Journal Afrika Statistika

ISSN 0852-0305

Journal Afrika Statistika
Vol. 7, 2012, pages 441–458.
DOI: http://dx.doi.org/10.4314/afst.v7i1.5

Bias-reduced estimation of Wang's two-sided deviation risk measure under Lévy-stable regime

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Received 7 December 2011; Accepted 28 October 2012

Abstract. Several risk measures, such as the distorted insurance premium and the two-sided deviation (TSD) measure, can be regarded as L-functionals with specific weight functions. In this paper, we focus on the TSD risk measure as we define a new estimator by using the bias-reduced estimators of extreme quantiles proposed by Li *et al.* (2010). A simulation study is carried out to compare, in terms of bias and mean squared error, the new estimator with that introduced recently by Necir and Meraghni (2010).

Résumé. Plusieurs mesures de risque, telles la prime d'assurance distordue et la mesure de déviation bilatérale (two-sided deviation: TSD), peuvent être considérées comme des L-fonctionnelles avec des fonctions poids spécifiques. Dans ce papier, on définit un nouvel estimateur pour la TSD en utilisant les estimateurs à biais rduits des quantiles extrêmes proposés par Li *et al.* (2010). Une étude de simulation est effedtuée dans le but de comparer, en termes de biais et d'erreur quadratique moyenne, le nouvel estimateur avec celui introduit par Necir and Meraghni (2010).

Key words: Bias reduction; High quantiles; Hill estimator; Lévy-stable distribution; L-statistics; Order statistics; Risk Measure; Second order regular variation, Tail index.
AMS 2010 Mathematics Subject Classification : 91B30; 62G32; 62G30; 62G05.

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1. Introduction

A Lévy-stable random variable (rv) X, denoted $X \sim S_\alpha(\sigma, \beta, \mu)$, is typically described by its characteristic function

$$\phi_X(t) := \mathbf{E}[\exp\{iXt\}] = \exp\{i\mu t - \sigma |t|^\alpha [1 - i\beta \operatorname{sgn}(t) z(t,\alpha)]\}, t \in \mathbb{R},$$

with $i^2 = -1$,

$$z(t,\alpha) := \begin{cases} \tan\left(\dfrac{\pi\alpha}{2}\right) & \text{if } \alpha \neq 1 \\ -\dfrac{2}{\pi}\log|t| & \text{if } \alpha = 1 \end{cases} \quad \text{and} \quad \operatorname{sgn}(t) := \begin{cases} -1 & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0, \end{cases}$$

where $0 < \alpha \leq 2$, $-1 \leq \beta \leq +1$, $\sigma > 0$ and $-\infty < \mu < +\infty$ are four parameters that completely characterize the stable distribution. The parameter α, called stability index, characteristic exponent or tail index, is the main one; it governs the tails of the distribution (the smaller α, the heavier the tails). The case $\alpha = 2$ corresponds to the well-known Gaussian law. β is the skewness parameter and σ and μ are the usual scale and location parameters. When $\alpha > 1$, the mean of X exists and is equal to the location parameter. In general the kth moment of a stable variable is finite if and only if $k < \alpha < 2$. When $\beta = 0$, the distribution is said to be symmetric about μ. For full details on the stable law, we refer to Samorodnitsky and Taqqu (1994) and Zolotarev (1986).

When $\alpha < 2$, the lower and upper tails of the cumulative distribution function (cdf) F of X are asymptotically equivalent to those of a Pareto distribution, i.e. they exhibit a power-law behavior. Indeed, from Property 1.2.15 of page 16 in Samorodnitsky and Taqqu (1994), there exist two non-negative constants $C_1$and C_2 such that, for $0 < \alpha < 2$,

$$F(-x) \sim C_1 x^{-\alpha} \text{ and } 1 - F(x) \sim C_2 x^{-\alpha}, \text{ as } x \to \infty. \tag{1}$$

This means that both distribution tails are regularly varying at infinity with index $(-\alpha) < 0$. In other words, we have as $x \to \infty$,

$$F(-x) = x^{-\alpha} L_1(x) \text{ and } 1 - F(x) = x^{-\alpha} L_2(x),$$

where L_1 and L_2 are slowly varying functions at infinity (i.e. $\lim_{t\to\infty} L_i(xt)/L_i(t) = 1$, $i = 1, 2$, for any $x > 0$). Furthermore, L_1 and L_2 satisfy what is known as the balance condition

$$\lim_{x\to\infty} \frac{L_1(x)}{L_1(x) + L_2(x)} = \frac{1+\beta}{2}.$$

Using the expansion (to the second order) of the stable distribution right tail and the relationship between both tails, respectively given on top of page 95 and in page 65 of Zolotarev (1986), we may write, for $1 < \alpha < 2$, as $x \to \infty$

$$F(-x) = c_L x^{-\alpha} + d_L x^{-\lambda} + o\left(x^{-\lambda}\right), \tag{2}$$

and

$$1 - F(x) = c_R x^{-\alpha} + d_R x^{-\lambda} + o\left(x^{-\lambda}\right), \tag{3}$$

with $\lambda = 2\alpha$ where c_L, c_R, d_L and d_R are real constants expressed in terms of the stable parameters α, β, σ and μ. This means that stable distributions belong to Hall's class of heavy-tailed distributions (see Hall, 1982 and Hall and Welsh, 1985), to be defined later by relation (11) below which is a special case of a more general second-order regular variation condition (see de Haan and Stadtmüller, 1996).

In general, stable variables suffer a crucial drawback: they do not have closed form expressions for their probability densities and cdf's, which severely hampers the estimation of their parameters. However, several useful numerical procedures based on classical estimation methods (sample quantiles, characteristic function, maximum likelihood) are proposed in the literature. For a complete survey of these methodologies, we refer those interested to Garcia *et al.* (2006). Furthermore, the heavy-tail feature allows the use of extreme value theory tools to make semi-parametric inference about the stable parameters. Indeed, the characteristic exponent α could be estimated by one of the existing tail index, estimators such as the very popular Hill estimator (Hill, 1975) and for parameters μ and σ, Peng (2001) and Meraghni and Necir Meraghni and Necir (2007) respectively proposed asymptotically normal estimators. For a discussion on the performance of Hill's estimator of α under the Lévy-stable regime, see Weron (2001).

This class of distributions was introduced by Paul Lévy during his investigations of the behavior of sums of independent rv's in the early 1920's (Lévy, 1925). They owe their importance in both theory and practice to the generalized central limit theorem which states that stable laws are the only possible limit distributions for properly normalized and centered sums of independent and identically distributed (iid) rv's. It is a rich class of probability distributions (with many important mathematical properties) that allow skewness and thickness of tails. As shown in early work by Mandelbrot (1963), the stable model is a good candidate to accommodate heavy-tailed financial series. It also proved to be appropriate for data sets in many types of physical and socioeconomic systems (see Chapter 1 in Zolotarev, 1986).

In the process of risk management, the main task is to properly measure the risk. It necessitates determining an appropriate price that must cover the riskiness due to possible losses. It is clear that the shape of the loss distribution (particularly the behavior of its tails) has a great influence on the strategy of computing the price of the risk. A risk measure is defined as a mapping from the set of all loss rv's to the non-negative real numbers. Artzner *et al.* (1999) put the axioms to be satisfied by any risk measure in order to have the desirable property of coherence. A large variety of risk measures are proposed in the literature, among which the two-sided deviation (TSD). Introduced by Wang (1998), it seems to be a suitable risk measure when dealing with financial data (such as asset log returns) which are modeled by rv's that can take any real values. If we let X be a real-valued rv with continuous cdf F, then Wang's TSD, denoted by Δ_r, is an average of the right-tail deviation and the left-tail deviation, respectively defined, for $0 < r < 1$, by

$$D_r^R[X] := \int_{-\infty}^{\infty} \left([1 - F(x)]^r - [1 - F(x)]\right) dx,$$

and

$$D_r^L[X] := D_r^R[-X] = \int_{-\infty}^{\infty} \left([F(x)]^r - F(x)\right) dx.$$

That is

$$\Delta_r = \Delta_r[X] := \frac{1}{2}\left(D_r^L[X] + D_r^R[X]\right), \ 0 < r < 1.$$

Changing variables and integrating by parts yield the following expression for Δ_r :

$$\Delta_r = \int_0^1 J_r(s) Q(s) ds, \ 0 < r < 1, \tag{4}$$

where

$$J_r(s) := \frac{r}{2}\left((1-s)^{r-1} - s^{r-1}\right), \ 0 < s < 1,$$

is a specific weight function and $Q(s) := \inf\{x \in \mathbb{R} : F(x) \geq s\}, \ 0 < s \leq 1$, is the quantile function pertaining to F. Note in passing that, for $t \downarrow 0$, the quantile $Q(1-t)$ is called high or extreme quantile.

Representation (4) shows the TSD in an L-functionnal form. Jones and Zitikis (2003) made a broad opening for developing statistical inferential results in the actuarial area, based on L-functionnals. They revealed a fundamental relationship between some risk measures and the classical L-statistics which themselves are considered as the natural estimates of L-functionnals. Indeed, let $(X_1, ..., X_n)$ be a sample, of size $n \geq 1$, drawn from a rv $X \sim S_\alpha(\sigma, \beta, \mu)$ and let $X_{1,n} \leq ... \leq X_{n,n}$ denote the corresponding order statistics. The non-parametric estimator of Δ_r, denoted by $\Delta_{n,r}$, is obtained by replacing $Q(s)$ in (4) by its empirical counterpart

$$Q_n(s) := \inf\{x \in \mathbb{R} : F_n(x) \geq s\}, \ 0 < s \leq 1,$$

corresponding to the empirical distribution function $F_n(x) := n^{-1}\sum_{i=1}^n \mathbb{I}(X_i \leq x), \ x \in \mathbb{R}$, with $\mathbb{I}(\cdot)$ denoting the indicator function. That is

$$\Delta_{n,r} := \sum_{i=1}^n a_{i,n}^{(r)} X_{i,n},$$

where, for $i = 1, ..., n$,

$$a_{i,n}^{(r)} := \frac{1}{2}\left[\left(1 - \frac{i-1}{n}\right)^r - \left(1 - \frac{i}{n}\right)^r - \left(\frac{i-1}{n}\right)^r + \left(\frac{i}{n}\right)^r\right].$$

For more details on L-statistics and their asymptotic properties, one refers to Chapter 19 of Shorack and Wellner (1986) and Jones and Zitikis (2003), Jones and Zitikis (2005).

Let $b_{n,r}$ and $e_{n,r}$ respectively denote the bias and the root of the mean squared error (RMSE) of $\Delta_{n,r}$. That is

$$b_{n,r} := \mathbf{E}[\Delta_{n,r} - \Delta_r] \text{ and } e_{n,r} := \sqrt{\mathbf{E}[\Delta_{n,r} - \Delta_r]^2}.$$

From Lemma 1 (see the Appendix), we have

$$b_{n,r} = 0 \text{ and } e_{n,r} = \frac{\delta_r}{\sqrt{n}}, \text{ for any } 0 < r < 1,$$

where

$$\delta_r^2 := \int_0^1 \int_0^1 (\min(s,t) - st) J_r(s) J_r(t) dQ(s) dQ(t). \tag{5}$$

This implies that, for any $0 < r < 1$, $\Delta_{n,r}$ is an unbiased estimator for Δ_r, with an asymptotically negligible RMSE (which is the standard deviation in this case), of convergence rate $n^{-1/2}$, provided that $\delta_r < \infty$. But, this is a very restrictive condition in the context of Lévy-stable distributions. Indeed, in Lemma 2 (see the Appendix), we show that $\delta_r = \infty$ for any $1 < \alpha < 2$ and $0 < r < 1$. It follows that the RMSE $e_{n,r}$ is infinite as well. Hence, we need to seek another approach to estimate Δ_r in order to handle the case.

Many other authors discussed the empirical estimation of L-functionals in the restrictive case of finite variances. Exploiting the extreme value theory, Necir and Meraghni (2010) proposed an alternative estimation method that extends the existing results to the more significant case where variances are infinite, which is more pertinent for dangerous risks in the areas of finance and insurance. They proposed estimators that are asymptotically normal regardless of the shape of the distribution tails. Next, we present the items that are necessary to the definition of the extreme value based TSD estimator (Necir and Meraghni, 2010).

Let $\ell = \ell_n$ and $m = m_n$ be two sequences of integers (called trimming sequences) satisfying

$$1 < \ell < n,\ 1 < m < n,\ \ell \to \infty,\ m \to \infty,\ \ell/n \to 0 \text{ and } m/n \to 0,\ \text{as } n \to \infty. \tag{6}$$

Then

$$\widehat{\alpha}_L^H = \widehat{\alpha}_L^H(\ell) := \left(\frac{1}{\ell} \sum_{i=1}^{\ell} \log^+(-X_{i-1,n}) - \log^+(-X_{\ell,n}) \right)^{-1}, \tag{7}$$

and

$$\widehat{\alpha}_R^H = \widehat{\alpha}_R^H(m) := \left(\frac{1}{m} \sum_{i=1}^{m} \log^+(X_{n-i+1,n}) - \log^+(X_{n-m,n}) \right)^{-1}. \tag{8}$$

with $\log^+(u) := \max(0, \log u)$, are two forms of Hill's estimator for the stability index α (Hill, 1975), respectively based on the left tail and right tail of the distribution. Hill's estimator of the tail index (also known as the extreme value index), originally defined for positive indices, has been thoroughly studied, improved and even generalized to any real index. As we can see, extreme value based estimators heavily rely on the numbers ℓ and m of lower and upper order statistics used in estimate computation. An optimal choice of these numbers involves the typical trade-off between bias and variance. Indeed, estimators $\widehat{\alpha}_L$ and $\widehat{\alpha}_R$ have, in general, substantial variances for small values of ℓ and m and considerable biases for large values of ℓ and m. Hence, a theoretical optimal choice of ℓ and m is to minimize the asymptotic mean squared errors of the estimators. Although there is no universal direction as how to optimally choose such numbers, several graphical and numerical procedures of sample fraction selection are available in the literature. For a discussion on this thorny issue, one may consult Necir and Meraghni (2010) and the references therein.

After observing that the asymptotic normality of $\Delta_{n,r}$ is not guaranteed for distributions with infinite variances, Necir and Meraghni (2010) use the extreme value theory to introduce an asymptotically normal estimator for Δ_r when F belongs to the domain of attraction of a Lévy-stable distribution. The proposed semi-parametric estimator $\widetilde{\Delta}_{n,r}^*$ is constructed as follows:

$$\widetilde{\Delta}^*_{n,r} = \int_0^{\ell/n} J_r(s)\, Q_L^W(s)\, ds + \int_{\ell/n}^{1-m/n} J_r(s)\, Q_n(s)\, ds + \int_{1-m/n}^{1} J_r(s)\, Q_R^W(s)\, ds,$$

where

$$Q_L^W(t) := -(nt/\ell)^{-1/\widehat{\alpha}_L^H} X_{\ell,n} \text{ and } Q_R^W(1-t) := (nt/m)^{-1/\widehat{\alpha}_R^H} X_{n-m,n}, \text{ as } t \downarrow 0,$$

are Weissman's estimators of the left and right high quantiles respectively. Note that both of functions $s \to J_r(s)\, Q_L^W(s)$ and $s \to J_r(s)\, Q_R^W(s)$ are regularly varying with tail indices $r - 1 - 1/\widehat{\alpha}_L^H$ and $r - 1 - 1/\widehat{\alpha}_R^H$ respectively, then for all large n, we have

$$\int_0^{\ell/n} J_r(s)\, Q_L^W(s)\, ds = -(1 + o(1)) \frac{(\ell/n)\, J(\ell/n)}{r - 1/\widehat{\alpha}_L^H} X_{\ell,n},$$

and

$$\int_{1-m/n}^{1} J_r(s)\, Q_R^W(s)\, ds = (1 + o(1)) \frac{(m/n)\, J(1-m/n)}{r - 1/\widehat{\alpha}_R^H} X_{m,n},$$

provided that

$$r - 1/\widehat{\alpha}_L^H > 0 \text{ and } r - 1/\widehat{\alpha}_R^H > 0. \tag{9}$$

On the other hand, for all large n, we have

$$J(\ell/n) = -(1 + o(1)) \frac{r}{2} (\ell/n)^{r-1} \text{ and } J(1-m/n) = (1 + o(1)) \frac{r}{2} (m/n)^{r-1}.$$

Then instead of $\widetilde{\Delta}^*_{n,r}$, we may use

$$\widehat{\Delta}_{n,r} := \frac{r}{2} \frac{(\ell/n)^r}{r - 1/\widehat{\alpha}_L^H} X_{\ell,n} + \sum_{j=\ell+1}^{n-m} a_{j,n}^{(r)} X_{j,n} + \frac{r}{2} \frac{(m/n)^r}{r - 1/\widehat{\alpha}_R^H} X_{n-m,n}. \tag{10}$$

In the context of insurance risks, that is when the loss is supposed to be a non negative rv, analogue studies were made as well (see, Necir *et al.*, 2007; Necir and Meraghni, 2009 and Brahimi *et al.*, 2011).

Let $b^+_{n,r}$ and $e^+_{n,r}$ respectively denote the bias and the RMSE of $\widehat{\Delta}_{n,r}$. From the asymptotic normality of $\widehat{\Delta}_{n,r}$ (Theorem 4.2 in Necir and Meraghni, 2010), we infer that, for any $1/\alpha < r < 1$, there exists a constant $\omega_r < \infty$ such that

$$(i)\ b^+_{r,n} \to 0;\ \ (ii)\ \sqrt{n} e^+_{n,r} / (\ell/n)^{-1/\alpha + 1/2} \to \omega_r, \text{ as } n \to \infty.$$

Assertion (i) implies that $\widehat{\Delta}_{n,r}$ is asymptotically unbiased. That is, the estimator $\widehat{\Delta}_{n,r}$ has negligible bias as $n \to \infty$. On the other hand, we deduce from assertion (ii) that, since $\alpha < 2$, the RMSE $e^+_{n,r}$ converges to zero with convergence rate $n^{-1/2} (\ell/n)^{1/\alpha - 1/2}$.

Assertion (i) is still a theoretical result, but from a practical point of view, when one deals with finite sample sizes, the estimator $\widehat{\Delta}_{n,r}$ presents a big bias. The reason is that $\widehat{\Delta}_{n,r}$ is based on Weissman's estimator of high quantiles for heavy-tailed distributions, known to be largely biased (Weissman, 1978). As a better alternative to Weissman's estimators, several estimators of extreme quantiles with reduced biases are proposed in the literature. For a survey, see, for instance, Feuerverger and Hall (1999), Beirlant *et al.* (2002), Beirlant *et al.* (2008), Gomes and Martins (2002), Gomes and Martins (2004), Caeiro *et al.* (2004), Caeiro *et al.* (2009), Peng and Qi (2004), Matthys *et al.* (2004), Gomes and Figueiredo (2006), Gomes and Pestana (2007).

Our task in this paper is to derive a new TSD estimator with reduced bias by applying the results of Peng and Qi (2004) and Li *et al.* (2010) who respectively introduced censored maximum likelihood (CML) based estimators for regular variation parameters and high quantiles. Our choice is motivated by the nice asymptotic properties of such results.

The rest of the paper is organized as follows. In Section 2, we briefly describe the bias reduction methods of Peng and Qi (2004) and Li *et al.* (2010) before constructing our new estimator for the TSD. In Section 3, we apply the algorithm of Reiss and Thomas (1997) to select the optimal numbers of extreme order statistics used in the computation of the TSD estimate. In Section 4, we perform a simulation study, by sampling from Lévy-stable distributions, to compare the newly proposed estimator with that previously introduced by Necir and Meraghni (2010). Section 5 is devoted to some concluding comments and remarks. Finally, some useful auxiliary results are gathered in the Appendix.

2. Bias-reduced estimator for the TSD

We start this section by a brief description of the method of Li *et al.* (2010) to reduce the bias in the estimation of high quantiles for heavy-tailed distributions. Let K be a cdf with tail belonging Hall's class of models, i.e.

$$1-K(x)=cx^{-\gamma_1}+dx^{-\gamma_2}+o\left(x^{-\gamma_2}\right), \text{ as } x\to\infty, \tag{11}$$

where $c>0, d\neq 0$ and $\gamma_2>\gamma_1>0$. Based on a sample $Z_1,...,Z_n$ from cdf K and the pertaining order statistics $Z_{1,n},...,Z_{n,n}$, Peng and Qi (2004) define the CML estimators $(\widehat{\gamma}_1,\widehat{\gamma}_2)$ of (γ_1,γ_2) as the solution of the following system (with the constraint $\gamma_2>\widehat{\gamma}_1$) :

$$\begin{cases} \dfrac{1}{k}\displaystyle\sum_{i=1}^{k}\dfrac{1}{G_i(\gamma_1,\gamma_2)}=1 \\ \dfrac{1}{k}\displaystyle\sum_{i=1}^{k}\dfrac{1}{G_i(\gamma_1,\gamma_2)}\left(\log^+(Z_{n-i+1,n})-\log^+(Z_{n-k,n})\right)=\gamma_2^{-1}, \end{cases}$$

where $k=k_n$ is an integer sequence such that $1<k<n$ and $k/n\to 0$ (as $n\to\infty$) and

$$G_i(\gamma_1,\gamma_2):=\frac{\gamma_1}{\gamma_2}\left(1+\frac{\gamma_1\gamma_2}{\gamma_1-\gamma_2}H(\gamma_1)\right)\left(\frac{Z_{n-i+1,n}}{Z_{n-k,n}}\right)^{\gamma_2-\gamma_1}-\frac{\gamma_1\gamma_2}{\gamma_1-\gamma_2}H(\gamma_1),$$

with

$$H(\gamma_1) := \frac{1}{\gamma_1} - \left(\frac{1}{k} \sum_{i=1}^{k} \log^+ (Z_{n-i+1,n}) - \log^+ (Z_{n-k,n}) \right).$$

If we denote the quantile function associated to cdf K by R, then it is easy to verify that (11) is equivalent to

$$R(1-t) = c^{1/\gamma_1} t^{-1/\gamma_1} \left(1 + \gamma_1^{-1} c^{-\gamma_2/\gamma_1} d t^{\gamma_2/\gamma_1 - 1} + o(1)\right), \text{ as } t \downarrow 0. \tag{12}$$

Substituting $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ for γ_1 and γ_2 in (12), we obtain the bias-reduced estimator for the quantile, $R(1-t)$, $t \downarrow 0$, proposed by Li *et al.* (2010) and that we denote by $R^{\mathbf{LPY}}(1-t)$ (the abbreviation **LPY** relates to the name initials of the authors of the paper).

$$R^{\mathbf{LPY}}(1-t) := \widehat{c}^{1/\widehat{\gamma}_1} t^{-1/\widehat{\gamma}_1} \left(1 + \widehat{\gamma}_1^{-1} \widehat{c}^{-\widehat{\gamma}_2/\widehat{\gamma}_1} \widehat{d} t^{\widehat{\gamma}_2/\widehat{\gamma}_1 - 1}\right), \ t \downarrow 0,$$

where $\left(\widehat{c}, \widehat{d}\right)$ are given by

$$\begin{cases} \widehat{c} := \dfrac{k}{n} \dfrac{\widehat{\gamma}_1 \widehat{\gamma}_2}{\widehat{\gamma}_1 - \widehat{\gamma}_2} Z_{n-k,n}^{\widehat{\gamma}_1} \left(\dfrac{1}{\widehat{\gamma}_2} - \dfrac{1}{k} \sum\limits_{i=1}^{k} \log^+ (Z_{n-i+1,n}) + \log^+ (Z_{n-k,n}) \right), \\ \widehat{d} := \dfrac{k}{n} \dfrac{\widehat{\gamma}_1 \widehat{\gamma}_2}{\widehat{\gamma}_2 - \widehat{\gamma}_1} Z_{n-k,n}^{\widehat{\gamma}_2} \left(\dfrac{1}{\widehat{\gamma}_1} - \dfrac{1}{k} \sum\limits_{i=1}^{k} \log^+ (Z_{n-i+1,n}) + \log^+ (Z_{n-k,n}) \right). \end{cases}$$

Now, we apply the results above to the case of a Lévy-stable distribution F with parameters α, β, σ and μ. Note that (2) and (3) may be respectively rewritten in terms of the left and right high quantiles $Q_L(t)$ and $Q_R(1-t)$. We have, as $t \downarrow 0$

$$Q_L(t) = -c_L^{1/\alpha} t^{-1/\alpha} \left(1 + \alpha^{-1} c_L^{-\lambda/\alpha} d_L t^{\lambda/\alpha - 1}\right) + o(1),$$

and

$$Q_R(1-t) = c_R^{1/\alpha} t^{-1/\alpha} \left(1 + \alpha^{-1} c_R^{-\lambda/\alpha} d_R t^{\lambda/\alpha - 1}\right) + o(1).$$

Hence, the bias-reduced estimators of $Q_L(t)$ and $Q_R(1-t)$ as $t \downarrow 0$, that we respectively denote by $Q_L^{\mathbf{LPY}}(t)$ and $Q_R^{\mathbf{LPY}}(1-t)$, are defined, for $t \downarrow 0$, by

$$Q_L^{\mathbf{LPY}}(t) := -\widehat{c}_L^{1/\widehat{\alpha}_L} t^{-1/\widehat{\alpha}_L} \left(1 + \widehat{\alpha}_L^{-1} \widehat{c}_L^{-\widehat{\lambda}_L/\widehat{\alpha}_L} \widehat{d}_L t^{\widehat{\lambda}_L/\widehat{\alpha}_L - 1}\right), \tag{13}$$

and

$$Q_R^{\mathbf{LPY}}(1-t) := \widehat{c}_R^{1/\widehat{\alpha}_R} t^{-1/\widehat{\alpha}_R} \left(1 + \widehat{\alpha}_R^{-1} \widehat{c}_R^{-\widehat{\lambda}_R/\widehat{\alpha}_R} \widehat{d}_R t^{\widehat{\lambda}_R/\widehat{\alpha}_R - 1}\right), \tag{14}$$

where

$$\begin{cases} \widehat{c}_L := \dfrac{\ell}{n} \dfrac{\widehat{\alpha}_L \widehat{\lambda}_R}{\widehat{\alpha}_L - \widehat{\lambda}_L} (-X_{\ell,n})^{\widehat{\alpha}_L} \left(\dfrac{1}{\widehat{\lambda}_R} - \dfrac{1}{\ell} \sum\limits_{i=1}^{\ell} \log^+ (-X_{i,n}) + \log^+ (-X_{\ell,n}) \right), \\ \widehat{d}_L := \dfrac{\ell}{n} \dfrac{\widehat{\alpha}_L \widehat{\lambda}_L}{\widehat{\lambda}_L - \widehat{\alpha}_L} (-X_{\ell,n})^{\widehat{\lambda}_L} \left(\dfrac{1}{\widehat{\alpha}_L} - \dfrac{1}{\ell} \sum\limits_{i=1}^{\ell} \log^+ (-X_{i,n}) + \log^+ (-X_{\ell,n}) \right), \end{cases} \tag{15}$$

and

$$\begin{cases} \widehat{c}_R := \dfrac{m}{n} \dfrac{\widehat{\alpha}_R \widehat{\lambda}_R}{\widehat{\alpha}_R - \widehat{\lambda}_R} X_{n-m;n}^{\widehat{\alpha}_R} \left(\dfrac{1}{\widehat{\lambda}_R} - \dfrac{1}{m} \sum_{i=1}^{m} \log^+ (X_{n-i+1,n}) + \log^+ (X_{n-m,n}) \right), \\ \widehat{d}_R := \dfrac{m}{n} \dfrac{\widehat{\alpha}_R \widehat{\lambda}_R}{\widehat{\lambda}_R - \widehat{\alpha}_R} X_{n-m;n}^{\widehat{\lambda}_R} \left(\dfrac{1}{\widehat{\alpha}_R} - \dfrac{1}{m} \sum_{i=1}^{m} \log^+ (X_{n-i+1,n}) + \log^+ (X_{n-m,n}) \right). \end{cases} \tag{16}$$

The CML estimators $\left(\widehat{\alpha}_L, \widehat{\lambda}_L\right)$ of (α, λ) are solutions of the system

$$\begin{cases} \dfrac{1}{\ell} \sum_{i=1}^{\ell} \dfrac{1}{G_i^L (\alpha, \lambda)} = 1 \\ \dfrac{1}{\ell} \sum_{i=1}^{\ell} \dfrac{1}{G_i^L (\alpha, \lambda)} \left(\log^+ (-X_{i,n}) - \log^+ (-X_{\ell,n}) \right) \; = \; \lambda^{-1}, \end{cases} \tag{17}$$

where

$$G_i^L (\alpha, \lambda) := \frac{\alpha}{\lambda} \left(1 + \frac{\alpha\lambda}{\alpha - \lambda} H^L (\alpha) \right) \left(\frac{X_{i,n}}{X_{\ell,n}} \right)^{\lambda - \alpha} - \frac{\alpha\lambda}{\alpha - \lambda} H^L (\alpha),$$

with

$$H^L (\alpha) := \frac{1}{\alpha} - \left(\frac{1}{\ell} \sum_{i=1}^{\ell} \log^+ (-X_{i,n}) - \log^+ (-X_{\ell,n}) \right).$$

Whereas $\left(\widehat{\alpha}_R, \widehat{\lambda}_R\right)$ are solutions of the system

$$\begin{cases} \dfrac{1}{m} \sum_{i=1}^{m} \dfrac{1}{G_i^R (\alpha, \lambda)} = 1 \\ \dfrac{1}{m} \sum_{i=1}^{m} \dfrac{1}{G_i^R (\alpha, \lambda)} \left(\log^+ (X_{n-i+1,n}) - \log^+ (X_{n-m,n}) \right) \; = \; \lambda^{-1}, \end{cases} \tag{18}$$

where

$$G_i^R (\alpha, \lambda) := \frac{\alpha}{\lambda} \left(1 + \frac{\alpha\lambda}{\alpha - \lambda} H^R (\alpha) \right) \left(\frac{X_{n-i+1,n}}{X_{n-m,n}} \right)^{\lambda - \alpha} - \frac{\alpha\lambda}{\alpha - \lambda} H^R (\alpha),$$

with

$$H^R (\alpha) := \frac{1}{\alpha} - \left(\frac{1}{m} \sum_{i=1}^{m} \log^+ (X_{n-i+1,n}) - \log^+ (X_{n-m,n}) \right).$$

Next, we introduce the bias-reduced estimator of Wang's TSD Δ_r. First, let us write Δ_r, defined in (4) , as

$$\Delta_r = \Delta_{L,n} + \Delta_{M,n} + \Delta_{R,n},$$

where

$$\Delta_{L,n} := \int_0^{\ell/n} J_r(s)\, Q(s)\, ds,\ \ \Delta_{R,n} := \int_{1-m/n}^{1} J_r(s)\, Q(s)\, ds, \tag{19}$$

and

$$\Delta_{M,n} := \int_{\ell/n}^{1-m/n} J_r(s)\, Q(s)\, ds.$$

As for $\widehat{\Delta}_{n,r}$, the middle term $\Delta_{M,n}$ is estimated by $\sum_{j=\ell+1}^{n-m} a_{j,n}^{(r)} X_{j,n}$. On the other hand, by replacing the lower high quantile $Q(s)$ as $s \downarrow 0$ and the upper high quantile $Q(s)$ as $s \uparrow 1$, in equations (19), by their respective estimators $Q_L^{\mathbf{LPY}}(s)$ and $Q_R^{\mathbf{LPY}}(s)$, and after straightforward calculations, we get

$$\widetilde{\Delta}_{L,n} := \frac{r}{2}\widehat{c}_L^{1/\widehat{\alpha}_L}(\ell/n)^{-1/\widehat{\alpha}_L+r}\left(\frac{\widehat{\alpha}_L}{r\widehat{\alpha}_L-1} + \widehat{d}_L\frac{\widehat{c}_L^{-\widehat{\lambda}_L/\widehat{\alpha}_L}(\ell/n)^{\frac{\widehat{\lambda}_L-1}{\widehat{\alpha}_L}+r-1}}{\widehat{\lambda}_L-1+\widehat{\alpha}_L(r-1)}\right),$$

and

$$\widetilde{\Delta}_{R,n} := \frac{r}{2}c^{1/\widehat{\alpha}_R}(m/n)^{-1/\widehat{\alpha}_R+r}\left(\frac{\widehat{\alpha}_R}{r\widehat{\alpha}_R-1} + \widehat{d}_R\frac{c^{-\widehat{\lambda}_R/\widehat{\alpha}_R}(m/n)^{\frac{\widehat{\lambda}_R-1}{\widehat{\alpha}_R}+r-1}}{\widehat{\lambda}_R-1+\widehat{\alpha}_R(r-1)}\right),$$

to respectively estimate $\Delta_{L,n}$ and $\Delta_{R,n}$. Finally, our bias-reduced estimator, for Wang's TSD Δ_r, has the following form:

$$\widetilde{\Delta}_{n,r} := \widetilde{\Delta}_{L,n} + \sum_{j=\ell+1}^{n-m} a_{j,n}^{(r)} X_{j,n} + \widetilde{\Delta}_{R,n}. \tag{20}$$

3. Optimal choices of the sample fractions ℓ and m

In this paper we adopt the algorithm of Reiss and Thomas (1997), who proposed a heuristic method of choosing the optimal number of upper extremes used in the computation of the tail index estimate. For Hall's model (11), this methodology selects the value $\widetilde{k}$ of k which minimizes the quantity

$$\frac{1}{k}\sum_{i=1}^{k} i^{\theta}\left|\widehat{\gamma}_{1_{i,n}} - \text{median}\left(\widehat{\gamma}_{1_{1,n}},...,\widehat{\gamma}_{1_{k,n}}\right)\right|,\ \ 0 \le \theta \le 1/2, \tag{21}$$

where $\widehat{\gamma}_{1_{i,n}}$ is an estimator of the shape parameter γ_1, based on the i upper extremes. Notice that $\widetilde{k} = \widetilde{k}(\theta)$ with respect to $\theta = 0,...,0.5$. For a discussion on the choice of θ, one refers to the paper of Neves and Fraga Alves (2004).

In our simulation study, we apply the procedure above once to the left tail of cdf F and then to its right tail in order to determine the optimal numbers $\widetilde{\ell}$ and $\widetilde{m}$ of lower and upper order statistics, to be used in the computation of $\widetilde{\Delta}_{n,r}$. On the light of the information provided by the simulation study, we choose $\theta = 0.3$. That is, we select $\widetilde{\ell}$ and $\widetilde{m}$ which respectively minimize

$$\frac{1}{\ell}\sum_{i=1}^{\ell} i^{0.3}\left|\alpha_{i,n}^{L} - \text{median}\left(\hat{\alpha}_{1,n}^{L}, ..., \hat{\alpha}_{\ell,n}^{L}\right)\right| \text{ and } \frac{1}{m}\sum_{i=1}^{m} i^{0.3}\left|\alpha_{i,n}^{R} - \text{median}\left(\hat{\alpha}_{1,n}^{R}, ..., \hat{\alpha}_{m,n}^{R}\right)\right|,$$

where $\hat{\alpha}_{i,n}^{L}$ and $\hat{\alpha}_{i,n}^{R}$ designate the estimators of α respectively based on the i lower and upper extremes.

Once $\widetilde{\ell}$ and $\widetilde{m}$ at hand, we calculate the corresponding $\left(\widetilde{\alpha}_L, \widetilde{\lambda}_L, \widetilde{\alpha}_R, \widetilde{\lambda}_R\right)$ (solutions of the two systems (17) and (18)) and $\left(\widetilde{c}_L, \widetilde{d}_L, \widetilde{c}_R, \widetilde{d}_R\right)$ (given in (15) and (16)), in order to finally compute $\widetilde{\Delta}_{n:r}$ given in (20). Finally,

4. Simulation study

In this section, we carry out a simulation study, by means of the statistical software R (Ihaka and Gentleman, 1996) to compare the performance of our new estimator $\widetilde{\Delta}_{n,r}$ with the estimator $\widehat{\Delta}_{n,r}$ previously introduced by Necir and Meraghni (2010). But, before we start the simulations, we would like to mention that, for the sake of computation time optimization, we limit the research of $\widetilde{\ell}$ and $\widetilde{m}$ to the integer intervals $(\ell^*, 2\ell^*)$ and $(m^*, 2m^*)$ respectively, where ℓ^* and m^* are initial values for ℓ and m, obtained by applying the algorithm of Cheng and Peng (2001) to Hill estimators of α, given in (8) and (7).

For the sake of simplicity, we consider the standard α-stable distribution $S_\alpha(1, 0, 0)$. We choose $(\alpha, r) = (1.7, 0.6)$ and $(1.5, 0.7)$, so that $1/\alpha < r < 1$ (see, (9)), and we compute the corresponding true values of the TSD which turn out to be 1.659 and 1.508 respectively. Then, we draw 200 samples, of size $n = 2000$, from a rv $X \sim S_\alpha(1, 0, 0)$, to derive both TSD estimators $\widehat{\Delta}_{n,r}$ and $\widetilde{\Delta}_{n,r}$. Our results are obtained by averaging over the number of replications. We repeat the same procedure with another 200 samples of size $n = 5000$. The simulation results are summarized in Tables 1 and 2.

$n = 2000$				$n = 5000$			
$\widehat{\Delta}_{n,r}$		$\widetilde{\Delta}_{n,r}$		$\widehat{\Delta}_{n,r}$		$\widetilde{\Delta}_{n,r}$	
Bias	0.553	Bias	0.093	Bias	0.301	Bias	0.019
Rmse	0.554	Rmse	0.189	Rmse	0.302	Rmse	0.082
m^*	182	$\widetilde{m}$	262	m^*	433	$\widetilde{m}$	773
α_R^H	2.588	$\widehat{\alpha}_R$	2.321	α_R^H	2.506	$\widehat{\alpha}_R$	2.092
ℓ^*	191	$\widetilde{\ell}$	271	ℓ^*	441	$\widetilde{\ell}$	781
α_L^H	2.330	$\widehat{\alpha}_L$	2.120	α_L^H	2.045	$\widehat{\alpha}_L$	1.920

Table 1. Comparison of the new TSD estimator and Necir and Meraghni TSD estimator, based on 200 samples from the standard Lvy-stable distribution with $\alpha = 1.7$, $r = 0.6$. The true value of the TSD is 1.659.

$n = 2000$				$n = 5000$			
$\widehat{\Delta}_{n,r}$		$\widetilde{\Delta}_{n,r}$		$\widehat{\Delta}_{n,r}$		$\widetilde{\Delta}_{n,r}$	
Bias	0.407	Bias	0.087	Bias	0.168	Bias	0.009
Rmse	0.408	Rmse	0.153	Rmse	0.172	Rmse	0.051
m^*	181	$\widetilde{m}$	251	m^*	452	$\widetilde{m}$	772
α_L^H	2.003	$\widehat{\alpha}_R$	1.751	α_R^H	1.782	$\widehat{\alpha}_R$	1.710
ℓ^*	180	$\widetilde{\ell}$	250	ℓ^*	449	$\widetilde{\ell}$	769
α_R^H	1.878	$\widehat{\alpha}_L$	1.717	α_L^H	1.747	$\widehat{\alpha}_L$	1.708

Table 2. Comparison of the new TSD estimator and Necir and Meraghni TSD estimator, based on 200 samples from the standard Lvy-stable distribution with $\alpha = 1.5$, $r = 0.7$. The true value of the TSD is 1.508.

The second part of our simulation study consists of a graphical comparison between the bias and RMSE of $\widetilde{\Delta}_{n,r}$ and $\widehat{\Delta}_{n,r}$, as the sample fractions increase. We start by 100 samples of size $n = 2000$. For $i \in \{1, 2, ..., 21\}$, we compute, by averaging over all the replications, the value of $\widetilde{\Delta}_{n,r}$ for the couple of sample fractions $(\ell(i), m(i))$, where

$$\ell(i) = \ell^* + (i-1) \times 10 \text{ and } m(i) = m^* + (i-1) \times 10,$$

Let us denote it by $\widetilde{\Delta}_{n,r}(i)$ and the resulting bias and RMSE by $\widetilde{b}_{n,r}(i)$ and $\widetilde{e}_{n,r}(i)$ respectively. Next, we plot in distinct panels the two sets of points $\left((i-1) \times 10, \widetilde{b}_{n,r}(i)\right)$ and $((i-1) \times 10,\ \widetilde{e}_{n,r}(i))$, and we add in each panel, a horizontal line representing the bias and the RMSE of $\widehat{\Delta}_{n,r}$, see Figures 1 and 3. We repeat the same procedure for another 100 samples of size $n = 5000$, with $i = 1, 2, ..., 51$, see Figures 2 and 4.
In Table 1 for $\alpha = 1.7$, the graphical optimal values of $\left(\widetilde{\ell}, \widetilde{m}\right)$ is $(271, 262)$ and $(781, 773)$ corresponding to $i = 9$ and $i = 35$, respectively for $n = 2000$, and $n = 5000$, by applying Reiss and Thomas's algorithm we get respectively, $(269, 277)$ and $(771, 772)$. The same things in Table 2, for $\alpha = 1.5$, the graphical optimal values $\left(\widetilde{\ell}, \widetilde{m}\right)$ is $(250, 251)$ and $(769, 772)$ corresponding to $i = 8$ and $i = 33$, respectively for $n = 2000$, and $n = 5000$. If we apply Reiss and Thomas's algorithm in the above steps we get $(247, 256)$ and $(760, 763)$ respectively.

5. Concluding remarks

The overall conclusion that we might make is that, on the light the tables and figures, the new TSD estimator $\widetilde{\Delta}_{n,r}$ performs better in the sense of bias and RMSE, but with larger sample fractions than the estimator $\widehat{\Delta}_{n,r}$ proposed by Necir and Meraghni (2010).
Through our simulation study, we found that for $\alpha = 1.7$, the values of Hill estimator of the stability index exceed the limit 2 of the Lévy-stable regime. This issue is discussed by Weron (2001) who noted that for $\alpha \leq 1.5$ Hill's estimation is quite reasonable but as α approaches 2, there is a significant overestimation when considering samples of typical size. For such values of α, a very large number of observations (one million or more) is needed in order to obtain acceptable estimates and avoid misleading inference on the characteristic exponent,

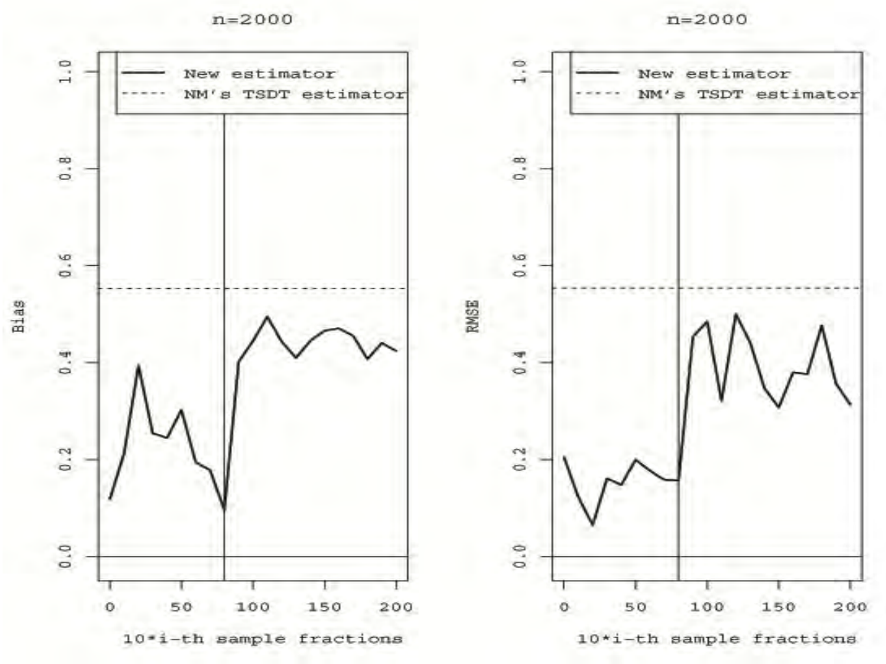

Fig. 1. Bias (left panel) and RMSE (right panel) of the new TSD estimator and Necir and Meraghni TSD estimator for $\alpha = 1.7$ and $r = 0.6$, based on 200 samples of 2000 stable observations.

because the true tail behavior of Lévy-stable distributions is only visible for extremely large data sets.

Acknowledgments. We thank the referee for her/his helpful comments.

Appendix A:

Lemma 1. *For any* $0 < r < 1$, *we have* $\mathbf{E}[\Delta_{n,r} - \Delta_r] = 0$ *and* $\sqrt{\mathbf{E}[\Delta_{n,r} - \Delta_r]^2} = \delta_r/\sqrt{n}$, *where* δ_r *is as in* (5).

Proof. Since $\Delta_{n:r}$ is an L-statistics, then from Shorack and Wellner (1986), page 662, we have

$$\Delta_{n,r} - \Delta_r \overset{d}{=} -\frac{1}{n}\sum_{i=1}^{n}\int_0^1 \left[1_{[U_i \leq t]} - t\right] J_r(t)\, dQ(t),$$

where $U_1, U_2, ...$ is a sequence of iid $(0,1)$-uniform rv's. Observe that

$$Y_i := \int_0^1 \left[1_{[U_i \leq t]} - t\right] J_r(t)\, dQ(t), \ i = 1, ..., n,$$

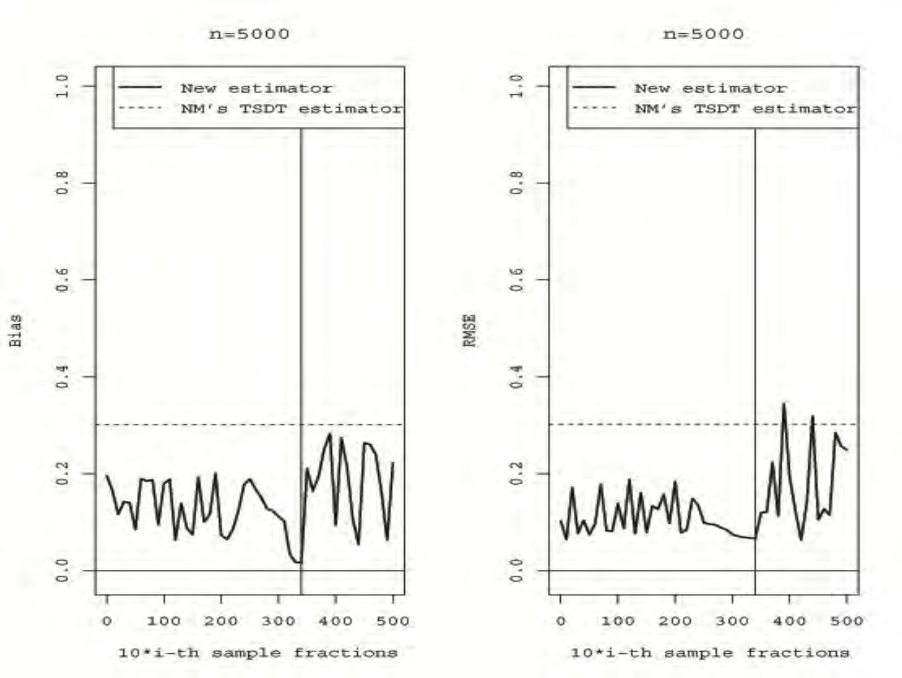

Fig. 2. Bias (left panel) and RMSE (right panel) of the new TSD estimator and Necir and Meraghni TSD estimator for $\alpha = 1.7$ and $r = 0.6$, based on 200 samples of 5000 stable observations.

are iid centred rv's, that is $\mathbf{E}[\Delta_{n,r} - \Delta_r] = 0$, with $\mathbf{E}[\Delta_{n,r} - \Delta_r]^2 = \mathbf{E}\left[Y_1^2\right]/n$. It is easy to verify that

$$\mathbf{E}\left[Y_1^2\right] = \int_0^1 \int_0^1 (\min(s,t) - st)\, J_r(s)\, J_r(t)\, dQ(s)\, dQ(t),$$

which is exactly δ_r^2. □

Lemma 2. *Suppose $F \sim S_\alpha(\sigma, \beta, \mu)$ for $1 < \alpha < 2$. Then for any $1 < r < 1$, we have $\delta_r = \infty$.*

Proof. Note that δ_r^2 may be rewritten into

$$\delta_r^2 = \int_0^1 G_r^2(s)\, ds - \left(\int_0^1 G_r(s)\, ds\right)^2,$$

where $G_r(s) := \int_0^s J_r(t)\, dQ(t)$ (see, e.g., equation (1.12) in Mason and Shorack, Mason and Shorack (1992)). It is clear that for any $0 < \varepsilon < 1$, we have $\int_0^1 G_r^2(s)\, ds \geq \int_0^\varepsilon G_r^2(s)\, ds$. On the other hand, from (1), we infer that $Q(t) \sim -C_1^{-1} t^{-1/\alpha}$, as $t \downarrow 0$, then

$$G_r(s) \sim (\alpha C_1)^{-1} \int_0^s t^{-1/\alpha - 1} J_r(t)\, dt, \text{ as } s \downarrow 0.$$

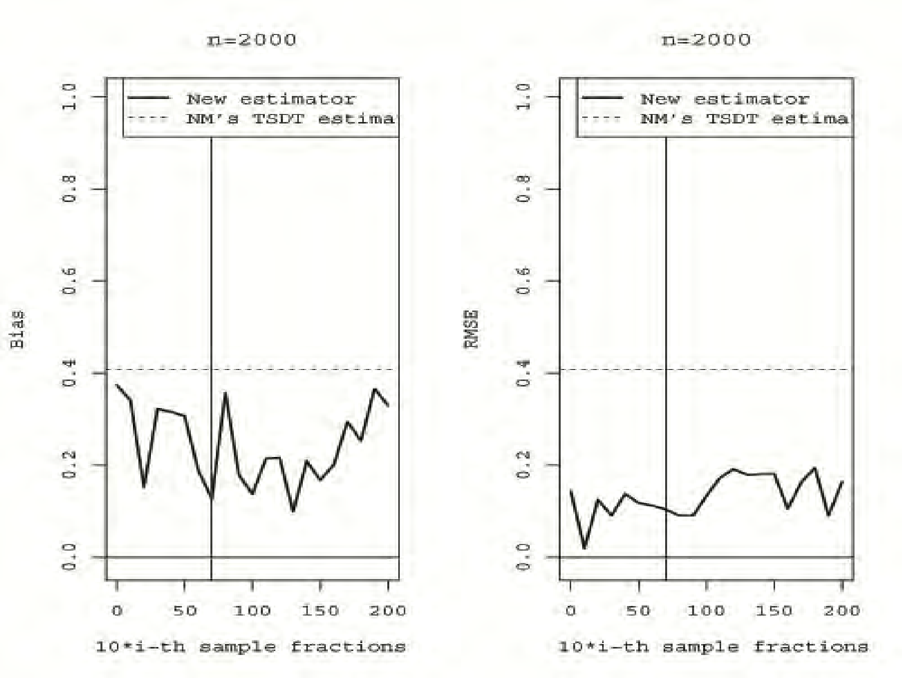

Fig. 3. Bias (left panel) and RMSE (right panel) of the new TSD estimator and Necir and Meraghni TSD estimator for $\alpha = 1.5$ and $r = 0.7$, based on 200 samples of 2000 stable observations.

Since $J_r(t) \sim -rt^{r-1}/2$, as $t \downarrow 0$, then it is readily checked that, for small $\varepsilon > 0$, we have

$$\int_0^{\varepsilon} G_r^2(s)\,ds \sim \begin{cases} \dfrac{r^2 \varepsilon^{2(r-1/\alpha-1)+1}}{4(r-1/\alpha)^2(2(r-1/\alpha)+1)}, & \text{if } 2(r-1/\alpha)-1 \geq 0, \\ \infty, & \text{if } 2(r-1/\alpha)-1 < 0. \end{cases}$$

This implies that $\delta_r^2 = \infty$ for any $0 < r < 1/2 + 1/\alpha$ which is verified for any $0 < r < 1$ and $1/2 < 1/\alpha < 1$. □

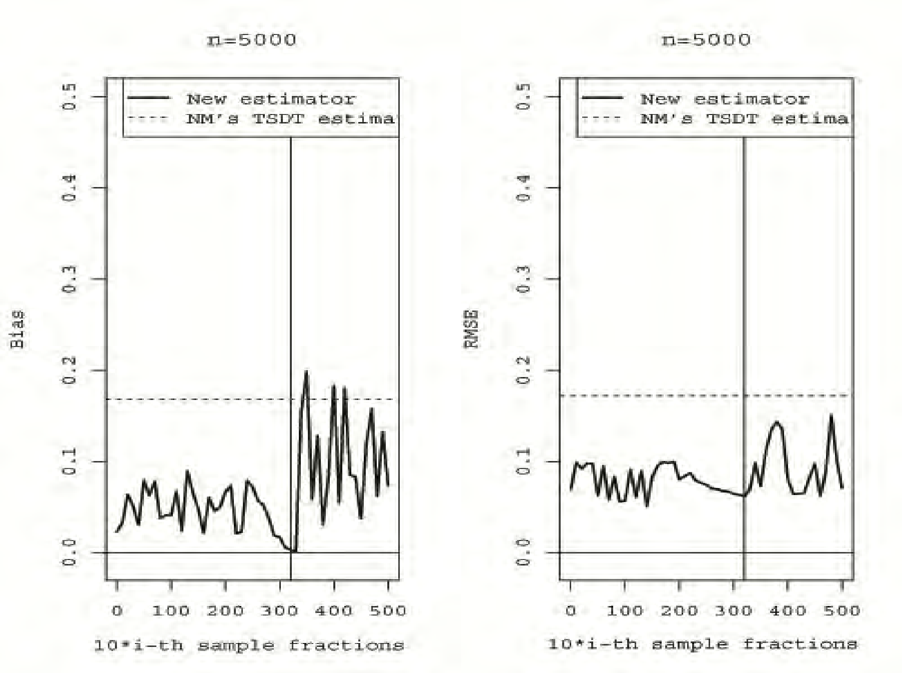

Fig. 4. Bias (left panel) and RMSE (right panel) of the new TSD estimator and Necir and Meraghni TSD estimator for $\alpha = 1.5$ and $r = 0.7$, based on 200 samples of 5000 stable observations.

References

Artzner, P., Delbaen, F., Eber, J.M. and Heath, D., 1999. Coherent measures of risk. *Math. Finance.* **9**(3), 203-228.

Beirlant, J., Diercks, G., Guillou, A. and Starica, C., 2002. On exponential representations of log-spacings of extreme order statistics. Extremes. **5**(2), 157-180.

Beirlant, J., Figueiredo, F., Gomes, M.I. and Vandewalle, B., 2008. Improved reduced bias tail index and quantile estimators. J. Statist. Plann. Inference. **138**, 1851-1870.

Brahimi, B., Meraghni, D., Necir, A. and Zitikis, R., 2011. Estimating the distortion parameter of the proportional-hazard premium for heavy-tailed losses. *Insurance Math. Econom.* **49**, 325-334.

Caeiro, F., Figueiredo, F. and Gomes, M.I., 2004. Bias reduction of a tail index estimator through an external estimation of the second order parameter. Statistics. **38**(6), 497-510.

Caeiro, F., Gomes, M.I. and Rodrigues, L.H., 2009. Reduced-bias tail index estimators under a third-order framework. Comm. Statist. Theory Methods. **38**(6-7), 1019-1040.

Cheng, S. and Peng, L., 2001. Confidence intervals for the tail index. Bernoulli **7**, 751-760.

Feuerverger, A. and Hall, P., 1999. Estimating a tail exponent by modelling departure from a Pareto distribution. Ann. Statist. **27**, 760-781.

Garcia, R., Renault, E. and Veredas, D., 2006. Estimation of Stable Distributions by Indirect Inference. Core Discussion Paper. 2006/112.

Gomes, M.I. and Figueiredo, F., 2006. Bias reduction in risk modelling: semi-parametric quantile estimation. *Test.* **15**(2), 375-396.

Gomes, M.I. and Martins, M.J., 2002. Asymptotically unbiased estimators of the tail index based on external estimation of the second order parameter. *Extremes.* **5**(1), 5-31.

Gomes, M.I. and Martins, M.J., 2004. Bias reduction and explicit efficient estimation of the tail index. *Journal of Statistical Planning and Inference.* **124**, 361-378.

Gomes, M.I. and Pestana, D., 2007. A simple second-order reduced bias' tail index estimator. J. Stat. Comput. Simul. **77**(5-6), 487-504.

de Haan, L. and Stadtmüller, U., 1996. Generalized regular variation of second order. *J. Austral. Math. Soc.* Ser. A **61**(3), 381-395.

Hall, P. 1982. On some simple estimators of an exponent of regular variation. J. Roy. Statist. Soc. Ser. B. **44**, 37-42.

Hall, P. and Welsh, A.H., 1985. Adaptive estimates of parameters of regular variation. *Ann. Statist.* **13**, 331-341.

Hill, B.M., 1975. A simple approach to inference about the tail of a distribution. *Ann. Statist.* **3**, 1136-1174.

Ihaka, R. and Gentleman, R., 1996. R: A Language for Data Analysis and Graphics. *Journal of Computational and Graphical Statistics.* **5**, 299-314.

Jones, B.L. and Zitikis, R., 2003. Empirical estimation of risk measures and related quantities. North American Actuarial Journal. **7**(4), 44-54.

Jones, B.L. and Zitikis, R., 2005. Testing for the order of risk measures: an application of L-statistics in actuarial science. *Metron.* **63**(2), 193-211.

Lévy, P., 1925. Calcul des probabilités. Paris, Gauthier-Villars.

Li, D., Peng, L. and Yang, J., 2010. Bias reduction for high quantiles. *Journal of Statistical Planning and Inference.* **140**, 2433-2441.

Mandelbrot, B., 1963. The variation of certain speculative prices. *Journal of Business.* **36**, 394-419.

Mason, D.M. and Shorack, G.R., 1992. Necessary and sufficient conditions for asymptotic normality of L-statistics. *Ann. Probab.* **20**, 1779-1804.

Matthys, G., Delafosse, M., Guillou, A. and Beirlant, J., 2004. Estimating catastrophic quantile levels for heavy-tailed distributions. Insurance Math. *Econom.* **34**, 517–537.

Meraghni, D. and Necir, A., 2007. Estimating the Scale Parameter of a Lévy-Stable Distribution via the Extreme Value Approach. *Methodology and Computing in Applied Probability.* **9**, 557-572.

Necir, A., Meraghni, D., and Meddi, F., 2007. Statistical estimate of the proportional hazard premium of loss. *Scand. Actuar. J*, **3**, 147-161.

Necir, A. and Meraghni, D. 2009. Empirical estimation of the proportional hazard premium for heavy-tailed claim amounts. *Insurance Math. Econom.* **45**(1), 49–58.

Necir, A. and Meraghni, D., 2010. Estimating L-functionals for Heavy-tailed Distributions and Applications. *Journal of Probability and Statistics.* Volume 2010, ID 707146. 34p.

Neves, C. and Fraga Alves, M.I., 2004. Reiss and Thomas' Automatic Selection of the Number of Extremes. *Computational Statistics and Data Analysis.* **47**, 689-704.

Peng, L., 2001. Estimating the mean of a heavy-tailed distribution. *Statistic and Probability Letters.* **52**, 255-264.

Peng, L. and Qi, Y., 2004. Estimating the first- and second -order parameters of a heavy-tailed distribution. *Aust. N. Z. J. Stat.* **46**(2), 305-312.

Reiss, R.D. and Thomas, M., 1997. *Statistical Analysis of Extreme Values with Applications to Insurance, Finance, Hydrology and Other Fields*, Third edition. Birkhäuser, Basel.

Samorodnitsky, G. and Taqqu, M.S., 1994. *Stable non-Gaussian random processes: Stochastic models with infinite variance.* Chapman & Hall, New York.

Shorack, G.R. and Wellner, J.A., 1986. *Empirical processes with applications to statistics.* New York: John Wiley & Sons.

Wang, S., 1998. An actuarial index of the right-tail risk. *North American Actuarial Journal.* **2**(2), 88-101.

Weissman, I., 1978. Estimation of parameters and large quantiles based on the k largest observations. *Journal of American Statistical Association.* **73**, 812-815.

Weron, R., 2001. Lévy-stable distributions revisited: Tail index > 2 does not exclude the Lévy-stable regime, *International Journal of Modern Physics.* C **12**, 209-223.

Zolotarev, V.M., 1986. *One-dimensional stable distributions.* Translated from the Russian by H. H. McFaden. Translation edited by Ben Silver. Translations of Mathematical Monographs, 65. American Mathematical Society, Providence, RI.