Uncovering Basic Four Learners' Ideas about the Concept of Equal-Sharing in Fractions

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Abstract

Over the years, understanding fraction-related concepts and applying them has presented a challenge to most learners, both young and old. Despite this, it is obvious that one cannot do away with the learning of fractions since it runs through almost all aspects of mathematics concepts. This study sought to explore 38 Basic Four learners' understanding of the equal-sharing concept related to fractions. An individual-test-assessment task was used as a data collection instrument by field assistants who administered the test to the learners in a one-on-one interview and the responses were observed and recorded. The results revealed that about 79% of the total class were found to be operating at the non-anticipatory strategy, 21% of them were showing emerging anticipatory strategies, while none showed dexterity in the anticipatory strategies that would have shown a more advanced knowledge of the concept of equal-sharing. The implications of these findings are discussed.

Keywords anticipatory strategy; emerging strategy; equal-sharing; fractions; mathematics

Introduction

Fractions are one of the first experiences learners meet in mathematics beyond the basic skills of addition, subtraction, multiplication and division (Hannich, 2009). Though fractions, to the majority of learners, is a complex concept and present a challenge to them, it is one of the pivotal concepts in advanced mathematics (Cramer, Post, & delMas, 2002). For example, Siegler, Fazio, Bailey and Zhou (2013) attribute low mathematics achievements and algebra knowledge in learners in high school to poor fraction knowledge obtained at the primary level. Additionally, Bentley and Bosse (2018) argue that the weak foundation in fractions permeates from primary school through to the tertiary with many learners never reaching mathematics proficiency. Furthermore, Bruce, Chang, Flynn and Yearley (2013) explain that learners' weak foundation in fractions can affect them from pursuing higher mathematics that may obstruct their occupational success. The numerous areas that require the use of fractional skills are an indication that we cannot escape fractions in our daily lives. It is, therefore, important that learners understand the various concepts related to fractions to support the smooth learning of other concepts that require knowledge of fractions and rational numbers. Despite several strategies adopted by teachers in teaching the concept, studies indicate that fractions remain a very difficult concept for learners in primary schools around the world (Kor, Teoh, Mohammed & Singh, 2019). This assertion buttresses the National Center for Educational Statistics (2009)'s survey that fractions are one of the most inexorable concepts of difficulty in mathematics for all learners.

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In Ghana, a study conducted by Anamuah-Mensah and Mereku (2005) suggested that Ghanaian learners perform very low in the area of fractions in most formal examinations. Similarly, Agbozo (2020) asserts that pre-service teachers see the concept of fraction algorithm as abstract and hence appeared difficult to them due to weak foundations established at the basic level. Given this, several studies have been conducted on fractions with most of them focusing on pre-service teachers' and/or in-service teachers' content knowledge of fractions (Asante & Mereku, 2012; Baah Duodu, Ennin, Borbye & Amoaddai, 2019) and specific models or strategies such as the use of the Cuisenaire rod in teaching fractions (Kwasi, 2020; Togah, 2020).

Statement of the Problem

Fractions, as earlier indicated, is one of the main pillars of mathematics. The basic concepts in mathematics are often seen as a headache to many basic school learners. It is worthy to note that, basic school learners' primary experiences with the concept and ideas of fractions and its algorithm must be well organised to build on the learner's existing knowledge (Sharp & Adams, 2010). Streefland (1991) suggests that teaching ought to provide realistic activities that assist children to shape their existing knowledge base. Per this assumption, there are serious implications on how teachers develop a concept since every child has their way of understanding the concept of fractions.

A fractional idea identified as one of the fundamental skills needed by learners to be able to learn other aspects of the concept of fractions is the concept of equal-sharing. For example, Lewis, Gibson, Kazemi and Lind (2015) uncovered U.S. learners' understanding of the concept of equal-sharing which brought to light the varied ways the participants understood this concept and ways of representation. The equal-sharing concept was chosen because studies indicate that, learners as young as 5 or 6 years old have some form of informal knowledge of partitioning situations (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993) which forms the basis for understanding the concept of fractions (Piaget, Inhelder & Szeminska, 1960). Again, the new Ghanaian curriculum introduces learners at Basic Four to comparing fractions, addition and subtraction of fractions, decimals and percentages (Ministry of Education [MoE], 2019) which are concepts that are built on the concept of equal-sharing.

Unfortunately, the literature review indicates that how learners conceptualise this foundational idea in fractions in Ghana has not received the needed attention. The purpose of the study, therefore, is to unveil the Basic Four learner's ideas about the concept of equal-sharing of fractions through an examination of the strategies and the kinds of representations they use. Through this study, an attempt is made to offer an insight into learners' thinking about equal-sharing tasks. Additionally, the findings from this study will provide ways to support and extend children's thinking. In this regard, the research questions underpinning this study are as follows:

1. What strategies do learners use in solving an equal-sharing task?
2. What kinds of representations and notations are used by learners in representing fractions?

Literature Review

The concept of equal-sharing

The equal-sharing concept means dividing a whole or a group of objects into equal parts. Therefore, by dividing an object or a group of objects equally, we get equal shares among the sharers. The equal parts obtained have to be the same in measurements like volume, dimensions, size, weights, numbers, and heights among others. Equal-sharing provides a potentially rich foundation for fraction comprehension that draws on learners' informally acquired resources and encourages those resources toward more advanced mathematical understanding through instruction (Empson, 1999). Empson and Levi (2011) assert that equal-sharing as a strategy adopted by children in conceptualising fractions has been well documented.

Learners' different mathematical interpretations are facilitated by equal-sharing contexts (Streefland, 1991), which is fundamental to cognitive transformation in collaborative activity. Partitioning, part-whole, and ratio are among the rational-number sub-constructs available for equal sharing (Kieren, 1976, 1988). Early equal-sharing strategies use partitioning as a crucial construct. When partitioning and considering the resulting parts and their size about a discrete unit, part-whole ideas develop. The number of persons sharing and the number of objects shared as a composite unit results in ratio-related constructs (Lamon, 1994; Streefland, 1991). Basically, in equal sharing, there are a total number of objects and several sharers to share the objects equally among themselves. The goal of equal sharing is to find out how many objects will each member receive equally. Studies also indicate that learners at the early stages resort to the use of informal strategies that are triggered by their understanding of partitioning and equal distribution to create fractional quantities (Krause, Empson, Pynes & Jacobs, 2016). Learners at the early stages of being introduced to fractions either cut, draw, colour, fold or split items into halves before reflecting on the results obtained (Lewis et al., 2015). Notwithstanding, it is reported that their sharing strategies begin to develop along an expected trajectory of being able to coordinate between the two quantities (number of objects to be shared and the total number of sharers) in advance (Empson & Levi, 2011).

The Ghanaian mathematics curriculum suggests that the concept of fractions be introduced to learners at primary one with learners expected to learn concepts such as "half of an object with learners conceptualising halves and fourths using concrete and pictorial representations (excluding fractional notations) at primary two (MoE, 2019). In primary three, however, learners are introduced to identifying and writing fractions in words and symbols. Learners in primary four per the new mathematics curriculum (MoE, 2019) are to be introduced to equivalent fractions, addition and subtraction of fractions, decimals and percentages. As argued earlier, equal-sharing is one of the key skills needed to be developed for other higher fractional concepts to be built on. Interestingly, at the primary level, the main approaches employed in helping learners to understand the concept of division (fractions) are equal-sharing and equal grouping. However, research indicates that children gain an in-depth understanding of fractions when equal grouping is taught before equal-sharing (Psarianos, 2019). Equal-sharing, therefore, is one of the fundamental aspects of fractions that are studied in the basic schools in Ghana.

Conceptual Framework

In this study, we draw on Empson and Levi's (2011) conceptualisation of the strategies learners used when solving equal-sharing tasks. According to the authors, learners adopt three main equal-sharing strategies. These strategies reflect the level of understanding by learners through their partitioning, representations and their symbolic notations. These strategies are non-anticipatory, emergent anticipatory strategies and anticipatory direct modelling. We explain each of these strategies.

Non-anticipatory strategies

Learners using this strategy only focus on one requirement of an equal sharing situation, that is, either the need to represent every share or the need for equal share among members. In so doing, they just go straight and start sharing without carefully considering how to go about sharing the objects among the sharers. Empson and Levi (2011) categorise learners using this strategy under two variations and that is either they are using the trial-and-error technique or the repeated halving technique. Those using the trial-and-error may begin by halving the items to see if those sharing will have equal shares when distributed. If that fails, they go on to divide the items into thirds, fourths, and fifths among others until they can share everything equally. A limitation with such a strategy is that learners may or may not consider the number of sharers in the initial stages. Another characteristic of learners using the non-anticipatory strategy is that they may or may not be able to use the appropriate terminology or symbolic notation to represent the fraction obtained.

Emergent anticipatory strategies (Additive coordination)

As the name connotes, this strategy is the emergence of the anticipatory thinking strategy. Learners operating at this level plan their partitioning before the activity. To Hunt and Empson (2015), learners operating with this strategy can anticipate the relationship between the number of sharers and the quantity to be shared. That is, they begin to bring together their previous two goals of making equal-sized parts and exhausting the whole (Hunt, Westenskow & Moyer-Packenham, 2017). In so doing, the learner begins by recognising that several objects must be shared among several sharers. Afterwards, the learner proceeds to use this technique by dividing each object to be shared into several parts proportional to the number of individuals who would be sharing it. When there are more sharers than the objects to be shared, the learner divides each object (or sometimes a small group of objects, such as a pair) into several portions equal to the number of sharers; this process is continued until there are no more objects to share. To quantify the share, the learner usually calls each portion a unit fraction within the whole (for example, one item divided into five parts produces "fifths") and counts the number of fractional parts each sharer receives. Some of the learners may resort to the use of addition to add the number of parts received by an individual before writing its symbolic notation.

Anticipatory direct modelling (Multiplicative coordination)

This strategy according to Lewis et al., (2015) is the most sophisticated. Anticipatory direct modelling, also called multiplicative coordination, is used by learners who have gained enough knowledge of multiplication, ratios, division and a fair idea that fractions can be quotients. Learners showing this level of strategy no longer create concrete models of each item to be shared but can relate the amount to be shared among the sharers. This is because they have a fair understanding of multiplication and can therefore use it to determine how multiple items would be shared. For example, given that six children share eight oranges problem, the learner using the anticipatory direct modelling strategy immediately says that each member gets eight-sixth because they get one-sixth of each of the orange. They then multiply the one-sixth by the eight oranges to identify how much each member obtains in total.

Research Methodology

An exploratory research design was adopted. The exploratory research design helps researchers to determine the nature of a problem and also to develop a better understanding of the problem. In an exploratory study, researchers are not in a haste to provide concrete solutions to the problem identified but explore various sources that are linked to the stated problem (Gratton & Jones, 2014). The census method was used to engage all 38 Basic Four pupils in the class for the study. There were 18 (47.37%) males and 20 (52.63%) females. Appropriate permission was sought from the head of the school and the consent of the parents of the participants. All 38 members of the class voluntarily assented to take part in the survey after explaining the nature of the study to them. Additionally, they were informed that they could withdraw their participation from the study at any point that they felt uncomfortable. Furthermore, to protect the identity of the participants from non-members of the research team, pseudonyms were used in coding their response sheets and throughout this paper.

An individual-test-assessment task (see box below) was used as a data collection

> *3 children are sharing 10 loaves of bread which are all of the same sizes. They are sharing so that each child gets the same amount. How many loaves of bread will one child get?*

instrument by field assistants who administered the test to learners in a one-on-one interview. As learners went about responding to the task, the field assistants interacted with them in an attempt to gain an in-depth understanding of how they were thinking about the given task and their use of representations in describing their work. A response sheet was used by the field assistants to record and keep track of the learners' ideas and use of representations throughout the interactive session. On average, the participants took about 20 minutes to complete the task. The data gathered with the learners' response sheets were analysed in themes based upon the above conceptual framework to answer the research questions.

The following vignette is an excerpt of the nature of the interaction that took place between learners and the Field Assistants and how the data collected were analysed.

Field Assistant 1: Wow! Renita, please tell me about what you just did.

Renita: Sir, the question says 3 children are sharing 10 loaves of bread so I divided each loaf of bread into three parts and gave each child a piece of the divided loaves of bread (happily pointing to the divisions on her Table).

Field Assistant 1: Okay! So, does everyone get the same amount?

Renita: Yes sir.

Field Assistant 2: So, if I am one of the children, what will I get?

Renita: (Pausing for a few seconds) 10 pieces.

Field Assistant 2: So how many loaves of bread is that altogether?

Renita: (Sounded confused and repeated) 10.

The vignette above shows a clear picture of sharing one item at a time and falls under what Empson and Levi (2011) describe as an emergent anticipatory strategy. With this strategy, the learner represents each loaf of bread and splits the first loaf into thirds because that is the number of sharers. In that sense, each person gets one-third of each loaf of bread. The splitting is repeated for each loaf of bread until all the loaves of bread are exhausted. The learner then adds the one-third each sharer received altogether to arrive at $3\frac{1}{3}$.

Another learner by the name of Duku also showed a different technique for his understanding of fractions. This is what transpired between Duku and the Field Assistants:

Field Assistant 3: So, tell me; how did you share the loaves of bread among the 3 children?

Duku: I first gave each child 1 piece of the loaf of bread and realised there are 7 more loaves of bread so I repeated the sharing by giving 1 piece to each child. There were 4 more loaves of bread so I shared 1 piece again and I was left with only 1 loaf of bread. I then divided the remaining 1 into 3 equal parts and gave a piece to each child. *(Pointing to the divisions on his paper).*

Field Assistant 2: So, if I am one of the children, can you point to what I get as my share?

Duku: This sir. *(Pointing to one of the drawings of the child and his respective number of loaves of bread beside him)*

Field Assistant 3: Okay! So how do we call this?

Duku: Three and half

Field Assistant 3: I see! So, how would you write that?

Duku: $3\frac{1}{3}$

Field Assistant 3: erh! And what do you say this fraction is called again?

Duku: Three and half sir.

The conversation between the Field Assistants and Duku provides enough evidence that Duku understands fractions but struggles with the use of appropriate fractional language in describing the fractional symbols.

Findings and Discussions

This section of the study presents the results and analysis of the study. The findings portrayed learners showing some elements of non-anticipatory and emergent anticipatory strategies and representations as indicated by Empson and Levi (2011). Additionally, there were a few of the learners who could not distinguish between the total numbers of objects to be shared from the number of sharers. In the following sections, we present the results for each research question. The first research question sought to find out the strategies learners use in solving tasks involving equal sharing. In the next section, we present the results on the number and percentages of learners beginning with those who adopted the non-anticipatory strategy.

Non-anticipatory strategies employed by learners

The analysis revealed that 30 (78.9%) of the participants showed evidence of what Empson and Levi (2011) describe as non-anticipatory strategies. For example, Owusu shared all the loaves of bread in a whole and when there was a leftover of one loaf of bread from his drawing, he picked an eraser and erased it. When he was asked why he did that, he responded that: "it will not be enough for each of them to get a full piece". An excerpt of this idea is shown in Figure 1 where the learner did not partition the leftover bread into thirds as would have been expected from a learner with adequate knowledge of partitioning. The learner not dividing the leftover bread because "it will not be enough" suggests a learner who is still thinking in terms of whole numbers and yet to develop fractional thinking.

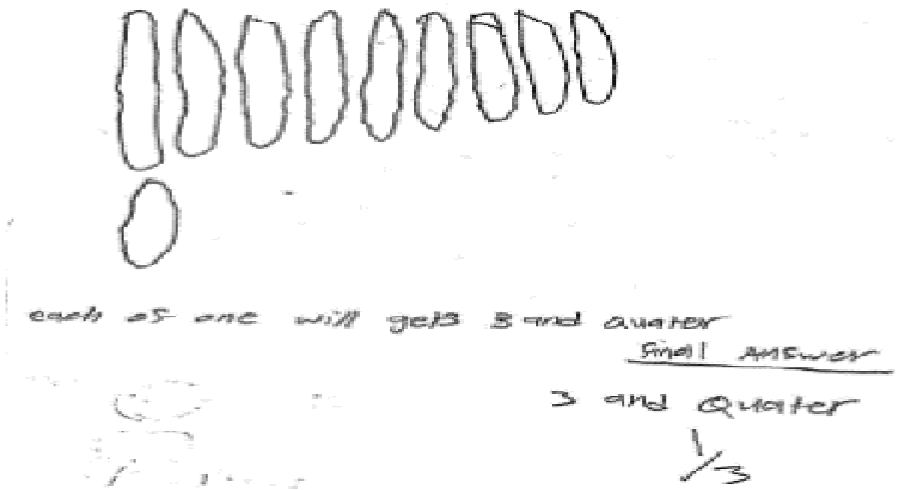

Figure 1: A response showing lack of knowledge in partitioning

Additionally, some of the participants were observed halving each loaf of bread without taking the number of sharers into account. From their illustrations, it was obvious they were using either the trial-and-error technique and/or the repeated halving strategies. In Figure 2, the learner used drawing to illustrate how the sharing was being done, in this case the bread with the hope of sharing all the given loaves of bread (see Figure 2).

Yet, another group of learners, from their drawings, did not seem to know what to do when it comes to the concept of fractions. Essel and Apreku, for example, showed no idea of the quantity of bread to be shared. In Figure 3, the number of loaves as depicted in the drawing on the left side is 12 which is

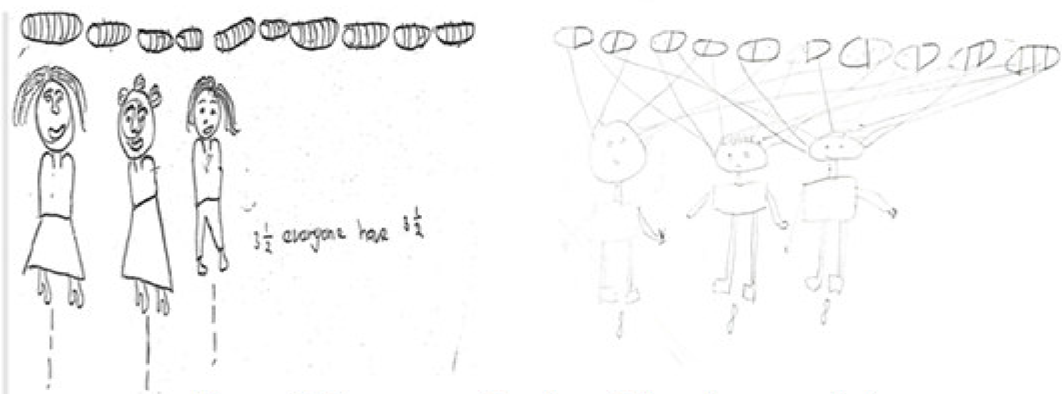

Figure 2: Responses showing trial-and-error technique

showing some fractional ideas. Unfortunately, the approach of first halving some of the bread and then dividing the remaining loaves of bread into thirds indicates an inefficient method and appears that the learning was engaged in trying various ways of partitioning more than the 10 given in the task. Also, the partition on the right side shows an attempt at halving but then, the drawing does not show how all the loaves of bread were shared leading to the answer of 3. In both cases, their drawings did not reflect anything related to the given task (see Figure 3).

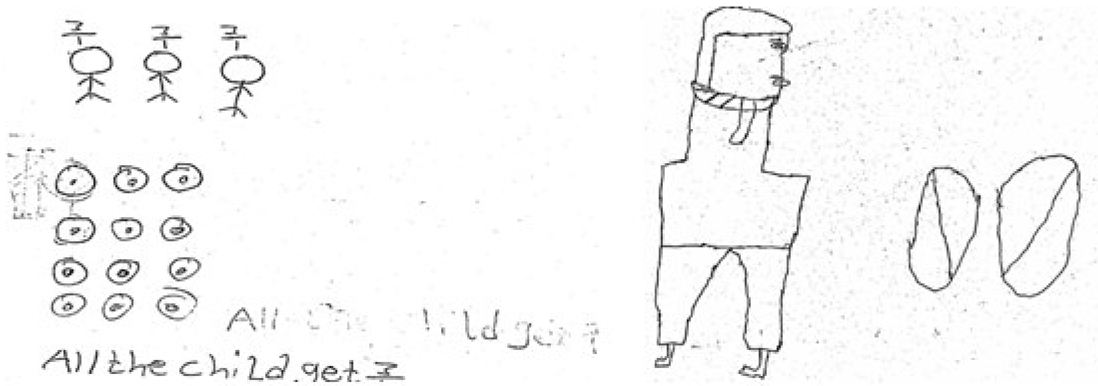

Figure 3: Responses showing no idea of fractions or sharing

Emergent anticipatory strategies employed by learners

Out of the 38 learners who took part in the study, as found that 8 (21%) demonstrated skills that presuppose that, there is an emerging idea of equal-sharing. According to Empson and Levi (2011), learners at the emergent anticipatory state can equally divide the number of objects proportional to the number of the sharers and later use additive coordination to determine each share.

Figure 4 gives an example of this approach as used by Renita. Renita, for example, partitioned each loaf of bread into three parts until every loaf of bread was exhausted.

Her strategy is an indication she understands the additive relationship in an equal-sharing task.

Such a strategy is called *sharing one item at a time.* However, when she was asked how much each member obtained, she replied "10" suggesting that she counted each third as if they were wholes. Figure 4 shows an example of a response indicating an emergent anticipatory state.

Another example of learners who used this strategy by distributing the wholes and then later dividing the last one into three parts are: Dapaah, Naana and Catherine. While their drawings show an attempt at partitioning into thirds, they named the resulting fraction as $3\frac{1}{2}$ as shown in Figure 5 highlighting a lack of coordination between their partitions and the named fraction. It appears that they consider every fraction as being halves, for example.

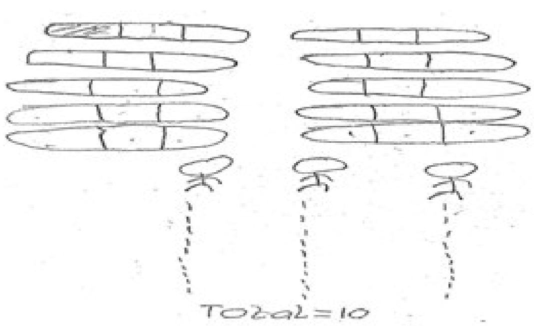

Figure 4: A response showing sharing one loaf at a time

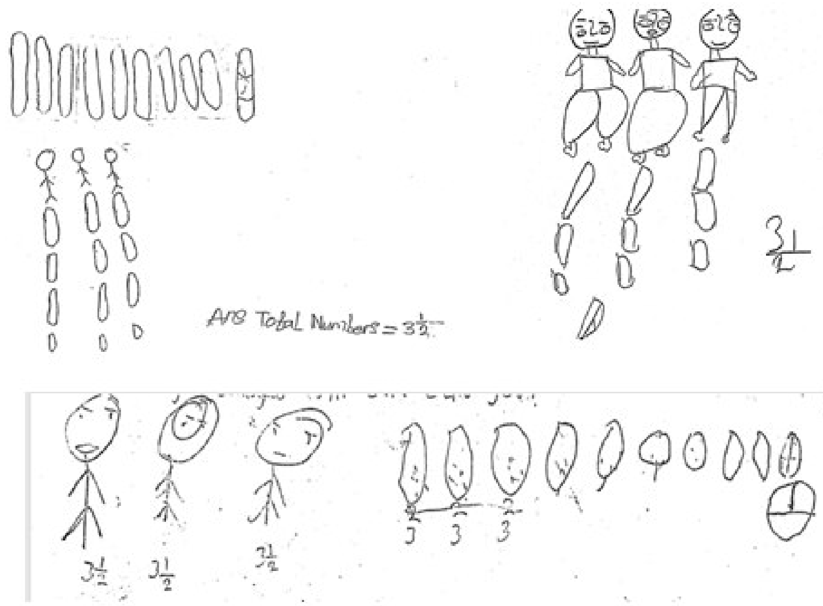

Figure 5: A response showing no coordination being partitioned and results obtained

The results for research question one indicate that learners' understanding of the equal-sharing concept is not fully developed. This assertion is being made as a result of the participants not ensuring that the partitions are indeed equal even though when probed, they claimed to have divided equally. Such a phenomenon, according to Lewis et al., (2015), is a result of underdeveloped spatial reasoning or fine motor skills. Though the majority of the learners were seen partitioning or dividing objects in a form of sharing, they did not pay attention to the equal-sharing concept since their drawings did not show that each share was indeed equal. It may seem to the learners that all that matters is sharing and not the core concept; equal-sharing. This finding is worrying since it paints a picture of learners who, despite having been taught fraction-related concepts such as naming of fractions, do not have a good grasp of what the concept means; idea of equal sharing.

Also, none of the learners demonstrated the third strategy of equal-sharing which Empson and Levi (2011) termed as multiplicative coordination or anticipatory strategy. Learners at this level are able to reflect, strategize and think critically in solving everyday life problems that involve fractions. This is because they have mastery of the concept of multiplication and can therefore use it to determine how multiple items would be shared without using any partitioning or concrete objects. Therefore, the resort to non-anticipatory strategies where they just zoom into drawing some objects and dividing without first of all, making a determination about the relationship between the number of objects and the sharers, has implication for them developing multiplicative reasoning required for proportional thinking and advanced mathematics topics. According to Lewis et al., the multiplicative coordination strategy is the most complicated and are more typical among high school learners since it involves the use of ratios and other higher fractional reasoning.

Going forward, it will be important for teachers to guide learners to identify the relationship between the number of objects to be shared (numerator) and the number of sharers (denominator) as well as the need for equal divisions (Empson & Levi, 2011).

Fraction language and notation

The second research question sought to find out the various representations that were used by the learners in solving the assigned task and their fractional language and notation usage. Learners' drawing, according to Lewis et al.,(2015), can reveal what learners comprehend about equal parts. Almost all the learners resorted to the use of drawing in partitioning the wholes. However, their drawings differed significantly in representing the shares and most often did not convey the idea of equal shares. As such, although the learners resorted to drawings to model the given task, their drawings revealed a partial understanding of the concept of equal sharing as the various partitions were unequal. Also, the loaves of bread drawn were unequal even though they are the same wholes being shared.

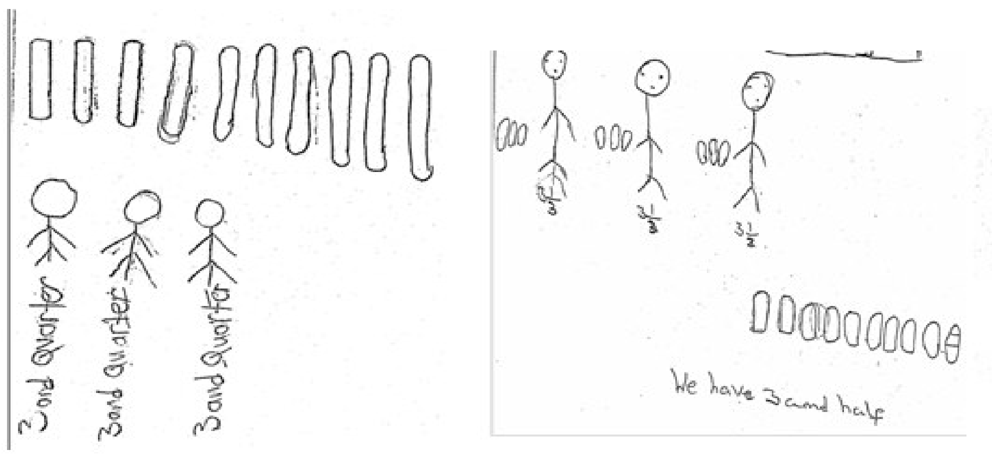

Figure 6: Response indicating lack of knowledge in fractional notations

Sharing can be considered intuitive but the use of the right language and fractional notation must be developed. Lewis et al., (2015) noted that children often use informal language and notation at the early stages of being introduced to fractions. In view of that, Empson and Levi (2011) advise teachers to introduce the fundamental logic underlying fraction language and notation to curb the observed phenomenon in this study. The use of language as another means of representing fractions was identified in the study as a major setback for the learners. The analysis showed that a significant number of 37 (97%) of the participants could not use the right language in describing the answers they got. Out of the eight (21%) participants who were able to write the correct answer, 7 of them could not use the right fractional language to describe their answer. For example, Duku called $3\frac{1}{3}$ as three and a half. This buttresses Hiebert (1988), that children often face challenges when shifting from the use of language and notation involving whole numbers to the use of language and notation when learning fractions. Again, it is argued that the learners in this study tended to associate every part of the whole as being either a half or a quarter of the whole although the whole may be divided into several parts. This finding is consistent with the findings by Lewis et al., (2015) who reported that learners' easily associate fractions as halves of objects or wholes as evident in Figure 6.

In Figure 6, it is observed that the notation used, "3 and quarter" and "3 and half" do not correspond to their respective drawings and give an impression of learners who view every fraction as either in fourths or halves.

Furthermore, some learners were unable to write the symbolic notation of their results.

The results from the learners' work showed that 30 (78.9%) participants were unable to write the correct fractional symbols for the results they obtained. This means only 8 (21%) were able to write the correct symbolic notation for their results. For example, in Renita's case (see Figure 4 above), she wrote 10 as her final answer after dividing each loaf of bread into three equal parts. It was obvious that she employed the use of additive coordination but lacked the skill in the computations, which could have led her to the correct answer. Further examples of learners who could not write the correct symbolic notation are shown in Figure 7. From the result obtained, it is evident that being able to model fraction tasks using drawing does not necessarily mean that they know how to name such fractions. The two are different skills that they need to learn; 1) how to model the task to depict the correct mathematical meaning of fractions and 2) how to correctly name the resulting fraction using the symbolic notation.

Conclusion and Recommendations

In order for teachers to be able to assist learners to understand mathematical concepts such as equal-sharing, they need to get a fair idea about learners' mathematical thinking (Jacobs & Empson, 2016). This study was therefore carried out to unveil how Basic Four learners conceptualise equal-sharing through the kinds of strategies they adopt and the representations used. The results from this study revealed that the majority of the learners used the non-anticipatory strategies. The few learners who demonstrated some level of emergent anticipatory strategies in attempting the task were found wanting when it came to representing their results in fractional notations or adding their results. This suggests that unless these learners have a better understanding of partitioning and fractional language and notation, they may face significant challenges learning other topics

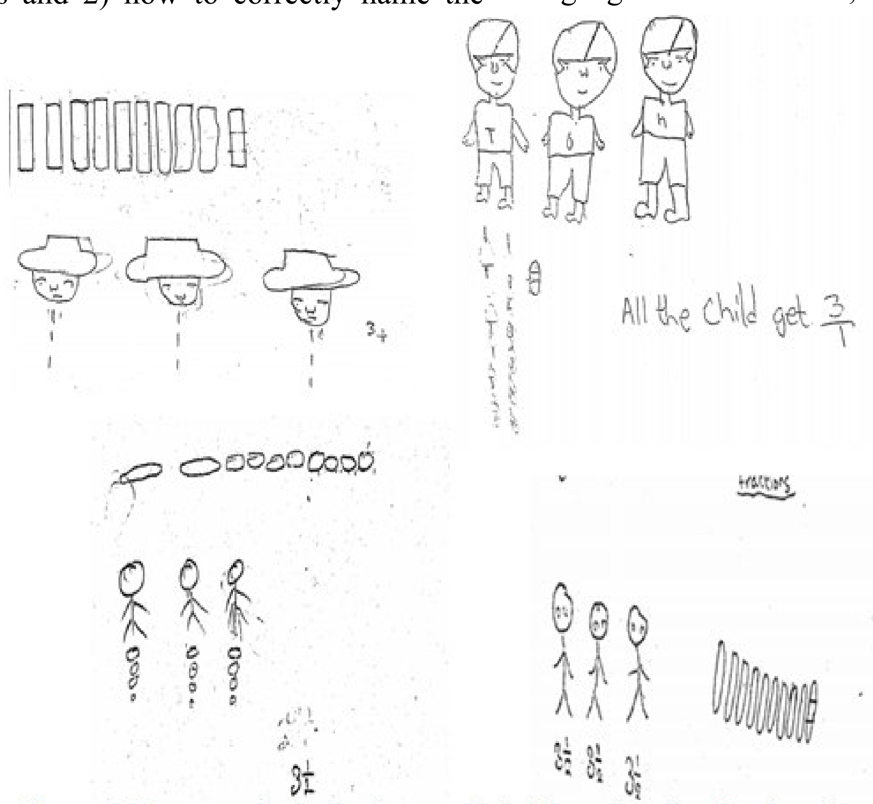

Figure 7: Response indicating learners' challenges in using fractional notations

such as ratio and proportion which require that they have multiplicative reasoning ability.

As aforementioned, knowledge about the equal-sharing concept of fractions is a major block for other aspects of fractions to be built on. Once learners are unable to understand the concept climbing higher in other complicated concepts in fractions and other mathematical concepts may be very difficult. It is, therefore, recommended that elementary school teachers identify individual learners' needs and design hands-on activities that meet each learners Zone of Proximal Development (Vygotsky, 1978: as stated in Mcleod, 2010) to help learners appreciate the concept of equal-sharing. When learners are able to comprehend the concept, they may be able to identify effective strategies and models of solving problems related to even higher thinking fractional questions.

Again, a lot was learned from the learners' drawings in terms of their partial understanding of the concept of equal-sharing and the nature of the challenge. It is, therefore, recommended that teachers encourage their learners to draw to model fraction tasks to elicit learners' ideas about a concept. This will enable the teachers plan an effective teaching strategy to address the errors identified from the learners' responses. Again, teachers at the basic level are to assist learners on the proper use of fractional notations since most of the learners in this study were found to associate every part of a whole as one-half or one-fourth.

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