A Study of Bases, Continuity and Homeomorphism in Hausdorff Topology

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ABSTRACT

This paper surveys the essential concepts of bases, continuity, and homeomorphism within the context of Hausdorff topology. We examine the characterization of Hausdorff spaces through various types of bases, including minimal generative bases, which serve to illuminate the intrinsic properties of these spaces. The interplay between bases and morphisms is explored, emphasizing how morphisms facilitate continuity of mappings between Hausdorff spaces and their quotient images. We investigate the implications of these relationships for determining homeomorphisms, establishing conditions under which two Hausdorff spaces can be deemed equivalent. Key results include the construction of quotient images using equivalence relations, as well as criteria for continuity and homeomorphism that are uniquely suited to Hausdorff spaces. The findings on bases also demonstrate that a topological space X is Hausdorff if and only if there exists a minimal generative base B_M for X such that for any subfamily B'_M of B_M that is also a base for X, every element of B_M can be expressed as a union of elements from B'_M. This result provides an important tool for studying the properties of Hausdorff spaces and their relationships with other topological spaces.

Mathematics Subject Classification: Primary 54A05; Secondary 54A20 .
Keywords: Hausdorff spaces, morphism, bases, quotient images

1 Introduction

The study of Hausdorff spaces, named after Felix Hausdorff [7], has been a subject of great attention for quite some time, generating interesting results over the years. For instance, Fell [5] determined some topology associated with Hausdorff topology for the closed subsets of a locally compact non Hausdorff space. Graves [6] considered the completeness of a Hausdorff space and revealed that the property of completeness carries over to certain spaces of functions whose functional value lie in a complete space (X, τ). Beginning with the neighborhood $U\pi\sigma(F)$ of (X, τ) consisting of all functions of G in τ such that $G(qi)$ is in $V\sigma(F(q_i))$ for each q_i in π with a proper definition of the relation $>$ for the complete space. Using the neighborhoods $W\alpha(F)$, it is shown that certain sub-spaces of (X, τ) are also complete. Further, the study showed that if (X, τ) is a topological space, then the space of all continuous functions f is complete and if (X, τ) has associated itself with a system of neighborhoods, then the space (X, τ) of all uniformly continuous functions of (X, τ) is complete.

Arens and Kelly[2] conducted a research on Characterizations of the space of continuous functions over a compact Hausdorff space. Accordingly, the first characterization proceeded by an investigation of the extreme points of the unit sphere Σ of the adjoint and space B^* of the Banach space B which was constructed . Indeed, if B is the class of continuous real-valued functions over a compact Hausdorff space X, then the $\|f\|$ $=sup_{x\in X}|f(x)|$. The Reisz-Markoff Saks representation [10] for linear continuous functionals on C can be used to prove the class of extreme point of Σ and can be divided into two disjoint closed sets, each of which is homeomorphic to X using the weak topology in B^*

Bing[3]connected countable Hausdorff spaces with countably many points. Urysohns gave an example of a connected Hausdorff space with only countably many points. Bing [3] used an example that the points of the space are rational points in the plane on or above the x-axis. That is if (a, b) is such a point and $\epsilon > 0, (a+b)+(r+0)$ then either $\mid r-(a+6/3, /2 \mid< \epsilon$ or $\mid r-(ab/3| > 2) \mid< \epsilon$ is a neighborhood. To construct geometrically a neighborhood with center (a, b), Bing [3] considered an equilateral triangle with base on the x-axis and apex at (a, b). If $b = 0$ and regard (a, b)as the triangle then (a, b) plus all rational points on the $x-$axis whose distances from a base vertex of the triangle are less than ϵ and is a neighborhood with (a, b). The space therefore satisfies the Hausdorff axiom and it has a property that for each neighborhood there is a point common to their closures and therefore the space is connected.

Ronse and Mohamed [11] used a new approach to discretize Hausdorff spaces such that the discretization of a compact Hausdorff distance is minimal. A mathematical description of Hausdorff discretizing sets which are related to the discretization by dilation was considered by Heijmans and Toet [8] in the cover of discretization studied by Andres [1]. Andres worked on the Hausdorff distance between a compact set and its maximal Hausdorff discretization. From this study its clear that the latter converges (for the Hausdorff metric) to the compact set when the spacing of the discrete grid tends to zero.

Ercan [4] attempted to give a better understanding of characterization of completely Hausdorff spaces. A topological space is said to completely Hausdorff [13] if for every $x, y \in X$ and $x \neq y$, then there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$. It was shown by Ercan [4] that a topological space is completely Hausdorff if and only if every compact subspace is C_b . A subspace Y of X is said to be C_b embedded if every bounded continuous function $f : Y \to \mathbb{R}$ has a continuous extension $\overline{f} : X \to \mathbb{R}$. Since every compact subspace of X is C_b embedded then X is compactly embedded. This therefore shows that the completeness of Hausdorff spaces and compactly embedded spaces brings out the notion of equivalence for Hausdorff spaces.

Despite the fact that studies concerning Hausdorff topological spaces is elaborate, the general construction and characterization of all classes Hausdorff spaces remain open. We therefore provide a survey of Bases, Continuity and

Homeomorphism in Hausdorff Topology. We consider classes of minimal generative bases and sub-bases and use their rich algebraic properties to generalize the patterns in Hausdorff spaces and their quotient images.

2 Quotient Images, Continuity, Homeomorphism in Hausdorff Topology

The following definition shall be useful in the sequel:

Definition 2.1. *A quotient map is a surjective continuous map between topological spaces that respects equivalence relation defined on one of the spaces. A topological space Y is a quotient image of X if there exists a surjection $f : X \to Y$ such that $V \subset C$ is open if and only it's pre-image $f^{-1}(V)$ is open in X.*

2.1 Construction of Quotient Maps/Images

In order to construct a quotient map, we need to define an equivalence relation. Let $(\mathbb{R}, d)$ be the metric topological space with the Euclidean topology d and consider an interval $[0, 1] \subset (\mathbb{R}, d)$. Two points x, y can induce an equivalence relation as $x \sim y$ if $x = y$ or if both are irrational. Now given two spaces, (X, τ) and (Y, ρ), the spaces are considered equivalent if there exist two distinct points $x \in X$ and $y \in Y$ inducing the relation equivalence.

Proposition 2.1. *Let $[0, 1] \subset (\mathbb{R}, d)$ be given and let X be the set of all equivalence classes with a topology generated by the set $U = [(1/2) - \epsilon, (1/2) + \epsilon]$ with $\epsilon > 0$. Define a map $f : [0, x] \to X; f(x) = [x]$ in $[0, 1]$ whereas $[x]$ denotes the equivalence class containing x. It is clear that the map is surjective as the equivalence class in X is also in $[0, 1]$. $f(x) \sim f(y)$ if and only if $x \sim y$. Thus f respects the equivalence relation.*

Proof. Let $U \subset X$ and $u \in U$ such that $u = [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$. Consider a surjection f with $f : [0, 1] \to X$. Since $[0, x] \subset [0, 1]$, then, $f[0, 1] = [x]$ thus

$$f : [0, 1] \to X \bigcap_{u \in U} u.$$

It is also clear that $f(0, 1) = (0, u)$ whenever $x \in [0, 1] \forall u \in X$. Hence $f(u) = u \in U$ (1)

Pre-multiplying equation (1) by f^{-1} gives: $f^{-1}(f(u)) = f^{-1}(u)$ implying that $x = f^{-1}(u) \in [0, 1]$ and therefore $f : [0, 1] \to X$. From this it is clear that $[0, 1]/ker f \cong Im X$. The images of X via f^{-1} lie in $[0, 1]$ and they are a representation of quotient images in the pre-image. This means that ker f =$\{0\}$.

Finally we need to show that f is continuous. Let U be an open subset of X. We need to show that $f^{-1}(u)$ is an open subset of $[0, 1]$. Since the topology generated by the set $U = \{[(1/2) - \epsilon, (1/2) + \epsilon]\}$, it is enough to show that $f^{-1}(U)$ is open for such a $\{u\}$

Let $x \in f^{-1}(U)$. If $f(x)$ belongs to U it would imply that $[x]$ intersects U. Thus there exists $\epsilon > 0$ such that $[(1/2) - \epsilon, (1/2) + \epsilon]$ intersects U. Since it is already known that $[x]$ is an equivalence class then there exists a point $y \in [x]$ such that $| y - 1/2 | < \epsilon$ and also since f maps y to [x] , then we have $f(y) \in U$. Thus for any point $x \in f^{-1}(u)$ there exists an $\epsilon > 0$ such that the interval $(x - \epsilon, x + \epsilon)$ is contained in $f^{-1}(U)$ implying that $f^{-1}(u)$ is open and therefore continuous. □

Example 2.1. *Let $(\mathbb{R}, d)$ be the metric topological space with the Euclidean topology d such that $d(x, y) < r\ \forall r \in \mathbb{R}^+$ if and only if $x \sim y$. This partitions $\mathbb{R}$ into equivalence classes that are balls of radius r centred at each point in $\mathbb{R}$. The quotient space obtained here consists of all these balls as points and we endow it with a metric induced.*

If $r = 1$. Then for any two points x, y in $\mathbb{R}$, we get

$$[0] = \{0\},\ [x] = y : d(x,y) < 1,\ [R \setminus \{x\}] = y : d(x,y) \geqslant 1$$

as the only equivalence classes.

Definition 2.2. *A topological space (X, τ) is called a T_1 **space** if it satisfies the T_1-Axiom.*

Proposition 2.2. *Let (X, τ) be a T_1 **space**. Then for every pair x, y of distinct points of X there exist subsets $G,\ H$ of X such that $x \in G$ and $x \notin H,\ y \in H$ and $y \notin G$. But $G \cap H \neq \Phi$.*

Proof. Let (X, τ) be T_1 and $p \in X$. We need to show that $\{p\}$ is closed. Hence, we must show that $\{p\}^c$ is open.
Let $q \in \{p\}^c$. So $q \neq p$. Since (X, τ) is T_1 there exists an open set G_q containing q but not p. That is, $q \in G_q$ but $q \neq p$. Now,

$$\{p\}^c = \cup\{q \in X : q \neq x\}.$$

So,

$$\{q\} \in G_q \subseteq \{p\}^c\ \forall q \neq p.$$

Thus

$$\cup\{q\} \subseteq \cup_q G_q \subseteq \{p\}^c.$$

Therefore $\{p\}^c = \cup_q G_q$ which is an open set for it is a union of open sets. Thus $\{p\}^c$ is open. That is, $\{p\}$ is closed.

Conversely, let $\{p\}$ be closed, for each $p \in X$, the space (X, τ) is T_1. Also, let $p,\ q \in X$ and $p \neq q$. Then, $\{p\}^c$ is open, containing q but not p. On the other hand $\{q\}^c$ is an open set containing q but not p. Thus, (X, τ) is T_1. □

Definition 2.3. *A topological space (X, τ) is said to be a T_2 -space or an Hausdorff space if for each pair of distinct points $p, q \in X; p \neq q$ there exist G, H such that $p \in G, q \in H$ and $G \cap H = \Phi$.*

Definition 2.4. *(X, τ) is said to be normal if for every pair of the disjoint closed sets E and F, there exists open sets G_1 and G_2 such that $E \subset G_1$ and $F \subset G_2$ and $G_1 \cap G_2 = \Phi$.*

Next, we present some results on continuity of maps between topological spaces.

Definition 2.5. *Let (X, τ) and (Y, ψ) be topological spaces and $f : X \to Y$ be a function. Let p be any fixed point in X. We say that f is $\tau - \psi$ continuous or continuous at the point p if each open set $H \in U$ containing $f(p)$ there is an open set containing p such that $f(u) \subseteq H$. That is, $U \subseteq f^{-1}(H)$.*

Proposition 2.3. *Let (X,τ) and (Y,ψ) be topological spaces and let $f: X \to Y$ be a function. Let B be base for ψ and S be a sub-base for ψ. Then the following statements are equivalent:*

(i) f *is* $(\tau - \psi)$ *continuous.*

(ii) $f^{-1}(B) \in \tau$ *for each* $\beta \in B$

(iii) $f^{-1}(S) \in \tau$ *for each* $s \in S$

Proof. $(i) \Rightarrow (ii)$
Assume that (i) holds, then by definition of continuity of f, the basis $B \subset \psi$ since ψ is open. Thus any $\beta \in B \Rightarrow \beta \in U$. Since f is continuous $f^{-1}(\beta) \in \tau$ which is (ii).
$(i) \Rightarrow (iii)$
Since $S \subset \psi$, each $s \in S$ belongs to ψ thus by (i) $f^{-1}(s) \in \tau$ for each $s \in S$ which is (iii).
Conversely $(iii) \Rightarrow (ii)$
By $(iii) f^{-1}(s) \in \tau$ for each $s \in S$. Let $H \in \psi$. Since S is a sub-base of ψ which can be written as

$$H = \cup_i (S_{i_1} \cap S_{i_2} \cap \ldots\ldots \cap S_{i_{n_i}} : n_i \in \mathbb{N}) \Rightarrow f^{-1}(H) = f^{-1}(\cup_i (S_{i_n} \cap \ldots\ldots \cap S_{i_{n_i}})).$$

Each $f^{-1}(S_{i_k} \in \tau)$. Thus, $f^{-1}(H) \in \tau$ which proves (i). □

Proposition 2.4. *Let (X,τ) and (Y,ψ) be topological spaces and let $f : X \to Y$ be a function. Then the following statements are equivalent:*

(i) f *is* $\tau - \psi$ *continuous*

(ii) $f(\overline{A}) \subseteq \overline{f(A)}$ *for every subset* A *of* X

(iii) $\overline{f^{-1}(B)} \subseteq f^{-1}\overline{(B)}$ *for every subset* B *of* Y.

Proof. For every closed subset F of Y , $f^{-1}(F)$ is a closed subset of X.
Now assume that $(i) \Rightarrow (ii)$ and $f(A)$ is a closed subset of Y. So by (i) it follows that $f^{-1}(\overline{f(A)})$ is a closed subset of X containing A

$$\therefore f^{-1}(\overline{f(A)}) \supseteq \overline{A} \text{ from the condition that suppose } M \supseteq A \Rightarrow M \supseteq \overline{A} \Rightarrow \overline{A} \subseteq f^{-1}((\overline{f(A)}) \text{ where } M = \overline{f(A)}.$$

Thus

$$f\overline{(A)} \subseteq f(f^{-1}(\overline{f(A)}) = \overline{f(A)} \text{ which proves (ii).}$$

$(ii) \Rightarrow (iii)$
By (ii), $f(\overline{(A)} \subseteq \overline{f(A)}$. Let $A = f^{-1}(B)$. Substituting in (ii), we get $f(\overline{f^{-1}(B)} \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B}(f(f^{-1}(B) \subseteq (B)$. Therefore, $f^{-1}\{f(\overline{f^{-1}(B)}\} \subseteq f^{-1}\overline{(B)}$. Now $f^{-1}(f(\overline{f^{-1}(B)})) \supseteq \overline{f^{-1}(B)}$ then $\overline{f^{-1}(B)} \subseteq \overline{f^{-1}(B)}$ which is (iii).
Finally, $(iii) \Rightarrow (i)$
By (iii) $f^{-1}(B) \subseteq f^{-1}\overline{(B)}$ for every $B \in Y$. To prove (i), let B be any closed subset of Y to show that $f^{-1}(B)$ is closed in (X,τ). Therefore, $\overline{B} = B$ and $\overline{f^{-1}(B)} \subseteq f^{-1}(B)$. But $\overline{f^{-1}(B)} \supseteq f^{-1}\overline{(B)}$ always.
Thus $\overline{f^{-1}(B)} \supseteq f^{-1}(B)$. That is, $f^{-1}(B)$ is a closed subset of (X,τ) which proves (i) □

Proposition 2.5. *Let (X, τ) and (Y, ψ) be topological spaces and $f : X \to Y$ be such that:*

(i) f is a bijection.

(ii) f is $(\tau - \psi)$ continuous.

(iii) f is also continuous

Then f is called a homeomorphism from the topological space (X, τ) to (Y, ψ) and f^{-1} is a homeomorphism from the topological space (Y, ψ) to (X, τ) and the two spaces (X, τ) and (Y, ψ) are said to be homeomorphic written $(X, \tau) \sim (Y, \psi)$

The open set properties using bases give rise to countability property in Hausdorff topology.

Definition 2.6. *Let (X, τ) be a topological space and $p \in X$. A family B_p of open sets containing p is called a **local base** at p if for each open set G containing p there is a member $\beta_p \in B_p$ such that $p \in B_p \subseteq G$.*

Now a topological space (X, τ) is said to be first countable if it satisfies the following axiom of countability:
At each point $p \in X$ there is at most countable family B_p of open sets containing p, that is, $\beta \in B \in G$.
In other words, each point $p \in X$ is associated with an **atmost countable local basis.**

Example 2.2.

1) *Take any metric space (X, ρ). Let $p \in X$. Take $B_p = \{\mathcal{N}(P : 1/n), n \in \mathbb{N}\}$. $B_p$is countable.*
If G is open and $p \in G$, then there exists a member $\beta \in B_p$ such that $x \in \beta_p \in G$
$\therefore$ *(X, ρ) is a first countable space.*

2) *Take X and the discrete topology $\tau = P(X)$. Each $\{p\}$ is open $\forall p \in X$. Take $B_p = \{\{p\}\}$. So B_p is at most countable.*
For each open G containing p, we have $p \in \{\{p\}\} \in G$. Therefore, $(X, P(xX))$ is first countable.

Definition 2.7. *A topological space (X, τ) is said to be **second countable** if it has an at most countable base.*

Example 2.3. *$(\mathbb{R}, \tau)$ where $\tau = d$ the usual topology on $\mathbb{R}$ is second countable.*
Reason: $\{\mathcal{N}(q : 1/n) : q \in \mathbb{Q}\}$ is an open at most countable neighbourhood.

Proposition 2.6. *Let X be a connected Topological space. Then X is connected if its Hausdorff but the converse is not true.*

Proof. Assume that X is not a connected Topological space. If $x \in X$, then we have a neighbourhood $N_x(x, r_1)$ and $N_y(y, r_2)$ such that $x \in N_x(x, r_1)$ and $y \in N_y(y, r_2)$.
From the definition of connectedness, if we have two open sets U and V then $U \cap V = \Phi$.
This implies that $N_x(x, r_1) \bigcap N_y(y, r_2) = \Phi$ implying that separation holds which is a contradiction. Hence it is a connected space.
Alternatively, suppose $x \in X$ and suppose that X is not connected. Consider $X = x, y, z$ and $\theta = (x, y)(y, z)$ such that θ is the base of the topology τ meaning that $\theta \in \tau$ and hence $(x, y) \cup (y, z) \in \tau$. This implies that x, y, z covers the entire space in τ This a contradiction and hence the space is connected.

Conversely, suppose X is a Hausdorff Topological space. There are two distinct points x and y such that $x, y \in X$; there exists two open sets G and H such that $x \in U$ and $y \in V$ and $U \cap V = \Phi$. This a contradiction for connectedness since a connected space, the space cannot be split into two disjoint. Hence the converse does not hold

□

Definition 2.8. *A first countable Hausdorff space X is a topological space whereby each point has a countable neighborhood and the space also obeys the separation axiom. That is every two distinct points have two disjoint open neighborhoods.*

Lemma 2.1. *Let X be a first countable Hausdorff space. Then X is a T_1 space. In other words every singleton set is closed.*

Proof. Let X be a first countable Hausdorff space and let $x \in X$ where x is an arbitrary point. We want to show that for every two distinct points say $x, y \in X$ there exists open sets U, V; $x \in U$ and $y \in V$ such that $x \notin V$ and $y \notin U$.

Since X is first countable, for each point $x \in X$ there exists a countable neighborhood basis $B_x = \{U_n\}_{n=1}^{\infty}$ at x where B_n is an open set containing the arbitrary point x. Similarly for each point $y \in X$ there exists a countable neighborhood basis $B_y = \{V_n\}_{n=1}^{\infty}$ at y where B_n is an open set containing the arbitrary point y. Since B_n is an open set containing x and y then there is a neighborhood N of x such that $B_n \subset N \; \forall \, n \in N$. Now consider the singleton sets of $\{x\}$ and $\{y\}$. Defining the two open sequences of open sets we will have;

1. For each n, let $U_n = X \setminus \{y\} \cup \{x\}$
2. For each n, let $V_n = X \setminus \{x\} \cup \{y\}$

Its clear from above that each U_n contains x but not y and same thing happens to V_n. We want to show that the two sequences are of open sets.

Since X is a Hausdorff space, then for every two distinct points $x, y \in X$ there exist two open disjoint sets U, V; $x \in U$ and $y \in V$ such that $U \cap V = \Phi$. Let $x, y \in B_n$ and $U \cap V = \Phi$. This implies that $X \setminus \{x\} \cap V = U \cap V = \Phi : \forall x, y \in X$, and $X \setminus \{y\} \cap U = U \cap V = \Phi : \forall x, y \in X$.

Since $X \setminus \{x\}$ is the set of union of points that do not intersect V_y it then follows that $X \setminus \{x\}$ is open. Similarly $X \setminus \{y\}$ is the set of union of points that do not intersect U_x it then follows that $X \setminus \{y\}$ is open.

Also since U contains x but not y and V contains y but not x for all $x, y \in X$, then x is T_1space. Therefore we have shown that for any point x in the first countable Hausdorff space the singleton set $\{x\}$ and $\{y\}$ are closed and hence satisfying the T_1 separation axiom. □

Theorem 1. *Every Hausdorff space is a T_1 space, but the converse is not true.*

Proof. To prove that every Hausdorff space is a T_1 space, we start by recalling the definitions of these topological properties. A topological space X is termed a **Hausdorff space** (or T_2) if, for any two distinct points x and y in X, there exist disjoint neighborhoods U of x and V of y. On the other hand, a space is called a T_1 **space** if for any two distinct points x and y, there exists a neighborhood of x that does not include y (and vice versa).

Now, let us assume X is a Hausdorff space and consider any two distinct points x and y in X. By the definition of a Hausdorff space, we can find neighborhoods U of x and V of y such that $U \cap V = \Phi$. This implies that y cannot be contained in the neighborhood U of x. Consequently, we have established that every neighborhood of x excludes y. Similarly, since the neighborhoods are disjoint, x cannot be included in the neighborhood V of y. Thus, every neighborhood of y excludes x. Therefore, we conclude that X satisfies the T_1 separation condition.

Next, we demonstrate that the converse is not true by providing a counterexample. Consider the **Sierpiński space**, which consists of two points, $\{a, b\}$, with the topology $\{\Phi, \{b\}, \{a, b\}\}$. In this topology, the singleton set $\{b\}$ is a neighborhood of b that does not contain a, thus satisfying the T_1 condition. However, there are no disjoint neighborhoods around the points a and b; any open set that includes a must also include b. This means that the space does not satisfy the Hausdorff condition.

In conclusion, while every Hausdorff space is indeed a T_1 space due to the ability to separate points with disjoint neighborhoods, the converse fails, as illustrated by the Sierpiński space, which is T_1 but not Hausdorff. □

Lemma 2.2. *Let (X, τ) be a Topological space and (Y, φ) be a Hausdorff space. If A is a non-empty subset of X; then $f : X \to Y$ is continuous.*

Proof. If $f : X \to Y$ is continuous, we expect that f is an injection and there exists an open subset $A \subset X$ such that $f(A) \subset Y$. Also $\{f^{-1}(y) : y \in Y\}$
$subset A.$

Since X is Hausdorff, Let $A = N_x(x, r)$ where $x \in X$. If there exists $G \subseteq X : x \in N_x \subseteq G$. Then A is □

Lemma 2.3. *Let X be a first countable Hausdorff space. Each minimal base $B \in X$ is closed and can be embedded.*

3 On Some Classes of Bases in Hausdorff Topology

3.1 Global Bases

Definition 3.1. *Let (X, τ) be a topological space. A sub-family B of τ is called a* ***base or basis*** *for the topology τ if each open set G of X (ie $G \in \tau$) can be expressed as a union of members of B. In other words for each $G \in \tau$ and $x \in G$, there exists some $\beta \in B$ such that $x \in B \subset G$.*

From the definition 3.1 above, we have two statements which are equivalent:

(i) Let G be a union of members of B : ie $G = \cup B_i : B_i \in B, i \in \mathbb{N}$.

(ii) Let $x \in G$. Then $x \in B_i$ for some i and $B_i \subset G$ that is $x \in B_i \subset G$ which is a statement.

Remark 3.1. *If B is a base for the topology τ of X, then B is itself a part of τ ie members of the base B are themselves open sets and subsets of (X, τ).*

Proposition 3.1. *Not every collection of subsets of X can serve as a base for a topology on X*

Proof. Consider $X = \{a, b, c\}$ and $\theta = \{\{a, b\}, \{b, c\}\}$. Claim that θ serves as a base for a topology τ on X then $\theta \in \tau$. Hence $\{a, b\} \cup \{b, c\} \in \tau$ and consequently $\{a, b\} \cap \{b, c\} = \{b\} \in \tau$. Thus this intersection $\{b\}$ must be expressed as a union of members from θ, which is impossible. Thus θ cannot serve as a base for a topology on X. □

Proposition 3.2. *Let X be a non void set and B be a collection of subsets of X. The following statements are equivalent: If B is a base for a topology on X, then B satisfies the following properties:*

(i) $\{\bigcup \beta : \beta \in B\} = X$

(ii) If β_1 and β_2 are members of B, then $\beta_1 \cap \beta_2$ is a union of members of B

Proof. Assume that B serves as a base for a topology τ on X. Then $B \in \tau$, so every $G \in \tau$ is a union of members of B. Now X is a member of τ, thus $X = \{\bigcup \beta : \beta \in B\}$ and this proves (i).

Next, let β_1 and $\beta_2 \in B$. So $\beta_1, \beta_2 \in \tau$ for $B \in \tau$. Therefore, $\beta_1 \cap \beta_2 \in \tau$ and B is a base for τ. Hence, $\beta_1 \cap \beta_2$ = a union of members of B which proves property (ii) □

Proposition 3.3. *Let (X, ρ) be a metric space, for each $p \in X$, $\epsilon > 0$ and let $\mathcal{N}(p : \epsilon)$ be the set $\{x \in X : \rho(x, p) < \epsilon\}$. Let B be a collection of all such sets $\mathcal{N}(p : \epsilon)$. Then B is a base for the metric topology on X.*

Proof. Now $\bigcap\{\beta : \beta \in B\} = \cup\{\mathcal{N}(p : \epsilon) : p \in X, \epsilon > 0\} \subset X$.

Conversely, if $x \in X$ and $\epsilon > 0$, then $x \in \mathcal{N}(x : \epsilon) \subset \cup\{\mathcal{N}(p : \epsilon) : p \in X, \epsilon > 0\}$ =$\cup\{\beta : \beta \in B\} \Rightarrow X \subset \{\beta : \beta \in B\}$. Thus $X = \{\beta : \beta \in B\}$
So the family B satisfies the requirement (i) to qualify to be a base.

To prove(ii):
Take any two members of B, say $\mathcal{N}_1(p : r_1)$ and $\mathcal{N}_2(q : r_2)$ where $p, q \in X$ and $r_1, r_2 > 0$ and its real.
If $\mathcal{N}_1 \cap \mathcal{N}_2 = \Phi$, then this is a void union of members of B. Otherwise let $\mathcal{N}_1 \cap \mathcal{N}_2 = \Phi$ and let $x \in \mathcal{N}_1(p : \epsilon) \cap \mathcal{N}_2(q : r_2)$
So $x \in \mathcal{N}_1$ and $x \in \mathcal{N}_2$. Therefore, $\rho(x : p) < r_1$ and $\rho(x : q) < r_2 \Rightarrow r_1 - \rho(x : p) = \delta_1 > 0$ and $r_2 - \rho(x : q) = \delta_2 > 0$.

Let $\delta = \min[\delta_1, \delta_2] \Rightarrow \delta > 0$, we claim that $\mathcal{N}(x : \delta) \subset \mathcal{N}_1(p : r_1), \mathcal{N}_2(q : r_2)$
We first prove that $\mathcal{N}(x : \delta) \subset \mathcal{N}_1(p : r_1)$. Let $y \in \mathcal{N}(x : \delta)$ that is, $\rho(y : x) < \delta$

$\therefore \rho(y : p) \leqslant \rho(y : x) + \rho(x : p)$[Triangle inequality in a metric space]
$< \rho + \rho(x : p)$
$\leqslant +\rho(x : p) = r_1$.

Therefore, $\rho(y : p) < r_1$ ie $y \in \mathcal{N}_1(p : r_1)$. Thus $y \in \mathcal{N}(x : \delta) \Rightarrow y \in \mathcal{N}_1(p : r_1)$ i.e. $\mathcal{N}(x : \delta) \subset \mathcal{N}_1(p : r_1)$.

We can similarly show that $\mathcal{N}(x : \delta) \subset \mathcal{N}_2(q : r_2)$. Thus $\mathcal{N}_1 \cap \mathcal{N}_2$ is a union of members of B which proves part (2) of(ii). Thus B is a base for a topology τ on X. □

Definition 3.2. *Let (X, τ) be a topological space. A family $S \subset \tau$ is called a* ***sub-base*** *for the topology τ if the collection of all finite intersection of members of S forms a base for τ*
Thus every open set $G \in \tau$ is a union of a finite intersection of members of S ie

$G = \cup(S_1 \cap S_2 \cap .... \cap S_n)$

Proposition 3.4. *Any nonvoid collection of subsets of a nonvoid set X serves as a sub-base for a unique topology called the topology generated by a. This topology is also the intersection of all the topologies on X that contain a.*

Proof. Let B be the collection of all finite intersections of members of a. We show that β is a base for a topology on X. Consider the void intersection of members of a. The latter is X thus $X \in \beta$ and $\Phi \in \beta$.

$\therefore \cup\{B : B \in \beta\} = X$.

Next, let B_1 and $B_2 \in \beta$, then by definition of β

$B_1 = S_1 \cap ... \cap S_{n_i} : S_i \in a$
$B_2 = S_1 \cap ... \cap S_{n_i} : S_i \in a$

Thus $B_1 \cap B_2 = (S_1 \cap ... \cap S_n) \cap (S_1' \cap .... \cap S_n') \in \beta$

$\therefore \beta$ is base for a topology on x. This topology is generated by a and a is a sub base for τ.

Next, denote this topology by $\tau(a)$. We need to show that $\tau(a)$ is the intersection of all topologies on X which contains the given collection a.

Let $\tau_\alpha : \alpha \in \Omega$ be the collection of all topologies on X containing a. Clearly this collection is non void for $p(x)$, the discrete topology contains a.

Now $\cap\tau_\alpha$ is a topology on X which contains a. So $\tau(a)$ is a topology containing a.

$\therefore \tau(a) \supseteq \cap\tau_\alpha : \alpha \in \Omega$ $\quad (i)$

To show the reverse inclusion : $\tau(a) \subseteq \cap\tau - \alpha : \alpha \in \Omega$ $\quad (ii)$

Let $G \in \tau(a) \Rightarrow G = i \cup (S_i \cap S_i') : S_i, S_i' \in a \subset \tau_\alpha \forall \alpha \in \Omega$

Since $S_i \cap S_i' \in \tau_\alpha \Rightarrow \cup(S_i \cap S_i') \in \tau_\alpha$ i.e. $G \in \cap\tau_\alpha : \alpha \in \Omega$

Thus $G \in \tau(a) \Rightarrow G \in \tau_\alpha : \alpha \in \Omega$ which proves (ii). Hence, $\tau(a) = \cap\{\tau_\alpha : \tau_\alpha \subseteq a\}$ □

Example 3.1. *Let $X = \{a, b, c, d\}, Z = \{\{a\}\{a, b\}, \{b, c\}\}$. Find the topology generated by Z*

Proof. First we find the set of all finite intersections of members of a. This gives the basis:

$\{X, \{a\}, \{a, b\}, \{b, c\}, \{b\}\} = \beta$.

Now, taking arbitrary union of members of β. This gives the topology generated by Z , that is $\tau(Z)$.

$\tau(a) = \{X, \Phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ □

3.2 Minimal Generative Bases and Sub-bases of First Countable Topological Spaces

We introduce the notion of minimal generative local bases and use them to characterize some classes of quotient images a kin to first countable Hausdorff topological spaces.

Definition 3.3. *A topological space (X, τ) on a base B which induces reducible conditions on general space, is called a minimal generative base.*

Example 3.2. *Consider the well known real line $\mathbb{R}$ induced with a set $B = \{(x, -x) : x \in \mathbb{R}\}$. Here we can consider arbitrary open sets, obtained from B with: $B = \{(x, -x) \cup \Phi : x \in \mathbb{R}\}$ and $\bigcup G_x = B$:*

$$x = \begin{cases} \frac{1}{n}\ 0 < x < 1; \\ n\ a > 1 \end{cases}$$

$$-x = \begin{cases} \frac{-1}{n}\ 1 < x < 0; \\ -n\ x < 1 \end{cases}$$

We can associate the set B above with a topology $T = \{\Phi, (x, -x)\}$

Lemma 3.1. *Let X be a Quotient image space and B_M a family of open sets. Then we say that B_M is the minimal generative base for the topology of X if:*

i B_M covers X

ii *If $A, B \in B_M$ then there exists a subfamily $\{B_i : i \in \mathbb{N}\}$ of B_M such that $A \cap B = \bigcup_{i\in\mathbb{N}} B_i$*

iii *If a subfamily $\{B_{i\in\mathbb{N}} \mid i \in N\} \subset B_M$ substantiates $\bigcup_{i\in\mathbb{N}} B_i \in B_M$ then, there exist $i_\circ \in \mathbb{N}$ such that $\bigcup_{i\in\mathbb{N}} B_i = B_{i_\circ}$*

Example 3.3. *. Let $B_M = \{(-x, x) \cup \Phi; \forall x \in \mathbb{R}\}$ and $\tau = \{(x, -x), \Phi, \mathbb{R}\}$. Then $\bigcup B_M = B_M \cup \tau = \mathbb{R}$. For any subsets $A, B \in B_M; A \subset B$ or $B \subset A$. Now assume that $A \subset B$. Then there exists a subfamily $\{B_i : B_i \subset A, i \in \mathbb{N}\}$ such A is a refinement of $A \subset B$ so that $A \cap B = A = \bigcup\{B_i : B_i \subset A, i \in \mathbb{N}\}$. If this sub family qualifies B_M to be a base, then $B_M = \bigcup\{B_i : i \in \mathbb{N}\}$ because clearly $\bigcup\{B_i : i \in \mathbb{N} \in B_M$ and $B_M \subset \bigcup\{B_i : i \in \mathbb{N}\}$. Thus the base B_M satisfies the conditions of the previous lemma 3.1)*

In the next result, we demonstrate the hereditary condition for global bases to be minimal generative.

Theorem 2. *Let X be a set and B_M a family of non-empty subsets of X. Then B_M is the minimal generative base for a topology of X if and only if:*

i *B_M covers X; that is $X = \bigcup_{M\in\mathbb{N}} B_M$*

ii *If $A, B \in B_M$, then there exists a subfamily $\{B_i : i \in \mathbb{N}\}$ of B_M such that $A \cap B = \bigcup_{i\in\mathbb{N}} B_i$.*

iii *If a subfamily $\{B_i : i \in I\}$ of B_M satisfies $\bigcup_{i\in\mathbb{B}} B_i$ then there exists $i_o \in \mathbb{N}$ such that $\bigcup_{i\in\mathbb{N}} B_i = B_{i_0}$*

Proof. Suppose B_M is a minimal generative base for a topology in the space X. Then the conditions (i) and condition (ii) of 2 have to hold. Assume a subfamily $B_{M_1} = \{B_i : i \in \mathbb{N}\}$ of B_M satisfies $\bigcup_{i\in\mathbb{N}} B_i \in B_M$ but $\bigcup_{i\in\mathbb{N}} B_i \neq B$ for each $B \subset B_M$.
Therefore, there exists a subset $B' \in B_M \setminus B_i$ such that $\bigcup_{i\in\mathbb{N}} B_i = B'$. That implies that $B_M \in B'$ is a base for the topological space X. This is a contradiction and thus condition (iii) holds. Conditions (i) and (ii) states that B_M is a base for a topological space X which is true from above.
If a subfamily B'_M of B_M is a base of the topological space X, then for every subset $B \in B_M$, there exists a subfamily $\{B_i : i \in \mathbb{N}\}$ ofB'_M such that $\bigcup_{i\in\mathbb{N}} B_i = B_{i_0}$. In relation to condition (iii), there exists $i_0 \in \mathbb{N}$ such that $\bigcup_{i\in\mathbb{N}} B_i = B_{i_0}$. Hence $B = B_{i_0} \in B'_M = B_M$. Therefore, B_M is minimal generative base for topological space X. □

Definition 3.4. *Let X be a topological space and B_M a family of non-empty subsets of X. Then for each subset $B \in B_M$,*

i *if there exists a subfamily $\{B_i : i \in \mathbb{N}\}$ of B_M such that $B \notin \{B_i : i \in \mathbb{N}\}$ and $\bigcup_{i\in\mathbb{N}} B_i = B$ then B is known as a union reducible element with respect to B_M;*

ii *If there exists a subfamily $B_i : i \in \mathbb{N}$ of B_M such that $B \notin \{B_i : i \in \mathbb{N}\}$ and, $\bigcap_{i\in\mathbb{N}} B_i = B$ then B is known as an intersection reducible element with respect to B.*

Remark 3.2. *A minimal generative local base for a first continuous topological space may fail to have a union reducible candidate and still poses the intersection reducible elements. This property has been applied before. See for example in [12] to qualify the minimality of an arbitrary base say B_N with a well known countable set which is a topological space with respect to a respected topology τ. Indeed the classification into reducibility using sub-bases usually defined over the intersections of union of bases all over for a consideration of only intersection reducibility for local minimal generative bases considered for the quotient of first countable Hausdorff topological spaces*

Next we characterize the necessary and sufficient conditions for minimal generative sub-bases.

Definition 3.5. *Let S be a sub-base for a topological space X for any subfamily S' of S. If S' is a sub-base for the topological space X, then S' = S.*

The following example helps us to illustrate the definition 3.5 ;

Example 3.4. *Let X =$\mathbb{R} \setminus Z$ and $S = \{(a, a+1) : a \in \mathbb{Z}\}$. Then, S is a partition of $\mathbb{R} \setminus \mathbb{Z}$. This therefore implies that S is a minimal generative sub-base for a topology of $\mathbb{R} \setminus \mathbb{Z}$.*

Proposition 3.5. *Let X be a set and S a family of non-empty subsets of X. If S is a minimal generative sub-base for a topology of X, then there doesn't exist a finite intersection reducible element in the minimal generative sub-base S.*

Proof. By definition 3.4 above, we need to show that if a finite subfamily $\{S_i : i \in \mathbb{N}\}$of S satisfies $\bigcap_{i\in\mathbb{N}} S_i \in S$, then there exists $i_0 \in \mathbb{N}$ such that $\bigcap_{i\in\mathbb{N}} S_i$ =S_{i_0} . Now assume that a finite subfamily $S = \{S_i \mid i \in \mathbb{N}\}$ of S satisfies $\bigcap_{i\in\mathbb{N}} S_i \in S$, but $\bigcap_{i\in\mathbb{N}} S_i \in I \neq S$ then for each subset $S \in S$. Therefore there exists a subset $S\prime \in S \setminus S_1$such that $\bigcap_{i\in\mathbb{N}} S_i = S\prime$. Thus this implies that $S \in \{S\}\prime$ is a sub-base for the topology of X of which is a contradiction. Therefore, the minimal generative sub-base S doesn't contain the finite intersection reducible elements.

From the following proposition, it is evident that a minimal generative sub-base is a minimal generative base with all finite intersection reducible elements removed. according to the following proposition. □

Proposition 3.6. *Let X be a topological space and B_M be a minimal base for the Topological Space. If there are no finite intersection reducible elements in B_M, then B_M is a minimal generative sub-base for the Topological Space X.*

Proof. Clearly, B_M can be viewed as a sub-base for the topological space$X$. Suppose a subfamily l of B_M is a sub-base for the topological space X. According to the definition of base and sub-base, B_M is a sub-base, B_M is a minimal base generated by S_M. Then for each subset $B \in B_M$, there exists a finite subfamily $\{S_i i \in I\}$ of S_M such that $\bigcap_{i\in\mathbb{N}} S_i = B \in B_M$. There exists $i_0 \in I$ such that $\bigcap_{i\in\mathbb{N}} S_i = S_{i_0}$, because the minimal base B_M does not have finite intersection reducible elements. Thus, $B = S_{i_0} \in S_M$, which means $S_M = B_M$. Therefore, B_M is a minimal generative sub-base for the topological space X

□

Proposition 3.7. *Suppose X is a set and is a Y topological space. Then if $f : X \to Y$ is a surjection map and S_M is a minimal generative sub-base for the topological space Y, the $f^{-1}(S_M) = \{f^{-1}(S) : S \in S_M\}$. is a minimal sub-base for a topology on X*

Proof. Given that S_M is a minimal generative sub-base of the topological space Y, then $Y = \bigcup S_M$ and we $X = \bigcup f^{-1}(S_M)$. Hence, $f^{-1}(S_M)$ is a sub-base for the topology τ on X. Suppose $f^{-1}(S_M)$ is not a minimal generative sub-base for the topological space (X, τ), then there exists a $S_0 \in S_M$ such that $f^{-1}(S_M) \setminus f^{-1}(S_0$ is a sub-base for the topological space (X, τ). Given that $S_M \setminus (S_0)$, is not a sub-base for the topological space Y, then there exists an open subset V of Y and $y \in V$ such that for any finite family $F_M \subset S_M \setminus (S_0), y \bigcap F_M \subset V$ does not hold. Take $x \in f^{-1}(y)$. Because $f^{-1}(V)$ is open in the topological space (X, τ) and $f^{-1}(S_M)\{f^{-1}(S_0)$ is a sub-base for (X, τ), there exists a finite family $F_M \subset S_M \setminus S_0$ such that $x \in \bigcap\{f^{-1}(F) : F \in F_M\} \subset f^{-1}(V)$. Hence, $y \in \bigcap\{F : F \in F_M\} = \bigcap F_M \subset V$. Thus, $f^{-1}(S_M)$ is a minimal generative sub-base for a topology on X □

Based on proposition 3.7, we obtain the following remark.

Remark 3.3. *Let Y be a quotient space of a topological spaces X and let $f : X \to Y$ be a quotient mapping. If a covering S_M is a minimal generative sub-base for the quotient space Y, then $f^{-1}(S_M) = \{f^{-1}(S) : S \in S_M\}$ is a minimal generative sub-base for the topological space X*

Proposition 3.8. *Let X and Y be topological spaces and a mapping be $f : X \to Y$ open and 1-1. If a covering S_M is a minimal generative sub-base for the topological space X, then $f(S_M) = \{f(S) : S \in S_M\}$ is a minimal generative sub-base for the subspace $f(X)$ of Y*

Proof. It's obvious that, $f(S_M) = \{f(S) : S \in S_M\}$ is a cover of $f(X)$. Because the mapping $f : X \to Y$ is open and a sub-base for the topological space $X, f(F_M)$ is a sub-base for the topological space $f(X)$. We now show that, $f(S_M)$ is minimal generative base. Now lets suppose that $f(S_M)$ not minimal generative sub-base for $f(X)$. Therefore there exists $S_0 \in S_M$ such that $\{f(S) : S \in S_M \setminus \{S_0\}\}$ is a sub-base for $f(X)$. Since S_M is a minimal generative sub-base for the topological space $X, S_M \setminus \{S_0\}$ is a sub-base for the topological space X. Therefore, there exists an open subset V_0 of X and $x \in V_0$ such that for any finite family $F_M \subset S_M \setminus \{S_0\}, \bigcap F_M \subseteq V_0$ if $x \in \bigcap F_M$. Now since the mapping is open, $f(V_0)$ is an open subset of $f(X)$. And since f is 1-1, $f(\bigcap F_M) \subseteq f(V_0)$ for any finite family $F_M \subset S_M \setminus S_0$ with $x \in \bigcap F_M$ Thus, $\bigcap\{f(F) : F \in F_M\} \subseteq f(V_0)$ for any finite family $F_M \subset S_M \setminus \{S_0\}$ with $x \in \bigcap F_M$. This is a contradiction that $f(S_M) \setminus \{f(S_0)\}$ is a sub-base for $f(x)$ □

Next some propositions have been presented as mapping properties.

Proposition 3.9. *Let $\{X_y\} : y \in \Gamma$ be a family of topological spaces. Then for each index such that $y \in \Gamma, \exists \pi_y : \pi_{y\in\Gamma} X \to X_y$ is a projective mapping and for each index $y \in \Gamma, S_{M_y}$ in that a minimal generative sub-base for topological space X, then $S_M = \{\pi_y^{-1}(S_y) : S_y \subset S_{M_y} \in \Gamma\}$ is a minimal generative sub-base for the product space $\pi_{y\in\Gamma} X_y$.*

Proof. It is clear that $S_M = \{\pi_y^{-1}(S_y) : S_y \subset S_{M_y} \in \Gamma\}$ is a sub-base for the product space $\prod_{y\in\Gamma} X_y$. Suppose S_M is not a sub-base of the product space $\pi_{y\in\Gamma} X_y$, there exist a $y_0 \in \Gamma$ and $S_{y_0} \in S_{M_{y_0}}$ such that $S_M \setminus \{\pi^{-1 y_0}(S_0)\}$ is a sub-base of the product space $\prod_{y\in\Gamma} X_y$. Hence, $S_{M_{y_0}} \setminus \{S_{y_0}\}$ is a sub-base of the space X_{y_0}. This is a contradiction that $S_{M_{y_0}}$ is a minimal generative sub-base for X_{y_0} □

Proposition 3.10. *If $\{X_y\} : y \in \Gamma$ be a family of pairwise disjoint topological spaces. For each index $y \in \Gamma$, if S_{M_y} is a minimal generative sub-base for a topological space (X_y, τ_y), then $S_M = \{S_y : S_y \in S_{M_y}, y \in \Gamma\}$ is a minimal generative sub-base for the sum of the spaces $\{X_y\}_{y_\Gamma}\}$*

Proof. If $(\oplus, X_y)$ is the sum of the spaces $\{X_y\}_{y\in\Gamma}$, let $B_{M_y} = \{\bigcap_{S\in\Gamma} S_y : S'_{M_y}\}$ that is a finite subfamily of S_{M_y} denote the base generated by S_{M_1}. It can be seen that $B_M = \{B_y : B_y \in B_{M_{y,y\in\Gamma}}\}$ is a base generated by S_M. For each open set $U \in \tau$ and each point $x \in U$, then there exists an index $y \in \Gamma$ such that $x \in X$. This therefore follows that $x \in U \cap X_y$ and there exists a subset $\bigcap S_{y\in S_{M_y}} S_y \in B_{M_y} \subset B_M$ such that $x \in \bigcap S_y$. Then $x \in S_{y\in S_M} S_y \subset U \cap X_y$. Therefore there exists a subset $\bigcap S_{y\in S_{M_y}} S_y \subset B_M$ such that $x \in \bigcap S_{y\in S_{M_y}} S_y \in B_{M_y} \subset B_M$ such that $x \in \bigcap S_{y\in S_{M_y}} S_y \subset U \cap X_y$ is open in $X_y, x \in \bigcap_{S_{y\in S_{M_y}}} S_y \subset U$ meaning that B_M is a base for the topological space $(\oplus_{y\in\Gamma}, X_y, \tau)$. Thus, S_M is a sub-base for the topological space $(\oplus_{y\in\Gamma}, X_y, \tau)$ Now suppose S_M is not a minimal generative sub-base for the topological

space $\oplus_{y\in\Gamma}X_y,\tau$. Then there exists $y_0 \in \Gamma$ and $S_{y_0} \in S_{M_{y_0}}$ such that $S_M \setminus (S_{y_0})$ is a sub-base of the topological space $\oplus_{y\in\Gamma}X_y,\tau$. Hence, $S_{M_y} \setminus \{S_y\}$ is a sub-base of X_{y_o}. This is a contradiction and hence S_M is a minimal generative sub-base for the Topological Space $\oplus_{y\in\Gamma}X_y,\tau$.
Let X be a Topological Space and Y be a subset of X which is a Topological subspace of X. If S_M is a minimal generative sub-base for the topological space X, then $S_M - y$ is a sub-base for the topological subspace Y. But $S_M - y$ may not be a minimal generative sub-base for the subspace. The following example is used to illustrate this point.

Example 3.5. *Let $a \in X$ and $X = \{a_1, a_2, \cdots a_9\}$*

$$S_M = \{\{a_1, a_2, a_4, a_5\}, \{a_2, a_3, a_5, a_6\}, \{a_4, a_5, a_7, a_8\}, \{a_5, a_6, a_8, a_9\}\}$$

be a sub-base for a topological space X. It is easy to see that S_M is a minimal generative sub-base for the topological space X. Take $Y = \{a_1, a_2, a_4, a_5, a_7\}$. According to the definition of a topological sub-space, we conclude that $S_M - y = \{\{a_1, a_2, a_4, a_5\}, \{a_2, a_5\}, \{a_4, a_5, a_7\}, \{a_5\}\}$ is a sub-base for the subspace Y. However, $S_M - y$ is not a minimal generative sub-base for the subspace Y, because $\{a_5\}$ is a finite intersection reducible elements with respect to S_{M_y}.

□

3.3 More Results on First Countable Hausdorff Spaces

We call a topological space X a first countable Hausdorff space if and only if each point in X has a unique minimal open neighborhood. Moreover, for the first countable Hausdorff space there exist a base B_M such that B_M is composed of all the minimal open neighborhoods of each point in X. With no doubt B_M is the unique minimal base for the quotient space X. By proposition 3.6 a minimal generative sub-base can be obtained from a minimal generative base.

Proposition 3.11. *For any first countable Hausdorff space, the minimal generative sub-base is not unique We can illustrate this the following using example;*

Example 3.6. *Consider $X = \{a_1, a_2, a_3, a_4\}$*

$$S_M = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_1, a_4\}, \{a_2, a_4\}, \{a_1, a_3\}\}$$

be a sub-base for the Topological Space $X(\tau)$.

From the topological space (X, τ), the following Minimal generative sub-bases can be derived.

$$F_M S_1 = \{\{a_1, a_2\}, \{a_3, a_4\}, \{a_2, a_4\}, \{a_1, a_3\}\}$$
$$F_M S_2 = \{\{a_2, a_3\}, \{a_1, a_4\}, \{a_3, a_4\}, \{a_1, a_4\}\}$$
$$F_M S_3 = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_1, a_4\}\}$$

Remark 3.4. *Let X be a topological space. If B_M is a base for the topological space X and S_M is a sub-base for the topological space X. $N_\tau(x) = N_{B_M}(x) N_{S_M(X)}$ can be derived according to the definitions of base and sub-bases. In addition, if (X, τ) is a first countable Hausdorff space, then $N_\tau(x)= \bigcap\{U \mid x \in U \in \tau\}$ is the unique minimal open neighbourhood of each point $x \in X$. . Now denote $N_\tau(x)$ by $N(x)$ and let $S_M = \{N(X) \mid x \in X\}$ which is the unique minimal generative base for the first countable Hausdorff space being a partition*

By our previous example, there exists an intersection reducible elements in a minimal generative base. As a special case, if a minimal generative base is a partition, then there is no any intersection reducible element. Hence when a minimal generative base is a partition, then there is a discrete topological space. To help us get a better understanding of a minimal generative base of first Countable space that is a partition a proposition has been provided.

Proposition 3.12. *Let X be a first countable Hausdorff space. For two points $x, y \in X$, define xRy if $N(X)$ =$N(y)$. Then the natural quotient space $X \setminus R$ is a discrete space if and only if the minimal generative base B_M is a partition for the first countable Hausdorff space X*

Proof. Suppose $X \setminus R$ be quotient mapping that is also natural and Suppose $X \setminus R$ is a discrete space, then for each point $x \in X, \{[x]\}$ is a singleton in $X \setminus R$ and $p^{-1}([x])$ is open in X. Thus $N(X \subset p^1([x])$. For each point $y \in p^{-1}([x]), xRy$ shows that $N(x) = N(y)$ that is $y \in N(x)$. Then $p^{-1}([x]) \subset N(x)$, so $p^{-1}([x]) = N(x)$. Thus for any two minimal open neighbourhoods $N(x), N(y) \in B_M$ and $N(x) \neq N(y)$ if $N(x) \cap N(y) \neq 0$, then there exists a point $a \in X$ such that $a \in N(x) \cap N(y)$. That is $a \in N(x)$ and $a \in N(y)$. So $a \in p^{-1}([x])$ and $a \in p^{-1}([y])$ that is xRaans yRa. Then xRy because R is an equivalence relation, which means $N(x) = N(y)$. Therefore, for any elements $N(x), N(y) \in B_M$, we have that $N(x) = N(y)$ or $N(x) \cap N(y) = \varnothing$ that is ,B_M is a partition
Because B_M is a partition, $N(x)$= $N(y)$ if $N(X) \cap N(y) = \varnothing$. Then for each point $y \in N(x), N(X) = N(y)$, that is xRy. Thus, $y \in p^{-1}([x])$ which implies that $N(x) \subset p^{-1}([x])$. If $yinp^{-1}([x])$, then xRy which means $N(X) = N(y)$. So $p^{-1}([x]) \subset N(X)$. Therefore, $p^{-1}([x])$= $N(x)$ is open in X. Since p is a quotient mapping that is also natural, then $\{[x]\}$ is open in $X \setminus R$. Hence $X \setminus R$ is a discrete space ☐

Next we present results about locally connected spaces and locally pathwise connected spaces

Proposition 3.13. *Let X be a first countable Hausdorff space:*

i *A space X is said to be locally connected space if and only if each point $x \in X$, there is a minimal open neighborhood of the point x that is a connected set*

ii *A space X is said to be locally pathwise connected space if and only if for each point $x \in X$, then the minimal open neighborhood of the point x is also pathwise connected set*

Proof. (i) Let X be a locally connected space. Then for each neighbourhood U of x such that $x \in X$, there exists a connected neighborhood V such that $x \in V \subset U$. This implies that the minimal open neighborhood of the point x is a connected set. Let $x \in X$, then the minimal open neighborhood $N(x)$ of the point x is a connected set. For each neighborhood U of the point $x, N(x) \subset U$. Thus, X is a locally connected space.
The proof for (ii) is similar to (i)

For our first countable Hausdorff space, based on the uniqueness of minimal base, two results about sub-base and minimal generative sub-base are presented. ☐

Lemma 3.2. *Let τ, τ' be two topologies on X such that X is a first countable Hausdorff space and $N_\tau(x)$= $N_{\tau\prime}(x)$ for all $x \in X$, then $\tau = \tau\prime$*

Proposition 3.14. *Let (X, τ) be first countable Hausdorff space. A covering S_M is a sub-base for the first countable Hausdorff space if the following conditions hold:*

i S_M *is a sub-base for a topological space* X, *each point in* x *has the minimal open neighborhood.*

ii $N_{S_M}(X)$= $N(x)$ *for each Point* $x \in X$

Proof. Suppose S_M is a sub-base for a topological space $(X, \tau\prime$ and $(X, \tau\prime)$ is a first countable space, since each point x in the topological space $(X, \tau\prime)$ has the minimal open neighborhood, then $N_{S_M}(x)$ for each point $x \in X$ and $(X\tau)$ is first countable Hausdorff space, by lemma 3.3, $\tau = \tau\prime$. Hence, S_M is a sub-base for the first countable Hausdorff space (X, τ). Suppose S_M is a generative sub-base for the first countable Hausdorff space (X, τ). Then each point in x in the first countable Hausdorff space (X, τ) has the minimal open neighbourhood. Hence for each point $x \in X$, $N_{S_M}(X) = N(x)$ for each point $x \in X$

□

Proposition 3.15. *Let* S_M *be sub-base for first countable Hausdorff space* (X, τ). S_M *is a minimal generative sub-base for the first countable Hausdorff space* (X, τ) *if and only if for any covering* $S'_M \subset S_M$, *if and only if the following conditions hold, then* $S_M = S_{M}\prime$.

i If S'_M *is a sub-base for a topological space* X, *then each point in* x *in the topological space* X *has the minimal open neighbourhood*

ii $N_{S'_M}(x) = N(x)$ *for each point* $x \in X$

Proof. Let a covering $S_{M}\prime \subset S_M$ be a sub-base for the first countable Hausdorff space $(X \in \tau)$. Then by proposition 3.13, conditions (i) and (ii) hold. Thus $S_{M}\prime$= S_M. Hence S_M is a minimal generative sub-base for (X, τ). Now suppose S_M is a minimal generative sub-base for the first countable Hausdorff space $(X, \tau,)$ then for any covering $S_{M}\prime \subset S_M$, if Conditions (i) and (ii) hold, by proposition 3.13 then S_M is a sub-base for the first countable Hausdorff space (X, τ). Thus, S'_M = S_M since S_M is a minimal generative sub-base for the first countable Hausdorff space (X, τ). □

Proposition 3.16. *Let* X *be a first countable Hausdorff space and* S_M *be a family of open sets of* X. *Then* S_M *is a minimal generative sub-base for the topology of* X *if and only if:*

i S_M *is a sub-base for the first countable Hausdorff space* X;

ii If a subfamily $\{S_i \mid i \in I\}$*of* S_M *satisfies* $\bigcap_{i \in I} S_i \in S_M$,*then there exists* $i \in I$ *such that* $\bigcap_{i \in I} S_i$ = S_{i_0}

Proof. Let S_M be a minimal generative sub-base for the first countable Hausdorff space. Suppose we assume a subfamily $S_{M_1} = \{S_i \mid i \in I\}$ of S_M satisfies $\bigcap_{i \in I} S_i \in S_M$, but $\bigcap_{i \in I} S_i \notin S_{M_1}$ i.e $\cap_{i \in I} S_i \in S_M \backslash S_{M_1}$. Then there exists a subset $S' \in S_M \backslash S_{M_1}$ such that $\cap_{i \in I} S_i = S'$. Then for each point $x \in S'$,

$$\begin{aligned} N_{S_M} &= \cap \left\{ S \in S_M \middle| x \in S \right\} \\ &= \left(\cap \left\{ S \in S_M \backslash S' \middle| x \in S \right\} \right) \cap S' \\ &= (\cap \{ S \in S_M \backslash S' \big| x \in S \}) \cap (\cap_{i \in I} S_i) \\ &= \cap \left\{ S \in S_M \backslash S' \middle| x \in S \right\} \\ &= N_{S_M \backslash \{S'\}}(x) \end{aligned}$$

Thus for each point $x \in X$, $N_{S_M}(x) = N_{S_M \setminus \{S'\}}(x)$. We can also say that, $S_M \setminus \{S'\}$ is a sub-base for the first countable Hausdorff space (X, τ'). According to proposition 3.13, $S_M \setminus S'$ is a sub-base for the first countable Hausdorff space (X, t) which is a contradiction.

We now prove that a generative sub-base S_M is minimal. Let S'_M of S_M be a subfamily of a sub-base for the first countable Hausdorff space (X, τ), but $S'_M \neq S_M$. Then there exists an open set $S' \in S_M$ such that $S' \notin S'_M$. For any subfamily $\{S_i \middle| i \in I\}$ of $S_M \setminus S'$, $\cap_i \in IS_i \neq S'$. Thus, there exist a point $x \in S'$ such that $N_{S'_M}(x) \neq N_{S_M}(x)$. This is a contradiction and thus $S'_M = S_M$. By the definition of minimal generative sub-base, S_M is a minimal generative sub-base for the first countable Hausdorff space (X, τ). □

Proposition 3.17. *Suppose S_M is a sub-base for a Topological Space (X, τ) and Y a Topological Space, for two points $x_1, x_2 \in X$, lets define $x_1 R x_2 = NS_M(x_2$. A mapping $\rho : X \to X/R$ is a natural quotient mapping. Now suppose a mapping $g : X/R \to Y$ is a bijection. If a mapping $f : X \to Y$ satisfies $f = g \circ p$, then the following holds:*

.i *For each subsets $S \in S_M, x_1 \in S$ implies $x_2 \in S$ for any points $x_1, x_2 \in X$ satisfying $f(x_1) = f(x_2$*

ii. *For any subset $S_1, S_2 \in S_M, f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$;*

iii. *$f(NS_M(x)) = \cap\{f(S) : x \in S \in S_M\}$ for each point $x \in X$;*

iv. *For each subset $S \in S_M, f^{-1}(f(S)) = S$;*

v. *$f^{-1}(f(NS_M(x)) = NS_M(x)$ for each point $x \in X$.*

Proof. (i) We first prove that $f(x_1) = f(x_2$ means that $NS_M(x_1) = NS_M(x_2)$ for any points $x_1, x_2 \in X$ because $f = g \circ p$ and $f(x_1) = f(x_2)$ implying that $g(p(x_1)) = g \circ p(x_1) = g \circ p(x_2)$.
Therefore $p(x_1) = p(x_2)$ means that $g(p(x_1)) = g \circ p(x_1) = g(p(x_2))$. Finally, for each subset $S \in S_M, x_1 \in S$ implies $x_1 \in S_M(x_1 \subset S$. $NS_M(x_1) = NS_M(x_2)$ since $f(x_1) = f(x_2)$. This implies that $NS_M(x_1) \subset S$. That is $x_2 \in S$.

(ii) We prove that $f(S_1) \cap f(S_2) = \Phi$ if $S_1 \cap S_2 = \Phi$. Assume by contradiction that $f(S_1) \cap f(S_2) \neq \Phi$. There then exists a point $y \in Y$ such that $f(S_1) \cap f(S_2)$ that is $y \in f(S_1)$ and $y \in f(S_2)$. Let there exists two points $x_1 \in S_1$ and $x_2 \in S_2$ such that $f(x_1) = f(x_2) = y$. By (i), $x_2 \in S_1$. Then $x_2 \in S_1 \cap S_2$. Which is a contradiction. Thus, $f(S_1) \cap f(S_2 = \Phi$. Next, we show that if $S_1 \cap S_2 \neq \Phi$, then $f(S_1 \cap S_2) = f(S_1) \cap f(S_2$. It's obvious that, $f(S_1 \cap S_2) \subset f(S_1) \cap f(S_2)$. For each point $y \in f(S_1) \cap f(S_2)$, there exists two points $x_1 \in S_1$ and $x_2 \in S_2$ such that $f(x_1)$=$f(x_2)$=y. By (3.17), $x_2 \in S_1$. Then $x_2 \in S_1 \cap S_2$, which means $y = f(x_2) \in f(S_1 \cap S_2)$. So $f(S_1) \cap f(S_2) \subset f(S_1 \cap S_2)$. That is $f(S_1 \cap S_2)$=$f(S_1) \cap f(S_2)$

(iii) For (iii) the proof is similar to (ii) .

(iv) It is clear that $S \subset f^{-1}(f(S))$ is always true. For each point $x \in f^{-1}(f(S) \subset S$. That is $f^{-1}(f(S)) = S$.

(v) The proof for (v) is similar to (iv).

□

Lemma 3.3. *Let X be a Topological Space and R an equivalence relation. If the mapping $p : X \to X/R$ is assumed to be a natural quotient mapping, then a mapping g of a quotient space X/R to a Topological Space Y is continuous if and only if the composition $g \circ p$ is continuous.*

Proposition 3.18. *Let S_M be a sub-base for first countable Hausdorff space (X, τ_x) and Y a Topological space. For two points $x_1, x_2 \in X$, define $x_1 R x_2$ if $NS_M(x_1) = NS_M(x_2)$. Let a mapping be defined as $p : X \to X/R$ a natural quotient mapping. Let a mapping $g : X/R \to Y$ be a continuous bijection. If a mapping $f : X \to Y$ is an open mapping satisfying $f = g \circ p$ and S_M is a minimal generative sub-base for the first countable Hausdorff space (X, τ_x), then $f(S_M)$ is a minimal generative sub-base for first countable Hausdorff space (Y, τ_y).*

Proof. Since g is a continuous bijection, then by lemma 3.3, f is an open and continuous surjection and $(Y, \tau y)$ is countable space. It is clear that, $f(S_M)$ is a covering of Y.
By Proposition 3.17 (iii), $f(NS_M(x)) = \cap\{f(S)|x \in S \in S_M\} = \cap f(S) \in f(S_M) = N_(S_M)(f(x))$ for each point $x \in X$. Because S_M is a minimal sub-base for the first countable Hausdorff space, then, $N_{S_M}(x) = N(x)$ for each point $x \in X$ and $NS_M(x)$ is open in $(X, \tau x)$. Then $N_{f(S_M)}(f(x)) = f(N_{S_M(x)})$ is open in $(Y, \tau y)$ because f is an open mapping. It is obvious that $N_{fS_M}(x)$ is the minimal set containing $f(x)$ for each point $x \in X$. Thus $N_{fS_M}(f(x))$ is the minimal open neighborhood of $f(x)$. Also, $N_{f(S_M}(f(x)) = f(N_{S_M(x)} = f(N(x)) = N(f(x))$ for each point $x \in X$. Thus, $f(S_M)$ is a sub-base for the described topological space $(Y, \tau y)$.
For any covering $f(S'_M) \subset f(f)$, suppose each point space Y generated by $f(S'_M)$ has the minimal open neighbourhood and $N_{f(S'_M}(y) = N(y)$ for each $y \in Y$. Then S'_M is a sub-base for the first countable Hausdorff space $(X, \tau x)$. If otherwise, there exist points $x, x' \in X$ such that the minimal open neighborhood of x doesn't exist or $N_{S_M(x')} \neq S_M(x')$. It is easy to see that there exist two points $y, y' \in Y$ such that $y = f(x)$ and $y' = f(x')$. From these proof, we are able to obtain that the minimal open neighborhood of y does not exist or $N_{f(S_M)(y')} \neq N(y')$. This is a contradiction. So $F_M S'$ is a sub-base for the first countable Hausdorff space $(X, \tau x)$. Because S_M is a minimal generative sub-base for the first countable Hausdorff space $(X, \tau x)$, by definition 3.5, $F_M S' = S_M$. Thus, $f(F_M S') = f(S_M)$. By proposition 3.13 , a minimal generative sub-base for the first countable space is (Y, τ_y). □

4 Conclusion

This paper characterized the properties of topological spaces using the properties of bases and various maps with restriction to Hausdorff topology.

5 Acknowledgment

The authors acknowledge the anonymous reviewers for their useful review comments that improved this manuscript.

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