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*Full Length Research Paper*

# On the matched pairs sign test using bivariate ranked set sampling: an application to environmental issues

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**The matched pairs sign test using bivariate ranked set sampling (BVRSS) is introduced and investigated. We show that this test is asymptotically more efficient than its counterpart sign test based on a bivariate simple random sample (BVSRS). The asymptotic null distribution and the efficiency of the test are derived. The Pitman asymptotic relative efficiency is used to compare the asymptotic performance of the matched pairs sign test using BVRSS versus using BVSRS. For small sample sizes, the bootstrap method is used to estimate P-values. Numerical comparisons are used to gain insight about the efficiency of the BVRSS sign test compared to the BVSRS sign test. Our numerical and theoretical results indicate that using BVRSS for the matched pairs sign test is substantially more efficient than using BVSRS. Illustration using palm trees data from sultanate of Oman is provided.**

**Key words:** Bootstrap method, bivariate ranked set sample, power of the test, P-value of the test, Pitman's relative efficiency, sign test.

## INTRODUCTION

In many environmental, agricultural and epidemiological studies, some times, it is necessary to use matched pairs when you comparing two cohorts of a study subjects, based on some lurking and confounding factors, in order to control the effects of those factors on the findings. "Matching as a technique for the control of confounding has great intuitive appeal and has been widely used over the years. Unlike randomization and restriction, which used to control confounding in the design stage of a study, matching is a strategy that must include elements of both design and analysis" (Hennekens and Buring 1987). For example, a two-year prospective study could be conducted to compare the effect of adding calcium supplements in manufactured milk for newborns to that in natural mother milk on baby's heights. A large group of new born identical twins could be considered: in one of the twins, milk with added calcium is provided; for the other twin, mothers are encouraged with some incentives to breast feed their babies for two years. After the two years, a matched pair's random sample could be drawn from the twins' population of babies; their heights would be measured along with other factors of interest. These types of studies produce data consisting of observations in a bivariate random sample: $\{(X_i, Y_i), i=1, 2, ..., n\}$, where there are *n* pairs of observations. Within each pair $(X_i, Y_i)$ a comparison is made, and the pair is classified as "+" if $X_i < Y_i$, "−" if $X_i > Y_i$ or "0" if $X_i = Y_i$. Here the measurement scale needs only to be ordinal. Other needed assumptions are (1) The bivariate variables $(X_i, Y_i)$, *i=1, 2,…, n,* are mutually independent. (2) The

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pairs $(X_i, Y_i)$ are internally consistent, in that if *P(+)>P(−)* for one pair $(X_i, Y_i)$, then *P(+)>P*(−) for all pairs. The same is true for P(+)<P(−) and P(+)=P(−).

The types of null hypotheses that can be tested using the matched pairs sign test are:

(1) $H_o: P(+) = P(-) = \frac{1}{2}$.

(2) $H_o: E(X_i) = E(Y_i)$, for all $i$, which is interpreted as $X_i$ and $Y_i$ have the same location parameter.

(3) $H_o:$ The median of $X_i$ equals the median of $Y_i$ for all $i$ (Conover 1980.)

The matched pairs sign test statistic, which denoted by $T_{BVSRS}$, for testing the above hypotheses, equals the number of "+" pairs, that is

$$T_{BVSRS} = \sum_{i=1}^{n} I(X_i < Y_i) \quad (1.1)$$

where $I(X_i < Y_i) = \begin{cases} 1 & \text{if } X_i < Y_i \\ 0 & \text{otherwise}. \end{cases}$

First, discard all tied pairs and let *n* equal the number of the remaining pairs. Depending on whether the alternative hypotheses is one-tailed or two-tailed, and if $n \leq 20$, then use the Binomial distribution with the values *n* and *p*=1/2 for finding the critical region of approximately size $\alpha$. For *n* larger than 20 and under the null hypothesis, $T \sim N(\frac{n}{2}, \frac{n}{4})$. Therefore, the critical region can be defined based on the normal distribution. It had been argued that $T_{BVSRS}$ is unbiased and consistent test statistic when testing $H_o: P(+) = P(-)$. However, for testing

$H_o: E(X_i) = E(Y_i)$, for all $i$ and

$H_o:$ The median of $X_i$ equals the median of $Y_i$ for all $i$

$T_{BVSRS}$ is neither unbiased nor consistent (Conover 1980.)

In most statistical applications the data used is assumed to consist of a simple random sample (SRS). Recently, it becomes obvious in many studies such as, agricultural, environmental and epidemiological studies; that quantification of sampling units with respect to the variable of interest is costly as compared with the physical acquisition of the unit. Cost savings of quantifying sampling units can be achieved by using ranked set sampling (RSS) methods which was introduced first by McIntyre (1952) without any mathematical prove, to estimate the population mean, and later called RSS by Halls and Dell (1966).

The RSS procedure can be described as follows: Randomly sample a group of sampling units from the target population. Then, randomly partition the group into disjoint subsets each having a pre-assigned size r. In most practical situations, the size *r* will be 2, 3 or 4. Rank the elements in each subset by a suitable method of ranking such as prior information, visual inspection or by the subject-matter experimenter himself etc. Then the *i-th* order statistic from the *i-th* subset, $X_{i(i)}$, $i$ = 1, …, *r,* will be quantified (actual measurement). Therefore, $X_{1(1)}, X_{2(2)}, \ldots, X_{r(r)}$ constitutes the RSS. This represents one cycle. The whole procedure can be repeated *m*-times as needed, to get a RSS of size *n = mr* for the theoretical aspects of RSS (Takahasi and Wakimoto, 1968; Dell and Clutter, 1972).

Stokes and Sager (1988) used RSS to estimate the cumulative distribution functions (cdf) using the empirical distribution function (edf) based on RSS (*F**). The procedure based on ranked set samples quantiles with applications was investigated by Chen (2000), Samawi (2001), Samawi and Al-Saleh (2004). An optimal ranked set sample scheme (ORSS) for inference on population quantiles was suggested by Chen (2001). Other authors have used the RSS sampling method to improve parametric and non-parametric statistical inference. For non-parametric methods, RSS was considered by Bohn and Wolfe (1992, 1994), Kvam and Samaniego (1994) and Hettmansperger (1995). Koti and Babu (1996) showed that the RSS sign test it provides a more powerful test than the SRS sign test. Barabesi (1998) provided a simpler and faster method for computing the exact distribution of the RSS sign test.

The optimality of the RSS sign test has been established by several researchers in the literature via Pitman asymptotic efficacy. It was shown that the median ranked set sample (MRSS) is the best among all possible sampling schemes in the ranked set sampling environment for the sign test procedure; for example see Öztürk (1999)and Öztürk and Wolfe (2000). However, to our knowledge, the optimality of the MRSS for the sign test has only been shown asymptotically. Samawi and

Abu- Dayyeh (2002) investigated the exact power and distribution function for finite sample sizes of the MRSS sign test. For more about univariate RSS and its variations, Kaur et al. (1995) and Patil et al. (1999).

Estimation of bivariate characteristics using bivariate ranked set sampling (BVRSS) was introduced by Al-Saleh and Zheng (2002). They indicated that this BVRSS procedure can be easily extended to multivariate RSS. Based on their description, a BVRSS can be obtained as follows:

Suppose (*X, Y*) is a bivariate random vector with the joint probability density function (p.d.f.) *f*(*x, y*).

1. A random sample of size $r^4$ is identified from the population and randomly allocated into $r^2$ pools each of size $r^2$, where each pool is a square matrix with *r* rows and *r* columns.

2. In the first pool, identify the minimum value by judgment w.r.t. the first characteristic *X*, for each of the r rows.

3. For the *r* minima obtained in Step 2, choose the pair that corresponds to the minimum value of the second characteristic *Y*, identified by judgment, for actual quantification. This pair, which resembles the label (1, 1), is the first element of the BVRSS sample.

4. Repeat Steps 2 and 3 for the second pool, but in step 3, the pair that corresponds to the second minimum value *w.r.t.* the second characteristic, *Y*, is chosen for actual quantification. This pair resembles the label (1, 2).

5. The process continues until the label (*r, r*) is resembled from the $r^2$-*th* (last) pool.

This will produce a BVRSS of size $r^2$. The procedure can be repeated *m* times to obtain a sample of size $n = mr^2$. Although, the BVRSS procedure can be extended to multivariate RSS (MVRSS). Our proceeding discussion will only be applied to the bivariate case (BVRSS) in order to simplify the presentation of the method. Moreover, Samawi et al. (2006) utilized BVRSS in sign test for one-sample bivariate location model. They show that BVRSS is more powerful procedure than BVSRS for this sign test.

In this paper we introduce the matched pairs sign test using BVRSS. Numerical comparisons between the performance of the BVRSS sign test and the performance of the BVSRS sign test via Pitman's asymptotic efficacy and asymptotic power are investigated. The exact distribution and the asymptotic null distribution and power of the BVRSS sign test are derived. It will be shown that BVRSS substantially improves the efficiency and the power of the sign test in the case of a matched pairs sample. We also introduce a bootstrap method for finding the P-value of the matched pairs test for small sample sizes. Illustration using real data is provided.

## Characteristics of BVRSS and some useful results

Let

$$[(X_{ijk}^z, Y_{ijk}^z),\ i=1,2,...,r,\ j=1,2,...,r,\ k=1,2,...,m\ \&z=1,...,r^2]$$

be $mr^4$ *i.i.d* ordered pairs from a bivariate probability density function, say f(x, y); (x, y) $\in \mathbb{R}^2$. Following the Al-Saleh and Zheng (2002) definition of BVRSS let

$$\left[(X_{[i](j)k}^z, Y_{(i)[j]k}^z),\ i=1,2,...,r;\ j=1,2,...,r;\ z=(j-1)r+i\ and\ k=1,2,...,m\right]$$

denotes such a sample from f(x, y). Let $f_{X_{[i](j)}^z, Y_{(i)[j]}^z}(x, y)$ be the joint p.d.f. of

$(X_{[i](j)k}^z, Y_{(i)[j]k}^z)$, k=1, 2, …, m. Then as in Al-Saleh and Zheng (2002),

$$f_{X_{[i](j)}^z, Y_{(i)[j]}^z}(x, y) = f_{Y_{(i)[j]}}(y)\frac{f_{X_{(j)}}(x) f_{Y|X}(y \mid x)}{f_{Y_{[j]}}(y)}$$

(1.2)

where $f_{X_{(j)}}$ is the density of the $j^{th}$ order statistic for a SRS of size *r* with marginal density of $f_X$ and $f_{Y_{[j]}}(y)$ be the density of the corresponding $Y$ − value given by

$$f_{Y_{[j]}}(y) = \int_{-\infty}^{\infty} f_{X_{(j)}}(x) f_{Y|X}(y \mid x)dx\ ,$$

while $f_{Y_{(i)[j]}}(y)$ is the density of the $i^{th}$ order statistic of an i.i.d.

sample of size *r* from $f_{Y_{[j]}}(y)$, i.e.

$$f_{Y_{(i)[j]}}(y) = c(F_{Y_{[j]}}(y))^{i-1}(1-F_{Y_{[j]}}(y))^{r-i} f_{Y_{[j]}}(y)$$

where $F_{Y_{[j]}}(y) = \int_{-\infty}^{y} (\int_{-\infty}^{\infty} f_{X_{(j)}}(x) f_{Y|X}(w \mid x)dx)dw$ and

$$c = \frac{r!}{(i-1)!\,(r-i)!}.$$

Putting these together, (1.2) can be written for any z ( the z notation will be dropped for simplicity) as

$$f_{X_{[i](j)}, Y_{(i)[j]}}(x,y) = c_1(F_{Y_{[j]}}(y))^{i-1}(1-F_{Y_{[j]}}(y))^{r-i}(F_X(x))^{j-1}(1-F_X(x))^{r-j} f(x,y)$$
(1.3)

where

$$c_1 = \frac{r!}{(i-1)!\,(r-i)!} \frac{r!}{(j-1)!\,(r-j)!}.$$

Again from Al-Saleh and Zheng (2002) we have the following results:

(1) $$\frac{1}{r^2}\sum_{j=1}^{r}\sum_{i=1}^{r} f_{X_{[i](j)}, Y_{(i)[j]}}(x, y) = f(x, y), \qquad (1.4)$$

(2) $$\frac{1}{r^2}\sum_{j=1}^{r}\sum_{i=1}^{r} f_{X_{[i](j)}}(x) = f_X(x), \qquad (1.5)$$

(3) $$\frac{1}{r^2}\sum_{j=1}^{r}\sum_{i=1}^{r} f_{Y_{(i)[j]}}(y) = f_Y(y). \qquad (1.6)$$

**Matched pairs sign test using BVRSS**

Using the BVRSS sample,

$$\left[ (X_{[i](j)k}, Y_{(i)[j]k}),\ i=1, 2, ..., r;\ j=1, 2, ..., r;\ \text{and } k=1, 2, ..., m \right]$$
, drawn from a

population with *p.d.f f(x,y),* the BVRSS sign test statistic can be defined as:

$T_{BVRSS}$ = the number of "+"; or; $= \#(X_{[i](j)k} < Y_{(i)[j]k})$ for all $i, j$, that is

$$T_{BVRSS} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{m} I(X_{[i](j)k} < Y_{(i)[j]k}) = \sum_{i=1}^{r}\sum_{j=1}^{r} T_{ij} \qquad (2.1)$$

where $T_{ij} = \sum_{k=1}^{m} I(X_{[i](j)k} < Y_{(i)[j]k})$. Clearly, $T_{ij}$, $i, j = 1, 2, ..., r$ are stochastically independent and each $T_{ij}$ has a binomial distribution with parameters $m$ and $p_{ij} = P(X_{[i](j)k} < Y_{(i)[j]k})$. Thus the exact distribution of $T_{BVRSS}$ is given by

$$P(T_{BVRSS} = t) = \sum_{L_{xy}}\prod_{i=1}^{r}\prod_{j=1}^{r}\binom{m}{l_{ij}} p_{ij}^{l_{ij}}(1-p_{ij})^{m-l_{ij}} \qquad (2.2)$$

for $t$=0, 1, 2, …, $mr^2$ ; where

$$L_{xy} = \{(l_{ij} : i, j = 1, 2, ..., r) : \sum_{i=1}^{r}\sum_{j=1}^{r} l_{ij} = t;\ 0 \le l_{ij} \le m,\ i, j = 1, 2, ..., r.$$

Unfortunately, the exact distribution in (2.2) depends on the given underlying bivariate distribution function even under the null hypothesis. Thus, finding the exact critical value and the P-value of the test requires knowledge of the underlying distribution function. Therefore, we will introduce a simple bootstrap algorithm for finding the P-value in the case of sample size *n*<20. For larger *n,* an asymptotic test procedure is introduced.

**Theorem 2.1:** Assuming no tied pairs

$(X_{[i](j)k} = Y_{(i)[j]k})$ for all $i, j$ and $k$, under the null hypothesis

$H_o:\ P(+) = P(-) = \frac{1}{2}$ and for fixed *r* and large *m* then

$$Z_{BVRSS} = \frac{\left(T_{BVRSS} - \frac{n}{2}\right)}{\sqrt{r^2 m(\frac{1}{2} - \frac{1}{r^2}\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij}^2)}}$$ has approximately $N(0, 1)$ ’

where $n=mr^2$,

$$V_o = r^2 m\left(\frac{1}{2}-\frac{1}{r^2}\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij}^2\right) \text{ and } p_{ij}=P(X_{[i](j)k} < Y_{(i)[j]k}).$$

**Proof**: Discard all tied pairs and let $n$ equal the number of pairs that are not ties. Then

We can write $T_{BVRSS}$ as a sum of i.i.d variables as follows: $T_{BVRSS}=\sum_{k=1}^{m} T_{ij}$, where $T_{ij}=\sum_{i=1}^{r}\sum_{j=1}^{r} I(X_{[i](j)} < Y_{(i)[j]})$. Note that

$$E(T_{ij})=\sum_{i=1}^{r}\sum_{j=1}^{r} E[I(X_{[i](j)} < Y_{(i)[j]})] = \iint I(X_{[i](j)} < Y_{(i)[j]})\sum_{i=1}^{r}\sum_{j=1}^{r} f_{X_{[i](j)},\,Y_{(i)[j]}}(x,\ y)dxdy$$

And by using (1.4),

$$E(T_{ij}) = r^2\iint I(x<y)f\ (x,\ y)dxdy = \frac{r^2}{2} < \infty.\ (r \text{ is fixed}).$$

Also,

$$Var(T_{ij})=\sum_{i=1}^{r}\sum_{j=1}^{r} Var[I(X_{[i](j)} < Y_{(i)[j]})]=\sum_{i=1}^{r}\sum_{j=1}^{r}\{E[I^2(X_{[i](j)} < Y_{(i)[j]})]-(E[I(X_{[i](j)} < Y_{(i)[j]})])^2\}$$

Thus, by using (1.4) again,

$$Var(T_{ij})=r^2\left(\frac{1}{2}-\frac{1}{r^2}\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij}^2\right)<\infty (r \text{ is fixed}).$$

Therefore, by using The Central Limit Theorem,

$$Z_{BVRSS}=\frac{\left(T_{BVRSS}-\frac{n}{2}\right)}{\sqrt{r^2 m\left(\frac{1}{2}-\frac{1}{r^2}\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij}^2\right)}} \text{ has approximately } N(0,\ 1)$$

A consistent estimator for $V$ is given by

$$\hat{V}_o = Var(T_{BVRSS}) = r^2 m\left(\frac{1}{2}-\frac{1}{r^2}\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{p}_{ij}^2\right), \text{ where}$$

$$\hat{p}_{ij}=\frac{\sum_{k=1}^{m} I(X_{[i](j)k} < Y_{(i)[j]k})}{m}.$$

Depending on the alternative hypothesis whether it is one-tailed or two-tailed and if $n \geq 20$, then the asymptotic test procedure is to reject the null hypothesis $H_o : P(+)=P(-)$ in favor of the alternative {e.g. $H_a : P(+)>P(-)$} if $Z_o=\frac{T_{BVRSS}-\frac{n}{2}}{\sqrt{\hat{V}_o}} > z_\alpha$, where $z_\alpha$ is the 100(1-$\alpha$)% quantile of the standard normal distribution.

### The asymptotic relative efficiency and power

The performance of the matched pairs sign test using BVRSS will be compared with the matched pairs sign test using BVSRS based on the criterion of Pitman's asymptotic relative efficiency (ARE). The Pitman's regularity conditions are satisfied for both $T_{BVRSS}$ and $T_{BVSRS}$ because all moments of the tests are in terms of probabilities, and hence are bounded above by 1. The Pitman's ARE of $T_{BVRSS}$ versus $T_{BVSRS}$ is defined as

$$ARE(T_{BVRSS}, T_{BVSRS}) = \frac{e^2(T_{BVRSS})}{e^2(T_{BVSRS})}, \qquad (3.1)$$

where the efficiency of a test statistics $T$ is given by $e\ (T)$ and

$$e(T) = \lim_{n\to\infty} \left.\frac{\frac{\partial E(T)}{\partial \theta}}{\sqrt{n\, var(T)}}\right|_{H_o}.$$

Using the above definition and noting that $F_D(0)=P(X<Y)=P(X-Y<0)=P(D<0)$, the efficiency

**Table 1**. Pitman's asymptotic relative efficiency $ARE(T_{BVRSS}, T_{BVSRS})$. The results for negative correlation coefficients are in bold and parenthesis.

| $r$ | $\rho = \pm 0.5$ | $\rho = \pm 0.9$ |
|---|---|---|
| 2 | 1.15 (**1.43**) | 1.04 (**1.59**) |
| 3 | 1.29 (**1.77**) | 1.09 (**2.38**) |

of $T_{BVRSS}$ and $T_{BVSRS}$ are obtained as

$$e(T_{BVRSS}) = \left.\frac{\sqrt{2} f_D(0)}{\sqrt{(1 - \frac{2}{r^2}\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij}^2)}}\right|_{H_o} \qquad (3.2)$$

and

$$e(T_{BVSRS}) = 2 f_D(0) \qquad (3.3)$$

respectively. Note that $\theta$ could be the central parameter of $D$ or the shifted parameter such that $P(X < Y + \theta) = 0.5$ under the null hypothesis.

Therefore, by (3.1), (3.2) and (3.3)

$$ARE(T_{BVRSS}, T_{BVSRS}) = \left.\frac{1}{2(1 - \frac{2}{r^2}\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij}^2)}\right|_{H_o} \geq 1$$

Moreover, by Theorem 2.1, the asymptotic power of testing the hypothesis $H_o: P(+) = P(-)$ versus the alternative {without loss of generality consider $H_a: P(+) > P(-)$} for $T_{BVRSS}$ and $T_{BVSRS}$ are defined by: $\beta_{BVRSS} = 1 - \Phi[(z_\alpha \sqrt{V_0} + n/2 - nP(X < Y))/\sqrt{V_a}]$, where $V_0$ ( as in Theorem 2.1) and $V_a$ are the variance of $T_{BVRSS}$ under the null and the alternative hypotheses respectively and

$$\beta_{BVSRS} = 1 - \Phi[(z_\alpha \sqrt{n/4} + n/2 - nP(X < Y))/\sqrt{nP(X < Y)(1 - P(X < Y))}]$$

Therefore, under the null hypothesis $\beta_{BVRSS} = 1 - \Phi(z_\alpha) = \alpha$ and

$\beta_{BVSRS} = 1 - \Phi(z_\alpha) = \alpha$. Note that

$$V_a = Var(T_{BVRSS}|_{H_a}) = r^2 m (P(X < Y) - \frac{1}{r^2}\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij}^2).$$

## Numerical comparisons

Assuming that the bivariate random variable (*X, Y*) has a bivariate normal distribution, $ARE(T_{BVRSS}, T_{BVSRS})$ for {$r$ = 2 and 3, and correlation coefficient ($\rho = \pm 0.5$, and $\pm 0.9$)} is computed. Also, the asymptotic power for {($r$ = 2, $m$ = 5), ($r$ = 2, $m$ = 6), and ($r$ = 3, $m$ = 3)}, shifted parameter of center of the two marginal distributions ($\theta = 0, 0.1, 0.5$, and $1$), level of significance {$\alpha = 0.05$} and correlation coefficient ($\rho = \pm 0.2$, $\pm 0.5$, and $\pm 0.9$)} is computed.

Table 1 shows Pitman's asymptotic relative efficiency $ARE(T_{BVRSS}, T_{BVSRS})$ and Table 2 and 3 show that asymptotic power of $T_{BVSRS}$ and $T_{BVRSS}$ respectively.

Assuming a bivariate normal underlying distribution function, Table 1 shows that the performance of $T_{BVRSS}$ is superior to $T_{BVSRS}$ via Pitman's asymptotic relative efficiency $ARE(T_{BVRSS}, T_{BVSRS})$. Also, it is clear that the $ARE(T_{BVRSS}, T_{BVSRS})$ increases as $r$ increases. It is clear that $ARE(T_{BVRSS}, T_{BVSRS})$ is higher when the correlation coefficient $\rho$ is negative and increases as $\rho$ negatively decreases away from 0. In practice this slight draw back in efficiency, when the correlation is positive, is not a draw back of using the test, because BVRSS sign test still more efficient than BVSRS sign test.

Table 2 gives evidence towards $T_{BVSRS}$ being unbiased and consistent in this case although such evidence is not a conclusive proof. The power of $T_{BVSRS}$ increases as the sample size increases and the shift parameter on the variable *Y* increases away from 0.

**Table 2.** Asymptotic power for $T_{BVSRS}$ when $\alpha = 0.05$. The results for negative correlation coefficients are in bold and are in parenthesis.

| *n*=$r^2$ *m* | $\theta$ | $\rho = \pm 0.2$ | $\rho = \pm 0.5$ | $\rho = \pm 0.9$ |
|---|---|---|---|---|
| | 0 | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) |
| *n*=20 | 0.1 | 0.0867 (**0.0786**) | 0.0983 (**0.0748**) | 0.1917 (**0.0712**) |
| | 0.5 | 0.3893 (**0.2993**) | 0.5273 (**0.2604**) | 0.9925 (**0.2245**) |
| | 1 | 0.8644 (**0.7200**) | 0.9726 (**0.6326**) | 1.0000 (**0.5468**) |
| | 0 | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) |
| *n*=24 | 0.1 | 0.0911 (**0.0819**) | 0.1044 (**0.0776**) | 0.2133 (**0.0735**) |
| (*r*=2, *m*=6) | 0.5 | 0.4431 (**0.3394**) | 0.5968 (**0.2938**) | 0.9981 (**0.2517**) |
| | 1 | 0.9187 (**0.7932**) | 0.9898 (**0.7073**) | 1.0000 (**0.6177**) |
| | 0 | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) |
| *n*=27 | 0.1 | 0.0942 (**0.0843**) | 0.1087 (**0.0796**) | 0.2290 (**0.0752**) |
| (*r*=3, *m*=3) | 0.5 | 0.4813 (**0.3684**) | 0.6435 (**0.3182**) | 0.9994 (**0.2716**) |
| | 1 | 0.9455 (**0.8367**) | 0.9953 (**0.7548**) | 1.000 (**0.6649**) |

**Table 3.** Asymptotic power for $T_{BVRSS}$ when $\alpha = 0.05$. The results for negative correlation coefficients are in bold and are in parenthesis.

| *n*=$r^2$ *m* | $\theta$ | $\rho = \pm 0.2$ | $\rho = \pm 0.5$ | $\rho = \pm 0.9$ |
|---|---|---|---|---|
| | 0 | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) |
| ***n*=20** | 0.1 | 0.0921 (**0.0843**) | 0.1130 (**0.0808**) | 0.2185 (**0.0774**) |
| **(*r*=2, *m*=5)** | 0.5 | 0.4521 (**0.3675**) | 0.5809 (**0.3321**) | 0.9940 (**0.3028**) |
| | 1 | 0.9212 (**0.8288**) | 0.9855 (**0.7733**) | 1.0000 (**0.7272**) |
| | 0 | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) |
| ***n*=24** | 0.1 | 0.0972 (**0.0884**) | 0.1199 (**0.0843**) | 0.2289 (**0.0805**) |
| **(*r*=2, *m*=6)** | 0.5 | 0.5127 (**0.4169**) | 0.6528 (**0.3762**) | 0.9988 (**0.3425**) |
| | 1 | 0.9586 (**0.8879**) | 0.9954 (**0.8396**) | 1.0000 (**0.7968**) |
| | 0 | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) | 0.0500 (**0.0500**) |
| ***n*=27** | 0.1 | 0.1060 (**0.0936**) | 0.1513 (**0.0910**) | 0.2357 (**0.0908**) |
| **(*r*=3, *m*=3)** | 0.5 | 0.6095 (**0.5101**) | 0.6734 (**0.5009**) | 1.0000 (**0.4566**) |
| | 1 | 0.9871 (**0.9000**) | 1.0000 (**0.9121**) | 1.0000 (**0.9013**) |

Table 3 shows that $T_{BVRSS}$ is more powerful than $T_{BVSRS}$ for all studied sample sizes and shifted parameter values. The superiority of $T_{BVRSS}$ over $T_{BVSRS}$ is clear for all values $\rho$ and all values of the set size *r*. There is evidence towards $T_{BVRSS}$ being unbiased and consistent in this case; such evidence is again not a conclusive proof. From Theorem 2.1 $T_{BVRSS}$ has a similar asymptotic distribution as $T_{BVSRS}$ but with smaller asymptotic variance. Therefore, it is safe to say that $T_{BVRSS}$ has similar asymptotic properties as $T_{BVSRS}$ for testing $H_o: P(+) = P(-)$, i.e. $T_{BVRSS}$ is also unbiased and a consistent test procedure. However, $T_{BVRSS}$ is more efficient and more powerful than $T_{BVSRS}$ .

Moreover, Table 3 shows that the asymptotic power of $T_{BVRSS}$ increases when set and sample sizes increase. Also, the asymptotic power of $T_{BVRSS}$ increases when $\rho$ >0 increases for all nonzero values of the shifted parameter $\theta$ and increases when $\rho$ <0 increases in absolute value for $\theta$ >0.1. However, it decreases slightly when $\rho$ <0 increases in absolute value for $\theta = -0.1$ .

## Bootstrap algorithm for estimating the p-value of the test

The distribution of our nonparametric test, $T_{BVRSS}$ , in (2.2) depends on the underlying bivariate distribution function. Thus, the exact P-value calculation for sample size *n*<20 is not feasible without knowing the underlying distribution. In this section we introduce a simple bootstrap method for calculating the P-value of the sign test for any given bivariate data. For general description of the bootstrap method of estimation see Efron and Tibshirani (1993). Suppose that a bivariate random sample of size n<20 is drawn from a population using the BVRSS or BVSRS sampling method.

This implies that $\{(X_i, Y_i), i = 1, 2, ..., n\}$ is a random sample. The bootstrap algorithm for approximating the bootstrap P-value of the test for testing the hypothesis $H_o : P(+) = P(-)$ versus the alternative {e.g. $H_a : P(+) > P(-)$} is :

1. Calculate the sample test statistic (say $T = \sum_{i=1}^{n} I(X_i < Y_i)$) from the original sample.

2. Estimate $\theta$ from the data; say $\hat{\theta}$. Shift $Y_i$ to $Y_i - \hat{\theta}$, I=1, 2, …, n.

3. Define $\hat{F}(x, y)$ by placing a mass probability $p_i = \frac{1}{n}$ on $(X_i, Y_i), i = 1, 2, ..., n$ .

4. Generate a resample $(X_i^*, Y_i^*), i = 1, 2, ..., n$ from $\hat{F}(x, y)$ .

5. Find $T_b^* = \sum_{i=1}^{n} I(X_i^* < Y_i^*)$ .

6. Repeat steps 3 and 4 *B* times.

Then the bootstrap P-value, $P^* = P(T^* \geq T \mid \hat{F}(x, y))$ , can be approximated by $P^* = \frac{1}{B} \sum_{b=1}^{B} I(T_b^* \geq T)$.

However, when the data is BVRSS, slight modification of the above algorithm is needed as follows:

1- Divide the sample into *r* mutually exclusive strata each contains *m* i.i.d order pairs label.
2- Independently from each stratum generate a resample with replacement of size *m* by placing a mass probability ($\frac{1}{m}$) on each original observation in that stratum.
3- Combine all *r* resamples and do similar steps like in (5) and (6) above.

## Illustration on date palm tree and final comments

Date palm is considered the most important fruit crop in the Sultanate of Oman and occupying nearly 50% of the cultivated land in Oman. It is estimated that 35,000 hectares of land are planted with date palms and 28,000 hectares with other crops, including 11,000 hectares planted with rotation crops. These statistics reflect the importance of date palm tree to the Omani people who have lived with this tree for centuries. The date palm has retained its value for the dwellers of the desert because of its adaptive characteristics to the environment and the wide range of its benefits. It provides the family with many of the life necessities.

Different literature at different times has cited variable estimation of the number of palm trees and yield quantity. The total number of date palm trees currently is estimated to be around seven million with a wide range of varieties. FAO (1982) report indicated that the estimated annual production of Omani dates 50,000 tons and the number of date palm trees was 1 million for the period 1961 to 1978. Currently the date palm trees are estimated to be higher than before due to the introduction of new and easier production practices along with new cultivar, which has increased the large scale farming of date palms. The number has risen to seven million trees.

Due to the variety of different types of palm trees in each farm, we would to test if there a significant difference on average between two types (Khalas and Khosab) of palm tree's production.

**Table 4.** Oman palm trees data (2003).

| RSS Sample (r=2, m=2) n=4 | | | | |
|---|---|---|---|---|
| Type of the tree | Khalas | | Khosab | |
| | Diameter (m) | Amount of date (X kg) | Diameter(m) | Amount of date (Y kg) |
| | 1.80 | 84.00 | 1.82 | 118.00 |
| | 2.00 | 99.00 | 2.00 | 160.00 |
| | 2.25 | 204.00 | 2.25 | 130.00 |
| | 2.30 | 30.00 | 2.25 | 150.00 |
| **SRS Sample n=4** | | | | |
| | 1.80 | 65.00 | 1.82 | 118.00 |
| | 2.00 | 120.00 | 2.00 | 106.00 |
| | 2.25 | 140.00 | 2.25 | 150.00 |
| | 2.30 | 135.00 | 2.25 | 132.00 |

Note that $T_{BVRSS}=3$ and $T_{BVSRS}=2$. The Bootstrap P-value of the tests using BVRSS and BVSRS are respectively 0.00048 (variance = 0.00023) and 0.00054 (variance = 0.00029). It is clear that we reject the null hypothesis using BVRSS and BVSRS samples. Therefore, Khalas trees on average produce significantly less date than Khosab trees.

In conclusion, whenever BVRSS can be obtained, it is recommended to be used instead of BVSRS for the bivariate matched pairs sign test.

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