = Δ(5,4) = Δ(5,6) = Δ(6,5) = Δ(5,8) = Δ(5,8) = Δ(8,5) = 4
and all other entries are zero, hence result.

Lemma 3.19 The maximum degree matrix for the zero-divisor graph of the ring $\mathbb{Z}_{12}$ is given by

$$M(G^{\mathbb{Z}_{12}}) = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 3 & 0 \\ 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 3 & 0 & 4 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \end{bmatrix}$$

Proof

Applying Definition 2.6 to figure 5, we have
δ(2) = δ(10) = 1; δ(3) = δ(9) = 2; δ(4) = δ(8) = 3; δ(6) = 4 and
Δ(2,6) = Δ(6,2) = Δ(4,6) = Δ(6,4) = Δ(6,8) = Δ(8,6) = Δ(6,10) = Δ(10,6) = 4;
Δ(3,4) = Δ(4,3) = Δ(3,8) = Δ(8,3) = Δ(4,9) = Δ(9,4) = Δ(8,9) = Δ(9,8)= 3
while all other entries are zero, establishing the result.

Lemma 3.20 The maximum degree matrix for the zero-divisor graph of the ring $\mathbb{Z}_{14}$ is given by

$$M(G^{\mathbb{Z}_{14}}) = \begin{bmatrix} 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 6 & 6 & 6 & 0 & 6 & 6 & 6 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \end{bmatrix}$$

Proof

Using Figure 6 and Definition 2.6,
δ(2) = δ(4) = δ(6) = δ(8) = δ(10) = δ(12) = 1; δ(7) = 6 and
Δ(2,7) = Δ(7,2) = Δ(4,7) = Δ(7,4) = Δ(6,7) = Δ(7,6) = Δ(7,8) = Δ(8,7) = Δ(7,10) = Δ(10,7) = Δ(7,12) = Δ(12,7) = 6
while all other entries are zero, proving the result.

Lemma 3.21 The maximum degree matrix for the zero-divisor graph of the ring $\mathbb{Z}_{15}$ is given by

$$M(G^{\mathbb{Z}_{15}}) = \begin{bmatrix} 0 & 4 & 0 & 0 & 4 & 0 \\ 4 & 0 & 4 & 4 & 0 & 4 \\ 0 & 4 & 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 4 & 0 \\ 4 & 0 & 4 & 4 & 0 & 4 \\ 0 & 4 & 0 & 0 & 4 & 0 \end{bmatrix}$$

Proof

Applying Definition 2.6 to figure 7 gives
δ(3) = δ(6) = δ(9) = δ(12) = 2; δ(5) = δ(10) = 4 and
Δ(3,5) = Δ(5,3) = Δ(3,10) = Δ(10,3) = Δ(5,6) = Δ(6,5) = Δ(6,10) = Δ(10,6) = Δ(5,9) = Δ(9,5) = Δ(9,10) = Δ(10,9) = Δ(5,12) = Δ(12,5) = Δ(10,12) = Δ(12,10)= 4

while all other entries are zero, establishing the result.

4. Energy of The Graphs

This section is where the three types of energies of the zero-divisor graphs of the rings are computed.

We start with adjacency energy which is shown in Theorems 4.1-4.7

Theorem 4.1 The adjacency energy of the zero-divisor of $\mathbb{Z}_6$; $E(G^{\mathbb{Z}_6}) = 2\sqrt{2}$.

Proof

From Lemma 3.1 the adjacency matrix of $G^{\mathbb{Z}_6}$ has the following eigenvalues, $\gamma_1 = 0$, $\gamma_2 = \sqrt{2}$, $\gamma_3 = -\sqrt{2}$ and n =3.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_6}) = \sum_{I=1}^{3} |\gamma_i| = |\sqrt{2}| + |-\sqrt{2}| = 2\sqrt{2}$

Theorem 4.2 The adjacency energy of the zero-divisor of $\mathbb{Z}_8$; $E(G^{\mathbb{Z}_8}) = 2\sqrt{2}$.

Proof

From Lemma 3.2 the adjacency matrix of $G^{\mathbb{Z}_8}$ has the following eigenvalues, $\gamma_1 = 0$, $\gamma_2 = \sqrt{2}$, $\gamma_3 = -\sqrt{2}$ and n =3. Applying definition 2.7 gives the energy as $E(G^{\mathbb{Z}_8}) = \sum_{I=1}^{3} |\gamma_i| = |\sqrt{2}| + |-\sqrt{2}| = 2\sqrt{2}$

Theorem 4.3 Adjacency energy of $G^{\mathbb{Z}_9}$; $E(G^{\mathbb{Z}_9}) = 2$.

Proof

From Lemma 3.3 the adjacency matrix of $G^{\mathbb{Z}_9}$ has the following eigenvalues, $\gamma_1 = 1$, $\gamma_2 = -1$, and n =2.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_9}) = \sum_{I=1}^{2} |\gamma_i| = |1| + |-1| = 2$

Theorem 4.4

The adjacency energy of the zero-divisor of $\mathbb{Z}_{10}$; $E(G^{\mathbb{Z}_{10}}) = 4$.

Proof

From Lemma 3.4 the adjacency matrix of $G^{\mathbb{Z}_{10}}$ has the following eigenvalues, $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$, $\gamma_4 = 2$, $\gamma_5 = -2$ and $n = 5$.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_{10}}) = \sum_{I=1}^{5} |\gamma_i| = |2| + |-2| = 4$

Theorem 4.5 The adjacency energy of the zero-divisor of $\mathbb{Z}_{12}$;
$E(G^{\mathbb{Z}_{12}}) = 2(\sqrt{4+2\sqrt{2}} + \sqrt{4-2\sqrt{2}})$.

Proof

From Lemma 3.5 the adjacency matrix of $G^{\mathbb{Z}_{12}}$ has the following eigenvalues , $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $\gamma_4 = -\sqrt{4+2\sqrt{2}}$, $\gamma_5 = \sqrt{4+2\sqrt{2}}$, $\gamma_6 = -\sqrt{4-2\sqrt{2}}$, $\gamma_7 = \sqrt{4-2\sqrt{2}}$ and n=7.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_{12}}) = \sum_{I=1}^{7} |\gamma_i| = \left|-\sqrt{4+2\sqrt{2}}\right| + \left|\sqrt{4+2\sqrt{2}}\right| \left|-\sqrt{4-2\sqrt{2}}\right| + \left|\sqrt{4-2\sqrt{2}}\right| = 2(\sqrt{4+2\sqrt{2}} + \sqrt{4-2\sqrt{2}})$.

Theorem 4.6 The adjacency energy of the zero-divisor of $\mathbb{Z}_{14}$; $E(G^{\mathbb{Z}_{14}}) = 2\sqrt{6}$ . .

Proof

From Lemma 3.6 the adjacency matrix of $G^{\mathbb{Z}_{14}}$ has the following eigenvalues , $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$, $\gamma_4 = 0$, $\gamma_5 = 0$, $\gamma_6 = -\sqrt{6}$, $\gamma_7 = \sqrt{6}$ and n=7.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_{14}}) = \sum_{I=1}^{7} |\gamma_i| = |-\sqrt{6}| + |\sqrt{6}| = 2\sqrt{6}$

Theorem 4.7 The adjacency energy of the zero-divisor of $\mathbb{Z}_{15}$; $E(G^{\mathbb{Z}_{15}}) = 4\sqrt{2}$.

Proof

From Lemma 3.7 the adjacency matrix of $G^{\mathbb{Z}_{15}}$ has the following eigenvalues , $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$, $\gamma_4 = 0$, $\gamma_5 = -2\sqrt{2}$, $\gamma_6 = 2\sqrt{2}$, and n =6.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_{15}}) = \sum_{I=1}^{6} |\gamma_i| = |-2\sqrt{2}| + |2\sqrt{2}| = 4\sqrt{2}$

Theorems 4.8 to 4.14 shows the Seidel energies of the zero-divisor graphs of the integers modulo *n*.

Theorem 4.8 The Seidel energy of zero-divisor graph of $\mathbb{Z}_6$; $E(G^{\mathbb{Z}_6}) = 4$.

Proof

From Lemma 3.8 the Seidel matrix of $G^{\mathbb{Z}_6}$ has the following eigenvalues, $\gamma_1 = 0$, $\gamma_2 = 2$, $\gamma_3 = -2$ and *n* =3.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_6}) = \sum_{I=1}^{3} |\gamma_i| = |2| + |-2| = 4$

Theorem 4.9 The Seidel energy of zero-divisor graph of $\mathbb{Z}_8$; $E(G^{\mathbb{Z}_8}) = 4$.

Proof

From Lemma 3.9 the Seidel matrix of $G^{\mathbb{Z}_8}$ has the following eigenvalues, $\gamma_1 = 0$, $\gamma_2 = 2$, $\gamma_3 = -2$ and *n* =3.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_8}) = \sum_{I=1}^{3} |\gamma_i| = |2| + |-2| = 4$

Theorem 4.10 The Seidel energy of zero-divisor graph of $\mathbb{Z}_9$; $E(G^{\mathbb{Z}_6}) = 2$.

Proof

From Lemma 3.10 the Seidel matrix of $G^{\mathbb{Z}_9}$ has the following eigenvalues, $\gamma_1 = \gamma_2 = 1$ and *n* =2.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_9}) = \sum_{I=1}^{2} |\gamma_i| = |1| + |-1| = 2$

Theorem 4.11 The Seidel energy; $E(G^{\mathbb{Z}_{10}}) = 8$; where $G^{\mathbb{Z}_{10}}$ is the zero-divisor graph of $\mathbb{Z}_{10}$.

Proof

From Lemma 3.11 the Seidel matrix of $G^{\mathbb{Z}_{10}}$ has the following eigenvalues , $\gamma_1 = 4$, $\gamma_2 = -1$, $\gamma_3 = -1$, $\gamma_4 = -1$, $\gamma_5 = -1$ and *n* =5. Applying Definition 2.7 gives the energy as

$E(G^{\mathbb{Z}_{10}}) = \sum_{I=1}^{5} |\gamma_i| = |4| + 4|-1| = 4 + 4(1) = 8$

Theorem 4.12 The zero-divisor graph of $\mathbb{Z}_{12}$ has a Seidel energy ; $E(G^{\mathbb{Z}_{12}}) = 7 + \sqrt{57}$.

Proof

From Lemma 3.12 the Seidel matrix of $G^{\mathbb{Z}_{12}}$ has the following eigenvalues , $\gamma_1 = 3$, $\gamma_2 = \frac{1}{2} + \frac{1}{2}\sqrt{57}$, $\gamma_3 = \frac{1}{2} - \frac{1}{2}\sqrt{57}$, $\gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = -1$ and *n* =7.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_{12}}) = \sum_{I=1}^{7} |\gamma_i| = |3| + \left|\frac{1}{2} + \frac{1}{2}\sqrt{57}\right| + \left|\frac{1}{2} - \frac{1}{2}\sqrt{57}\right| +4|-1| = 7 + \sqrt{57}$

Theorem 4.13 The Seidel energy of $G^{\mathbb{Z}_{14}}$; $E(G^{\mathbb{Z}_{14}}) = 12$, where $G^{\mathbb{Z}_{14}}$ is the zero-divisor graph of $\mathbb{Z}_{14}$

Proof

From Lemma 3.13 the Seidel matrix of $G^{\mathbb{Z}_{14}}$ has the following eigenvalues , $\gamma_1 = 6$, , $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = -1$ and *n*=7.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_{14}}) = \sum_{I=1}^{7} |\gamma_i| = |6| + 6|-1| = 12$

Theorem 4.14 The Seidel energy of the zero-divisor graph of $\mathbb{Z}_{15}$, $E(G^{\mathbb{Z}_{15}}) = 10$.

Proof

From Lemma 3.14 the Seidel matrix of $G^{\mathbb{Z}_{15}}$ has the following eigenvalues ,$\gamma_1 = 5$, $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = -1$ and *n* =6.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_{15}}) = \sum_{I=1}^{6} |\gamma_i| = |5| + 5|-1| = 10$

The maximum degree energy of the zero-divisor graphs is shown in Theorems 4.15 to 4.21

Theorem 4.15 The maximum degree energy of $G^{\mathbb{Z}_6}$; $E(G^{\mathbb{Z}_6}) = 4\sqrt{2}$.

Proof

From Lemma 3.15 the maximum degree matrix of $G^{\mathbb{Z}_6}$ has the following eigenvalues, $\gamma_1 = 0$, $\gamma_2 = 2\sqrt{2}$, $\gamma_3 = -2\sqrt{2}$. and *n* =3.

Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_6}) = \sum_{I=1}^{3} |\gamma_i| = |2\sqrt{2}.| + |-2\sqrt{2}| = 4\sqrt{2}$.

Theorem 4.16 The maximum degree energy of $G^{\mathbb{Z}_8}$; $E(G^{\mathbb{Z}_8}) = 4\sqrt{2}$.

Proof

From Lemma 3.16 the maximum degree matrix of $G^{\mathbb{Z}_8}$ has the following eigenvalues, $\gamma_1 = 0$, $\gamma_2 = 2\sqrt{2}$, $\gamma_3 = -2\sqrt{2}$ and n =3.
Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_8}) = \sum_{I=1}^{3} |\gamma_i| = |2\sqrt{2}| + |-2\sqrt{2} = 4\sqrt{2}$

Theorem 4.17 The maximum degree energy of $G^{\mathbb{Z}_9}$; $E(G^{\mathbb{Z}_9}) = 2$.

Proof

From Lemma 3.17 the maximum degree matrix of $G^{\mathbb{Z}_9}$ has the following eigenvalues, $\gamma_1 = 1$, $\gamma_2 =$ -1 and n =2.
Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_9}) = \sum_{I=1}^{2} |\gamma_i| = E(G^{\mathbb{Z}_9}) = \sum_{I=1}^{2} |\gamma_i| = |1| + |-1| = 2$

Theorem 4.18 The maximum degree energy of $G^{\mathbb{Z}_{10}}$; $E(G^{\mathbb{Z}_{10}}) = 16$.

Proof From Lemma 3.18 the maximum degree matrix of $G^{\mathbb{Z}_{10}}$ has the following eigenvalues , $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $\gamma_4 = 8$, $\gamma_5 = -8$ and n =5.
Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_{10}}) = \sum_{I=1}^{5} |\gamma_i| = |8| + |-8| = 16$

Theorem 4.19 The zero-divisor graph of $\mathbb{Z}_{12}$ has a maximum degree energy ; $E(G^{\mathbb{Z}_{12}}) = 2(\sqrt{50 + 2\sqrt{337}} + \sqrt{50 - 2\sqrt{337}})$.

Proof From Lemma 3.19 the maximum degree matrix of $G^{\mathbb{Z}_{12}}$ has the following eigenvalues , $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $\gamma_4 = -\sqrt{50 + 2\sqrt{337}}$, $\gamma_5 = \sqrt{50 + 2\sqrt{337}}$, $\gamma_5 = -\sqrt{50 - 2\sqrt{337}}$, $\gamma_5 = \sqrt{50 - 2\sqrt{337}}$ and n =7.
Applying Definition 2.7 gives the energy as $E(G^{\mathbb{Z}_{12}}) = \sum_{I=1}^{7} |\gamma_i| = \left|-\sqrt{50 + 2\sqrt{337}}\right| + \left|\sqrt{50 + 2\sqrt{337}}\right| + \left|-\sqrt{50 - 2\sqrt{337}}\right| + \left|\sqrt{50 - 2\sqrt{337}}\right| = 2(\sqrt{50 + 2\sqrt{337}} + \sqrt{50 - 2\sqrt{337}})$

Theorem 4.20 The maximum degree energy of the zero-divisor graph of $\mathbb{Z}_{14}$; $E(G^{\mathbb{Z}_{14}}) = 12\sqrt{6}$.

Proof From Lemma 3.20 the maximum degree matrix of $G^{\mathbb{Z}_{14}}$ has the following eigenvalues , $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 0$, $\gamma_6 = 6\sqrt{6}$ $\gamma_7 = -6\sqrt{6}$ and n=7.
Applying Definition 2.7 gives the energy as E $(G^{\mathbb{Z}_{14}}) = \sum_{I=1}^{7} |\gamma_i| = |6\sqrt{6}| + |-6\sqrt{6}| = 12\sqrt{6}$

Theorem 4.21 The maximum degree energy of $G^{\mathbb{Z}_{15}}$; $E(G^{\mathbb{Z}_{15}}) = 16\sqrt{2}$.

Proof From Lemma 3.21 the maximum degree matrix of $G^{\mathbb{Z}_{15}}$ has the following eigenvalues , $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$, $\gamma_5 = 8\sqrt{2}$, $\gamma_5 = -8\sqrt{2}$ and n =6. Applying Definition 2.7 gives the energy as E$(G^{\mathbb{Z}_{15}}) = \sum_{I=1}^{7} |\gamma_i| = |8\sqrt{2}| + |-8\sqrt{2}| = 16\sqrt{2}$.

RESULTS

Table 2. Energies of Zero-Divisor Graphs of the integers modulo n

S/N	Rings	Adjacency Energy	Seidel Energy	Maximum Degree Energy
1	$\mathbb{Z}_6$	$2\sqrt{2}$	4	$4\sqrt{2}$
2	$\mathbb{Z}_8$	$2\sqrt{2}$	4	$4\sqrt{2}$
3	$\mathbb{Z}_9$	2	2	2
4	$\mathbb{Z}_{10}$	4	8	16
5	$\mathbb{Z}_{12}$	$2(\sqrt{4+2\sqrt{2}}+\sqrt{4-2\sqrt{2}})$	$7+\sqrt{57}$	$2(\sqrt{50+2\sqrt{337}}+\sqrt{50-2\sqrt{337}})$
6	$\mathbb{Z}_{14}$	$2\sqrt{6}$	12	$12\sqrt{6}$
7	$\mathbb{Z}_{15}$	$4\sqrt{2}$	10	$16\sqrt{2}$

CONCLUSION

This paper shows the adjacency matrix, seidel matrix and the maximum degree matrix of the zero-divisor graph of the first seven rings of integers modulo *n;* $\mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_{10}, \mathbb{Z}_{12}, \mathbb{Z}_{14}$ *and* $\mathbb{Z}_{15}$ which are not integral domains. Each of the matrices generated above is a square matrix ($n \times n$) with n as the cardinality of the set of zero-divisors of the ring. All the matrices are symmetric, this is due to the fact that the definition of zero divisor also defines a symmetric relation on the ring. Energies for the zero-divisor graphs of the rings are computed from the three types of matrices generated. It has already been established that if the energy of a graph is rational then it must be an even integer [6], the result of computations done here show all the rational energies are even in conformity with that finding.

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