



The price of portfolio selection under tail conditional expectation with consumption cost and transaction cost

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Abstract. One of the ways of managing the risk that can arise from the changes in the relationship between assets and liabilities is by asset-liability management. Recently, Value-at-risk (VaR) and tail conditional expectation (TCE) have also emerged as standard tools for measuring and controlling the risk of trading portfolios. The limits of TCE can be transformed into the limits of VaR and conversely in some dynamical setting, TCE is preferable to VaR for being coherent. In this paper we obtain a portfolio selection model for an institution's assets- liabilities under the TCE with consumption cost and transaction cost. A set of partial differential equations are derived and closed form solution proffered, when there is no transaction cost.

Résumé. Jusqu'à récemment, les mesures de risque dénommées Valeur Au Risque (VaR) et l'Espérance Conditionnelle de Queue (ECQ), étaient utilisées pour évaluer et contrôler les risques en gestion de portefeuille. Ces deux mesures sont équivalentes dans les cas limites. De plus, dans une approche dynamique, l'ECQ est préférable à la VaR en raison notamment de sa cohérence. Une autre approche de la gestion des portefeuilles concerne l'évaluation du risque potentiellement dû aux changements qui peuvent intervenir dans la relation entre les actifs et les passifs, communément dénommée gestion de l'actif-passif. Dans ce papier, nous proposons un modèle de sélection de portefeuille d'actifs-passifs d'une institution sous contrôle du TCE en présence des coûts de consommation et de transaction. Une famille d'équations différentielles partielles est proposée et une solution fermée est trouvée, lorsqu'il n'y a pas de coût de transaction.

Key words: Risk management; Tail conditional expectation; Asset-liability control; Value-at-risk; consumption cost; Transaction costs.

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1. Introduction

Financial risk management is vital to the survival of financial institutions and the stability of the financial system. A fundamental task in risk management is to measure the risk entailed by a decision, such as the choice of a portfolio (Lan *et al.*, 2010). Recently, the substitution of variance as a risk measure in the standard Markowitz (1952) mean-variance problem has been emphasized, because it makes no distinction between positive and negative deviations from the mean. Variance is a good measure of risk only for distributions that are (approximately) symmetric around the mean such as the normal distribution or more generally, elliptical distributions (Frey and Embrechts, 2006). In cases such as in portfolio containing option as well as credit portfolio (i.e wealth distributions that are highly skewed), it is reasonable to consider asymmetric risk measures since individuals are typically loss averse. Ait-Sahalia (2001) studied asset allocation when the conditional moments of returns are partly predictable. Rather than first model the return distribution and subsequently characterize the portfolio choice, the determined directly the dependence of the optimal portfolio weights on the predictive variables. Asset-liability control is a means of managing the risk that can arise from changes in the relationship between asset and liabilities.

In this regard, Value-at-Risk, VaR, a downside (negative) risk measure, has also emerged as the industry standard with regulatory authorities enforcing its use in risk measurement and management, Jorion (2001). Despite its widespread acceptance, VaR is known to possess unappealing features. Artzner *et al.* (1999) proposed an axiomatic foundation for risk measures, by identifying four properties that a reasonable risk measure should satisfy and provided a characterization of the risk measures satisfying these properties, which they called coherent risk measures. Tail conditional expectation TCE is one of such so-called coherent risk measures (Rockafellar and Uryasev, 2001). Going by these axioms, VaR is not coherent. TCE provides a more conservative measure of risk than VaR for the same level of degree of confidence (Landsman and Valdez, 2003). Therefore, the TCE is preferable to the VaR in many applications and has recently received growing attentions in the insurance and finance literature.

Let risk Z be a non-negative random variable with cumulative distribution F , where Z may be referred to as a claim for an institution's asset or liability. Given $0 < \alpha < 1$, then z_α , determined by $\bar{F}(z_\alpha) = 1 - F(z_\alpha) = \alpha$ and denoted by $VaR_Z(1 - \alpha)$ is called the value at risk VaR with a degree of confidence $1 - \alpha$. The conditional expectation of Z given by $Z > z_\alpha$ denoted by $TCE_Z = \mathbb{E}(Z|Z > z_\alpha)$ is called the a tail conditional expectation (TCE) of Z at VaR_{z_α} .

Notice that

$$TCE_Z(z_\alpha) = z_\alpha + \mathbb{E}(Z - z_\alpha|Z > z_\alpha),$$

where $(Z - t|Z > t)$ is known as the residual lifetime in reliability and the excess loss (liability) in finance (Shaked and Shanthikumar, 1994).

The $TCE_Z(z)$ function is increasing in $z > 0$ or equivalently, $TCE_Z(z)$ is decreasing in $\alpha \in (0, 1)$, since

$$\frac{d}{dz}(z - \mathbb{E}(Z - z|Z > z)) \geq 0.$$

However in some dynamical settings, it is possible to transform a *TCE* limit into an equivalent VaR limit, and conversely (Cuocu, 2008).

Yiu (2004) has successfully controlled risky investment by imposing *VaR* as a dynamic constraint with a model that applies to the *VaR* constraint over time and emphasized the repeated re-calculations of the *VaR* like in practice. He expressed the belief that other risk measures imposed in the same way will achieve similar results. We close that gap here by experimenting with the *TCE* constraint and extending the utility maximization to cover consumption and terminal wealth.

Akume (2009) studied the dynamic portfolio and consumption choice of a trader subject to a risk limit specified in terms of TCE. The tail conditional expectation is calculated for short intervals of time and imposed as risk constraint dynamically (Akume, 2010).

On the other hand, Davis and Norman (1990) had studied an optimal consumption and investment decisions for an investor who has available a bank account paying a fixed rate of interest and a stock whose price is a log-normal diffusion. They showed that the optimal buying and selling policies are the local times of the two-dimensional process of bank and stock holdings at the boundaries of a wedge-shaped region determined by solving a nonlinear free boundary problem. Osu (2011), obtain the optimal price of an institution's assets-liabilities under the *TCE* with no transaction cost. This paper extends Osu (2011) by the application of TCE to obtain the price of portfolio selection with consumption cost and transaction cost. Furthermore, we give a two boundary equations in the asset region and the liability region. In the no-transaction region, asset region and liability region, two horizon-dependent boundaries are characterized to obtain a partial differential equation with close form solution.

The rest of this paper is structured as follows: In section 2, we model the financial market and describe the portfolio dynamics. Section 3, derives the Value-at-Risk (*VaR*) and tail conditional expectation (*TCE*) constraints. While section 4 makes precise the portfolio selection problem to be solved.

2. Formulation of the Problem

We assume the institution operates on a market of one riskless bank with constant interest rate r and n different stock. The evolution of stock prices is described by an n -dimensional Wiener process $W(t)$ on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $(\mathcal{F}_t) = \sigma \{W(s); 0 < s < t\}$:

$$\begin{aligned} dB(t) &= rB(t)dt, \\ dS(t) &= \mu_i S(t) + \sigma_i S(t)dW(t), i = 1, \dots, n. \end{aligned}$$

Here $\sigma_t = (\sigma_{i,j}(t))_{i \leq j \leq n}$ is an $m \times m$ positive definite matrix representing the covariance structure, $\sigma' \sigma$. Where σ' is the transpose of σ . The institution has initially x_0 Naira invested in the bank and (x_1, \dots, x_n) Naira invested in stock 1, ..., n . It can control its portfolio composition by buying and selling arbitrarily large or small amounts of stock from its bank account at any time.

The institutions portfolio selection strategy θ is described by the control processes $C(t)$ and $L(t)$, where $C(t)$ (the institution net cash flow at time t) and $L(t)$ (the market value of the institution's liabilities at time t) are *f-adapted vector processes*.

The dynamics of the control system (Osu and Ihedioha, 2011) is governed by the differential equations

$$dB(t) = rB(t)dt - (1 + \alpha) dC(t) (1 + \lambda) dL(t) \quad (1)$$

and,

$$dS(t) = S(t) \left[\left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma dW(t) \right] + dC(t) - dL(t), \quad (2)$$

with boundary conditions $B(t) = B_t$, $B(0) = B_0$, and $S(t) = S_t$, $S(0) = S_0$.

Defined a wealth process $h(t)$ as a sum additive random and multiplicative terms thus:

$$h(t) = \begin{cases} (1 - \lambda) S(t) + B(t), & \text{with probability } q \\ (1 - \lambda) S(t)B(t), & \text{with probability } 1 - q \end{cases} \quad (3)$$

where λ is a stochastic positive factor with probability distribution $\pi(\lambda)$, such that with probability q the integral form of (1) and (2) combined is

$$h(t) = h(0) + \int_0^t \left[rB + (1 - \lambda) \left(\mu + \frac{\sigma^2}{2} \right) \right] ds + (1 - \lambda) \sigma \int_0^t dW(s) - (1 + \alpha) C(t). \quad (4)$$

The processes, $C(t)$, $L(t)$ and hence $h(t)$ are right continuous with left limit at each $t \geq 0$. For each available strategy (C, L) , we can associate a feasible set of controls of the long term performance functional

$$\mathfrak{F}_z(C, L) = \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}_x [\ln (h(t))],$$

with $z = (C, L)$, $z \in \mathbb{R}_+^2$.

The objective is to optimize the long-run rate of growth

$$V(x, y) = \sup_{(C, L) \in \mathcal{B}} \mathfrak{F}_z(C, L).$$

\mathcal{B} is a class of pair $(x, y) \in z$, where x and y are the initial endowment of the riskless and risky asset respectively.

Let (C, L) be any feasible policy. These set of controls can be approximated by a sequence of continuous processes (C_n, L_n) , such that for h_n the net wealth corresponding to them, we have;

$$\liminf_{t \rightarrow \infty} \frac{1}{t} [\ln h(t)] \leq \lim_n \liminf_{t \rightarrow \infty} \frac{1}{t} [\ln h_n(t)].$$

Thus, we can softly assume (C_n, L_n) such that for the wealth corresponding to them, we have $(C_0 = L_0 = 0)$.

Applying Ito's formula, the relationship between $\ln h(t)$ and the processes $B(t)$ and $S(t)$ is (Rodriguez-Pedraza, 2005);

$$\begin{aligned} \ln h(t) = & \int_0^t \frac{1}{h(s)} \left[rB(s) + (1 - \lambda) \left(\mu + \frac{\sigma^2}{2} \right) S(s) \right] ds \\ & + (1 - \lambda) \sigma \int_0^t \frac{S(s)}{h(s)} dW(s) - (\lambda - \alpha) \int_0^t \frac{dC(s)}{h(s)} \end{aligned} \quad (5)$$

when $\lambda \rightarrow 0$ and $\alpha \rightarrow 0$, obtain the case of no consumption and transaction costs as in Osu (2011).

Let $\lambda = \alpha$, Equation (5) becomes;

$$\ln h(t) = \int_0^t \frac{1}{h(s)} \left[rB(s) + (1 - \lambda) \left(\mu + \frac{\sigma^2}{2} \right) S(s) \right] ds + (1 - \lambda) \sigma \int_0^t S(s) dW(s),$$

or

$$h(t) = h_0 \exp \int_0^t \left[rB(s) + (1 - \lambda) \left(\mu + \frac{\sigma^2}{2} \right) S(s) \right] ds + (1 - \lambda) \sigma \int_0^t S(s) dW(s),$$

where $h_0 > 0$ denotes the initial value of the portfolio.

This implies,

$$h(t + \tau) = \exp \left\{ \int_t^{t+\tau} \left[rB(s) + (1 - \lambda) \left(\mu + \frac{\sigma^2}{2} \right) S(s) \right] ds + (1 - \lambda) \sigma \int_t^{t+\tau} S(s) dW(s) \right\}.$$

For a given $\tau > 0$, $h > 0$, and $S \in \mathbb{R}^n$, we have

$$h_{t+\tau} = h(t) \exp \left(\left(rB(s) + (1 - \lambda) \left[\mu + \frac{\sigma^2}{2} \right] S(s) \right) \tau + (1 - \lambda) \sigma S (w(t + \tau) - w(t)) \right).$$

Now for a given probability level $\lambda \in (0, 1)$ and a given horizon $\tau > 0$, the VaR at time t of a portfolio $s \in S$, denoted by $VaR_t^{\lambda, s}$ is given by:

$$VaR_t^{\lambda, s} = \inf \{ L \geq 0, \mathbb{P} (h_t^s - h_{t+\tau} (h_t^s, S_t)) \geq L | f_t < \lambda \} = \left(Q_t^{\lambda, s} \right)^-, \quad (6)$$

where

$$\left(Q_t^{\lambda, s} \right)^- = \sup \{ L \in \mathbb{R}^n : \{ \mathbb{P} (h_t^s - h_{t+\tau} (h_t^s, S_t)) \geq L | f_t < \lambda \} \}$$

is the quantile of the projected asset gain over the interval $(t, t + \tau)$ and $z^- = \max [0, -z]$. In other words, $VaR_t^{\lambda, S}$ is the liability over the next period of length τ which would be exceeded only with a (small) conditional probability λ if the current price S_t was kept unchanged.

The fact that $VaR_t^{\lambda, S}$ is computed under the assumption that the current portfolio is kept unchanged reflects the actual practice and the fact that the financial institutions monitoring

their traders do not typically know the traders' future portfolio choices over VaR horizon. The measure of VaR in (6) only requires the knowledge of the current portfolio value, the current asset value and the conditional distribution of asset returns.

The TCE of a price $s \in S$ is defined by;

$$TCE_t^{\lambda,s} = \left(\frac{\mathbb{E} [h_t^s - h_{t+\tau}(h_t^s, S_t)] h_t^s - h_{t+\tau}(h_t^s, S_t) \geq -Q_t^{\lambda,s} | f_t}{\lambda} \right)^+,$$

where $z^+ = \max\{0, z\}$.

3. Value-at-Risk and tail conditional expectation constraints

3.1. Value-at-Risk and tail conditional expectation constraints when, $\lambda = \alpha$

Lemma 1. Let $N(x)$ and $N^{-1}(x)$ denote the normal distribution and inverse normal distribution functions with

$$\epsilon = (1 - \lambda) \left[\mu + \frac{\sigma^2}{2} \right]$$

and

$$\phi = (1 - \lambda).$$

Then,

$$VaR_t^{\lambda,s} = h_t^s [1 - \exp(rB(s) + \epsilon S(s)) \tau + \phi N^{-1}(\lambda) S_t \sigma \sqrt{\tau}]^+$$

and

$$TCE_t^{\lambda,s} = h_t^s \left[1 - \exp \frac{(rB(s) + \epsilon S(s)) \tau + NN^{-1}(\lambda) + \sigma \sqrt{\tau} S_t}{\lambda} \right]^+.$$

Proof. We have

$$\begin{aligned} & \mathbb{P}(h_{t+\tau}(h_t, S_t - h_t) \leq |f_t,) \\ &= \mathbb{P} \left(\exp \left(rB(s) + \epsilon S(s) + \phi S(w(t+\tau) - w(t)) \leq 1 + \frac{L}{h_t} \middle| f_t, \right) \right) \\ &= \mathbb{P} \left(\sigma S(w(t+\tau) - w(t)) \leq \frac{\ln \left(1 + \frac{L}{h_t} \right)^+ - (rB(s) + \epsilon S(s)) \tau}{\phi} \middle| f_t, \right) \\ &= N \left(\frac{\ln \left(1 + \frac{L}{h_t} \right)^+ - (rB(s) + \epsilon S(s)) \tau}{\phi} \right). \end{aligned}$$

The last equation is due to the fact that the random variable $\phi \sigma S(w(t+\tau) + w(t))$ is conditionally normally distributed.

Thus

$$\mathbb{P}(h_{t+\tau}(h_t), S_t - h_t) \leq |f_t) \leq \lambda$$

$$\Leftrightarrow N \left(\frac{\ln \left(1 + \frac{L}{h_t} \right)^+ - (rB(s) + \epsilon S(s)) \tau}{\phi} \right)$$

$$\Rightarrow L \leq h(t) \left[\exp \left\{ (rB(s) + \epsilon S(s)) \tau + \phi N^{-1}(\lambda) S_t \sigma \sqrt{\tau} \right\} - 1 \right]^+,$$

which implies;

$$Q_t^{\lambda,s} = h_t^s \left\{ \exp \left[(rB(s) + \epsilon S(s)) \tau + \phi N^{-1}(\lambda) S_t \sigma \sqrt{\tau} \right] - 1 \right\}^+.$$

Therefore,

$$VaR_t^{\lambda,s} = h_t^s \left\{ 1 - \exp \left((rB(s) + \epsilon S(s)) \tau + \phi N^{-1}(\lambda) S_t \sigma \sqrt{\tau} \right) \right\}^+.$$

Also,

$$\begin{aligned} & \mathbb{E} \left\{ (h_t^s - h_{t+\tau}(h_t^s, S_t)) [h_t^s - h_{t+\tau}(h_t^s, S_t)] - Q_t^{\lambda,s}(t) | f_t \right\} \\ &= h_t^s \mathbb{E} \left\{ - \exp \left((rB(t) + \epsilon S_t) \tau + \phi \sigma S_t (w(t+\tau) - w(t)) \frac{\phi \sigma S_t (w(t+\tau) - w(t))}{\sigma \sqrt{\tau} S_t} \leq \phi N^{-1}(\lambda) | f_t \right) \right\} \\ &= h_t^s \left\{ \lambda - \exp \left((rB(t) + \epsilon S_t) \tau \right) \int_{-\infty}^{\phi N^{-1}(\lambda)} \frac{1}{2\pi} \exp \left(- \frac{(x - \sigma \sqrt{\tau} S_t)^2}{2} \right) dx \right\} \\ &= h_t^s \left\{ \lambda - \exp \left((rB(t) + \epsilon S_t) \tau \right) N \left(\phi N^{-1}(\lambda) - \sigma \sqrt{\tau} S_t \right) \right\}. \end{aligned}$$

Particularly,

$$0 \leq VaR_t^{\lambda,s} \leq TCE_t^{\lambda,s} \leq h_t^s \text{ and } VaR_t^{\lambda,0} \leq TCE_t^{\lambda,0} = 0.$$

We seek the asset and liability allocation that maximizes (over admissible (C_t, L_t)) the expected utility of terminal wealth at time T and liability over the entire horizon $[0, T]$, for a risk averse institution that limits its risk by imposing an upper bound on the TCE .

In mathematical terms the stochastic asset-liability control problem with transaction under a TCE constraint is:

$$\max_{(C,L) \in \mathcal{B}} \mathbb{E}(U(h_t^s))$$

subject to the wealth process

$$\begin{aligned} h_t^s &= h_0 \exp \int_0^t [rB(s) + \epsilon S(s)] ds + \phi \int_0^t S(s) dW(s), \\ \ln \left(\frac{1 - \overline{TCE}(h_t^s, t)}{h_t^s} \right) - (rB(t) + \epsilon S_t) \tau + N(\phi N^{-1}(\lambda) S_t \sigma \sqrt{\tau}) &\leq 0 \end{aligned}$$

and the TCE constraint for fixed $\Delta t > 0$ given by

$$TCE_t^{\lambda,s} \leq \rho(h, t), \forall t \in [0, \tau].$$

Where

$$\rho(S_t, t) = 1 - \exp \left((rB(t) + \epsilon S_t) \tau + N(\phi N^{-1}(\lambda) S_t \sigma \sqrt{\tau}) \right)$$

and

$$\hat{\rho}(S_t, t) = 1 - \exp(rB(t) + \phi S_t) \tau \frac{N(\phi N^{-1}(\lambda) - \sigma \sqrt{\tau} S_t)}{\hat{\lambda}}.$$

With probability $1 - q$, we have

$$h(t) = (1 - \lambda)S(t)B(t)$$

or

$$S(t) = \frac{h(t)}{(1 - \lambda)B(t)}.$$

This is based on the classical function which implies that the price $S(t)$ of the risky asset equals the ratio of multiple of the wealth process $h(t)$ to the price of bond $B(t)$.

Applying the *TCE* constraint while maximizing the institution's logarithmic utility over asset-liabilities throughout the investment horizon and over the terminal wealth, we have:

$$\max_{(C,L) \in \mathcal{B}} \mathbb{E}(U(h_t^s)),$$

subject to the wealth process;

$$\begin{aligned} h_t^s &= h_0 \exp \int_0^t \left[rB(s) + \beta \frac{h(s)}{B(s)} \right] ds + \sigma \int_0^t \frac{h(s)}{B(s)} dW(s), \\ \ln \left((1 - \lambda) - \frac{\overline{TCE}(h_t^s, t)}{(1 - \lambda)S(t)B(t)} \right) - \left(rB(t) + \beta \frac{h(t)}{B(t)} \right) \tau \\ &\quad + \ln N \phi N^{-1}(\lambda) \frac{h(t)}{B(t)} \sigma \sqrt{\tau} \leq 0, \end{aligned}$$

with $\beta = [\mu + \frac{\sigma^2}{2}]$.

3.2. Value-at-Risk and tail conditional expectation constraints when, $\lambda \neq \alpha$

Using (5) with $\epsilon = (1 - \lambda) [\mu + \frac{\sigma^2}{2}]$, $\phi = (1 - \lambda)$, $\beta = [\mu + \frac{\sigma^2}{2}]$ and $\eta = (\lambda + \alpha)$, we obtain:

$$h(t) = h_0 \exp \int_0^t [rB(s) + \epsilon S(s)] ds + \phi \int_0^t S(s) dW(s) + \eta C(t)$$

and

$$h(t + \tau) = \exp \left\{ \int_0^{t+\tau} [rB(s) + \epsilon S(s)] ds + \phi \int_0^{t+\tau} S(s) dW(s) \right\} + \eta C(t)$$

for any $\tau > 0$.

So that

$$h_{t+\tau} = h(t) \exp((rB(s) + \epsilon S(s)) \tau + \phi \sigma S(w(t + \tau) - w(t))) + \eta C(t).$$

Lemma 2. *The value-at-risk VaR and Tail conditional expectation are given as follows:*

$$VaR_t^{\alpha,s} = h_t^s [1 - \exp(rB(s) + \epsilon S(s)) \tau + \eta C(t) + \phi N^{-1}(\alpha)]^+$$

and

$$TCE_t^{\alpha,s} = h_t^s \left[1 - \exp \frac{(rB(s) + \epsilon S(s)) \tau + \eta C(t) + N(\phi N^{-1}(\alpha))}{\alpha} \right]^+,$$

where $N(x)$ and $N^{-1}(x)$ denote the normal distribution and inverse normal distribution functions.

Proof. Following similar method of proof as in lemma 1, we have

$$\begin{aligned} & \mathbb{P}(h_{t+\tau}(h_t, S_t - h_t) \leq |f_t,) \\ &= \mathbb{P} \left(\exp(rB(s) + \epsilon S(s)) \tau + \eta C(t) + \phi \sigma S(w(t+\tau) - w(t)) \leq 1 + \frac{L}{h_t} \middle| f_t \right) \\ &= \mathbb{P} \left(\sigma S(w(t+\tau) - w(t)) \leq \frac{\ln \left(1 + \frac{L}{h_t} \right)^+ - \exp(rB(s) + \epsilon S(s)) \tau - \eta C(t)}{\phi} \middle| f_t, \right) \\ &= N \left\{ \frac{\ln \left(1 + \frac{L}{h_t} \right)^+ - (rB(s) + \epsilon S(s)) \tau - \eta C(t)}{\phi} \right\}. \end{aligned}$$

The last equation is due to the fact that the random variable $\phi(w(t+\tau) - w(t))$ is conditionally normally distributed with mean 0 and variance $S(t, \phi^2) \tau$.

Thus

$$\begin{aligned} & \mathbb{P}(h_{t+\tau}(h_t), S_t - h_t \leq |f_t) \leq \alpha \\ & \Leftrightarrow N \left\{ \frac{\ln \left(1 + \frac{L}{h_t} \right)^+ - (rB(s) + \epsilon S(s)) \tau - \eta C(t)}{\phi} \right\} \\ & \Rightarrow L \leq h(t) [\exp \{ (rB(s) + \epsilon S(s)) \tau + \eta C(t) + \phi N^{-1}(\alpha) \sigma S_t \sigma \sqrt{\tau} \} - 1]^+ \end{aligned}$$

which implies:

$$Q_t^{\lambda,s} = h_t^s \{ \exp [(rB(s) + \epsilon S(s)) \tau + \eta C(t) + \phi N^{-1}(\alpha) S_t \sigma \sqrt{\tau}] - 1 \}^+.$$

Therefore,

$$VaR_t^{\lambda,s} = h_t^s \left\{ 1 - \exp(rB(s) + \epsilon S(s) + \eta C(s)) \tau + N^{-1}(\alpha) \sigma \sqrt{\tau S_t} \right\}^+.$$

Similarly

$$\mathbb{E} \left\{ (h_t^s - h_{t+\tau}(h_t^s, S_t)) [(h_t^s - h_{t+\tau}(h_t^s, S_t))] z - Q_t^{\lambda,s}(t) | f_t \right\}$$

$$\begin{aligned}
 &= h_t^s \mathbb{E} \left(- \exp \left((rB(s) + \epsilon S(s)) \tau + \eta C(t) \right. \right. \\
 &\quad \left. \left. + \phi S_t (w(t + \tau) - w(t)) \frac{\phi S_t (w(t + \tau) - w(t))}{\sigma \sqrt{\tau S_t}} \leq N^{-1}(\alpha) | f_t \right) \right) \\
 &= h_t^s \left\{ \alpha - \exp \left((rB(s) + \epsilon S(s)) \tau + \eta C(t) \right) \int_{-\infty}^{N^{-1}(\alpha)} \frac{1}{2\pi} \exp \left(-\frac{(x - \sqrt{\tau S_t})^2}{2} \right) dx \right\} \\
 &= h_t^s \left\{ \alpha - \exp \left((rB(s) + \epsilon S(s)) \tau + \eta C(t) \right) N \left(\phi N^{-1}(\alpha) - \phi \sigma \sqrt{\tau S_t} \right) \right\}.
 \end{aligned}$$

The stochastic asset-liability control problem with transaction under a *TCE* constraint is now:

$$\max_{(C,L) \in \mathcal{B}} \mathbb{E}(U(h_t^s))$$

subject to the wealth process

$$\begin{aligned}
 &\exp \int_0^t [rB(s) + \epsilon S(s)] ds + \eta C(t) + \phi \int_0^t S(s) dW(s), \\
 &\ln \left(\frac{1 - \overline{TCE}(h_t^s, t)}{h_t^s} \right) - rB(s) + \epsilon S(s) + \eta C(t) + N(\phi N^{-1}(\alpha) \sigma \sqrt{\tau S(t)}) \leq 0.
 \end{aligned}$$

For fixed $\Delta t > 0$, the *TCE* constraint is given by,

$$TCE^{\alpha, s} \leq \rho(h, t), \quad \forall t \in [0, \tau],$$

where

$$\rho(S_t, t) = 1 - \exp \left(rB(t) + \epsilon S_t + \eta C(s) \tau + N(\phi N^{-1}(\alpha) \sigma \sqrt{\tau S(t)}) \right)$$

and

$$\hat{\rho}(S_t, t) = 1 - \exp \left(rB(t) + \epsilon S_t + \eta C(s) \tau + \frac{N(\phi N^{-1}(\alpha) \sigma \sqrt{\tau S(t)})}{\hat{\alpha}} \right)$$

The application of the *TCE* constraint now gives:

$$\max_{(C,L) \in \mathcal{B}} \mathbb{E}(U(h_t^s))$$

subject to the wealth process;

$$\begin{aligned}
 &h_t^s = h_0 \exp \int_0^t \left[rB(s) \beta \frac{h(s)}{B(s)} + \eta(s) \right] ds + \sigma \int_0^t \frac{h(s)}{B(s)} dW(s), \\
 &\ln \left((1 - \lambda) - \frac{\overline{TCE}(h_t^s, t)}{(1 - \lambda) S(t) B(t)} \right) - \left(rB(t) + \rho \frac{h(t)}{B(t)} \right) \tau + N(\phi N^{-1}(\alpha) \sigma \sqrt{\tau}) \frac{h(s)}{B(s)} \leq 0.
 \end{aligned}$$

4. The Price of Portfolio Selection under TCE with Consumption cost and no Transaction Cost

The asset-liability problem is:

$$J(C, B, S, t; T) = \max_{C_t, B_t, S_t > 0} \mathbb{E} \left[\int_0^T e^{-\rho\tau} \frac{C_{t+1-\gamma}}{1-\gamma} d\tau + e^{-\rho T} \frac{(B_T + S_T)^{1-\gamma}}{1-\gamma} \right].$$

Subject to:

$$\begin{aligned} dB(t) &= rB(t) - dC(t) + (1 - \lambda) dL(t) \\ dS(t) &= S(t) \left[\left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma dW(t) \right] + dC(t) - dL(t). \end{aligned}$$

The value function should also satisfy the terminal condition, that all the stock holding must be transformed to cash at time T :

$$J(C, B, S, t; T) = \frac{(B_T + S_T)^{1-\gamma}}{1-\gamma}. \tag{7}$$

Assumption 1. The value function $J(C, B, S, t; T)$ is once continuously differentiable in B and twice continuously differentiable in S . The two boundary equations in the asset region and the liability region are as given below. In the no-transaction region, to obtain the HJB, apply Ito's lemma:

$$dJ = \left(\frac{dJ}{dt} + (rB - C) \frac{dJ}{dB} + \mu S \frac{dJ}{dS} + \frac{1}{2} S^2 \frac{d^2 J}{dS^2} \right) dt + \sigma S \frac{dJ}{dS} dZ,$$

to the Bellman equation (7). We have:

$$\frac{C^{1-\gamma}}{1-\gamma} J_t + J_B (rB - C) + J_S \mu S + \frac{1}{2} J_{SS} \sigma^2 S^2 - \rho J = 0, : 0 \leq \frac{B}{S} \leq P,$$

$$J_B = (1 - q) J_S, : \frac{B}{S} > P,$$

$$(1 - q) J_B = J_S, : \frac{B}{S} < 0.$$

Substituting optimal consumption into HJB equation $C^* = (J_B)^{-1/\gamma}$ yields:

$$\frac{\left(\frac{-1}{J_B^\gamma} \right)^{1-\gamma}}{1-\gamma} + J_t + \left(rB - J_B^{\frac{-1}{\gamma}} \right) J_B + \mu S J_S + \frac{1}{2} \sigma^2 S^2 J_{SS} - \rho J = 0; : 0 \leq \frac{B}{S} \leq P. \tag{8}$$

The value function $J(C, B, S, t; T)$ is homogeneous of degree $1 - \gamma$ for all positive numbers in (B, S) , as shown in Fleming and Soner (1993)xxx. Define $h = \frac{B}{S}$, for a new value function

$$f : (-\infty, +\infty) \times [0, T] \rightarrow \mathbb{R}.$$

Homogeneity gives:

$$J(C, B, S, t; T) = S^{1-\gamma} f(C, h, t; T).$$

The no-transaction region, asset region and liability region thus can be characterized by two horizon-dependent boundaries $O(t; T)$ and $P(t; T)$. We derive the new value function and its derivatives with respect to h and t by applying the chain rule, as follows:

$$J = S^{1-\gamma} f, J_t = S^{1-\gamma} f_t, J_B = S^{-\gamma} f_h, J_S = (1 - \gamma) S^{-\gamma} f - B S^{-\gamma-1} f_h,$$

$$J_{SS} = \gamma(1 - \gamma) S^{-\gamma-1} f + 2\gamma B S^{-\gamma-2} f_h + B^2 S^{-\gamma-3} f_{hh}.$$

Substituting the new value function and its derivatives with respect to h into the modified HJB equation and the two boundary equations on the asset region and the liability region, one obtains a system of partial differential equations (PDEs).

On the no-transaction region:

$$\frac{\gamma}{1-\gamma} f_h^{1-\frac{1}{\gamma}} + \frac{1}{2} f_{hh} \sigma^2 h^2 + f_h (\gamma \sigma^2 - (\mu - r]) h + f \left((1 - \gamma) \left(\mu - \frac{\gamma \sigma^2}{2} \right) - \rho \right) + f_t = 0,$$

$$O(t; T) \leq h \leq P(t; T). \tag{9}$$

For

$$\frac{\gamma}{1-\gamma} \gamma f_h^{\frac{-1}{\gamma}} = (\gamma - 1) \iota h,$$

(9) becomes:

$$\frac{1}{2} f_{hh} \sigma^2 h^2 + f_h (\gamma \sigma^2 - (\mu - r) + (\gamma - 1)\mu) h + f \left((1 - \gamma) \left(\mu - \frac{\gamma \sigma^2}{2} \right) - \rho \right) + f_t = 0,$$

$$O(t; T) \leq h \leq P(t; T). \tag{10}$$

On the asset region:

$$\left(\frac{1}{1-q} + h \right) f_h(h, t; T) = (1 - \gamma) f(h, t; T); : h > P(t, T). \tag{11}$$

On the liability region:

$$(1 - q + h) f - h(h, t; T) = (1 - \gamma) f(h, t; T); : h < O(t, T). \tag{12}$$

In addition, the following terminal condition must be satisfied

$$f(h, t; T) = \frac{(h_T + 1)^{1-\gamma}}{1 - \gamma}. \tag{13}$$

Theorem 1. *Let $f(h)$ be the price of the asset-liability with h as the prevailing money market account-stock ratio. Let $f(h)$ be twice continuously differentiable, the solution of the time-homogeneous value function equation (11) with:*

$$f(0) = 0, \text{ and } f'(h) = 0 \tag{14}$$

is given by

$$f(h) = c(\sigma h)^{\lambda_1} + \frac{h}{k - \frac{\xi(t)}{\sigma}}, \tag{15}$$

with

$$\frac{\hat{h}}{k - \frac{\xi(t)}{\sigma}} + c\lambda_1 (\sigma\hat{h})^{\lambda_1} = 0 \quad (16)$$

and

$$f(h) = \frac{1}{\xi(t) - \sigma k} \left\{ \frac{\sigma^{1+\lambda_1} h^{1-\lambda_1}}{\lambda_1} + \sigma h \right\}, \quad (17)$$

where \hat{h} is the expected optimal money market account-stock ratio for a period t , c is a constant and;

$$\lambda_1 = - \left[\frac{\xi}{\sigma} - \frac{1}{2} \right] + \left\{ \left[\frac{\xi}{\sigma} - \frac{1}{2} \right]^2 + 2k \right\}^{\frac{1}{2}} \quad (18)$$

$$\lambda_2 = - \left[\frac{\xi}{\sigma} - \frac{1}{2} \right] - \left\{ \left[\frac{\xi}{\sigma} - \frac{1}{2} \right]^2 + 2k \right\}^{\frac{1}{2}}, \quad (19)$$

are the positive and negative characteristic roots of (8) respectively.

Proof. Let $\xi = (\gamma\sigma^2 - (\mu - r) + (\gamma - 1)\mu)$ and $k = \left\{ (\gamma - 1) \left(\mu - \frac{\gamma\sigma^2}{2} \right) + \rho \right\}$, then (10) reduces to the ode (with the conditions in Osu and Okoroafor, 2007)

$$\frac{\sigma^2 h^2}{2} f_{hh} + \xi h f_h - k f = -h. \quad (20)$$

By the method of change of independent variables using Euler's substitution and solving by variation of parameters, the solutions are obtained. An important relationship derived under the optimal condition is that the discount rate is proxy of the systematic volatility factor in the economy. So that the discounted rate gains from a unit investment at \hat{h} equals the optimal unit \bar{h} of ratio of money market account to stock for the expected optimal money market account to stock ratio \hat{h} . Therefore, by (14), we have

$$f(\hat{h}) = c (\sigma\hat{h})^{\lambda_1} + \frac{\hat{h}}{k - \frac{\xi}{\sigma}} = \bar{h}. \quad (21)$$

Solving for C in (16) and (21) and equating the results gives:

$$\hat{h} = \frac{\lambda_1 \left[k - \frac{\xi}{\sigma} \right]}{\lambda_1 - 1}. \quad (22)$$

Note: When the drift parameter ξ is large enough so that $\frac{\xi}{\sigma} > \frac{1}{2}$ then the right hand side of (18) is approximated by first order Taylor's expansion as $\frac{k}{\frac{\xi}{\sigma} - \frac{1}{2}}$, thus, $\frac{\xi}{\sigma} - \frac{1}{2}$ and we obtain $\lambda_1 = 1$ and the optimal money market account-stock ratio of (22) becomes indeterminate.

5. conclusions

We have considered in this paper the Price of Portfolio Selection under Tail Conditional Expectation with Consumption cost and Transaction Cost. The dependence of expected return and TCE on the initial portfolio, in particular when transaction costs are high, is largely removed by the introduction of the TCE constraint. In answer to the second question our analysis shows that both expected return and risk as measured by TCE are brought to some intermediate level when intermediate transactions are made possible subject to a TCE constraint. Without the TCE constraint, intermediate transactions aiming at maximising expected log-returns lead to higher returns and higher risk when initial portfolios are low return. Notice that on the asset and liability regions, $\frac{1}{1-q} + h = 1 - q + h$, which gives $q = 0$ or $q = 2$. Equation (11) has the solution, $f(h) = k \left(\frac{1}{1-q} + h \right)^{1-\gamma}$ which becomes $f(h) = k(1+h)^{1-\gamma}$ for $q = 0$ and $f(h) = k(h-1)^{1-\gamma}$ for $q = 2$. For $k > 0$, $f(h)$ increases as $q \rightarrow 0$ and decreases as $q \rightarrow 2$. Thus if there is no transaction cost, growth rate of the value function is higher than when there is transaction cost. On the other hand, equation (12) has a solution; $f(h) = k(1-q+h)^{1-\gamma}$. Generally, the value of the investment increases (decrease) in the asset region and decreases (increases) in the liability region for some values of $0 < q < 2$, but are equal when $q = 0$ or $q = 2$.

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