Journal Afrika Statistika
ISSN 0825-0305

Journal Afrika Statistika
Vol. 6, 2011, pages 346–370.
DOI: http://dx.doi.org/10.4314/afst.v6i1.4

Another look at Second order condition in Extreme Value Theory

Gane Samb Lo[†] **and Adja Mbarka Fall**[‡]

[†] LSTA, UPMC, France and LERSTAD, Université Gaston Berger de Saint-Louis, SENEGAL
[‡] MAPMO, Université d'Orléans, France and LERSTAD, Université Gaston Berger de Saint-Louis, SENEGAL

Received 01 June 2011; Accepted 01 November 2011

Abstract. This note compares two approaches both alternatively used when establishing normality theorems in univariate Extreme Value Theory. When the underlying distribution function (df) is the extremal domain of attraction, it is possible to use representations for the quantile function and regularity conditions (RC), based on these representations, under which strong and weak convergence are valid. It is also possible to use the now fashion second order condition (SOC), whenever it holds, to do the same. Some authors usually favor the first approach (the SOC one) while others are fond of the second approach that we denote as the representational one. This note aims at comparing the two approaches and show how to get from one to the other. The auxiliary functions used in each approach are computed and compared. Statistical applications using simultaneously both approaches are provided. A final comparison is provided.

Résumé. Cet article compare deux approches couramment et alternivement utilisées en vue d'établir des résultats de normalité asymptotique en Théorie des Valeurs Extrmes. Lorsque la fonction de répartition (fr) est dans le domaine d'attraction exremal, il est possible d'utiliser des hypothses basées sur les représentations des quantiles, et sous lesquelles des résultats de convergence forte, faible et/ou de loi sont établis. Il est aussi possible d'utiliser une méthode devenue standard, dite celle du second ordre. Chacune est associée à des fonctions dites auxilliaires, servant à exprimer les conditions de validité des résultats asymptotiques. L'une de ces deux méthodes est utilisée selon les auteurs. Dans ce papier, nous exposons une étude comparative et montrons comment passer de l'une à l'autre par le biais des fonctions auxilliaires. Cette étude permet une lecture comparative des articles selon l'approche utilisée. Deux exemples, le processus des grands quantiles et le processus de Hill fonctionnel, sont proposés comme exemples statistiques.

Key words: Extreme value theory; Quantile functions; Quantile representation; Theorem of Karamata; Slowly and regularly variation; Second order condition; Statistical estimation; Asymptotic normality.

AMS 2010 Mathematics Subject Classification : Primary 62E20, 62F12, 60F05. Secondary 60B10, 60F17.

Gane Samb Lo: gane-samb.lo@ugb.edu.sn, ganesamblo@ufrsat.org
Adja Mbarka Fall: amfall@ufrsat.org

1. Introduction

Statistical modelling based on the univariate Extreme Values Theory usually requires regularity conditions of the underlying distributions. Since the work of de Haan and Stadtmüller [7] on the the so-called Second Order Condition (SOC), using this SOC has became fashion in research papers so that this way of doing is the mainstream one, led by de Haan (see for instance [4], [12], [6]). However the second order condition does not always hold as we will show it (see (1)), although a large class of distribution functions fulfills it. Yet, there exists an other approach, that is the representational one, based on the Karamata representation for a slowly varying function. In this view, each distribution function F in the extremal domain may be represented a couple of functions $p(s)$ and $b(s)$, $s \in (0,1)$, to be precised in Theorem 1. This approach is the one preferred by many other authors, for instance Csörgő, Deheuvels, and Mason, [2], Lo [9], Hall (see [10],[11]) etc. This latter in particularly adapted for the use on the Gaussian approximations like that of Csörgő-Csörgő-Horvàth-Mason [1].

This motivates us to undertake here a comparative study of the second order condition in the two approaches and provide relations and methods for moving from one to the other. We give specific statistical applications using simultanuously the two ways. The paper is to serve as a tool for comparitive reading of papers based on the two approaches.

The paper is organized as follows. In Section 1, we introduce the second order condition in the frame of de Haan and Stadmüller [7] using quantile functions. In Section 3, we recall the representational scheme and link them to the second order condition. Precisely, we express the second order condition, when it holds, through the couple of functions (p, b) associated with a *df* attracted to the extremal domain. The results are then given through the *df* $G(x) = F(e^x)$, $x \in R$, that is the most used in statistical context. In Section 4, we settle a new writing the SOC for the quantiles while the auxiliary functions of that condition, denoted as s and S, are computed for a large number of $df's$. In Section 6, we deal with applications in statistical contexts. The first concerns the asymptotic normality of the large quantiles process and the second treats the functional Hill process. In both cases, we use the two approaches. We finish by comparing the two methods at the light of these applications.

2. The second order condition

2.1. Definition and expressions.

Consider a *df* F lying in the extremal domain of attraction of the Generalized Extreme Value (GEV) distribution, that is

$$G_\gamma(x) = exp(-(1+\gamma x)^{-1/\gamma}), for 1 + \gamma x > 0,$$

denoted denoted $F \in D(G_\gamma)$, and let $U(x) = (1/(1-F))^{-1}$, where for any nondecreasing and right-continuous function $L : R \mapsto [a,b]$, with $a < b$,

$$L^{-1}(t) = \inf\{x \geq t, L(x) \geq t\}, a \leq t \leq b,$$

is the generalized inverse of L. One proves (see [6], p. 43) that there exists a positive function $a(t)$ of $t \in R$, such that

$$\forall (x > 0), \lim_{t\to\infty} \frac{U(xt) - U(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma} = D_\gamma(x),$$

where $(x^\gamma - 1)/\gamma$ is interpreted as $\log(x)$ for $\gamma = 0$. Now, by definition, F is said to satisfy a second order condition ([7]) if and only if there exists a function $A(t)$ of $t \in R$ with a constant sign such that

$$\lim_{t\to\infty} \frac{\frac{U(xt)-U(t)}{a(t)} - \frac{x^\gamma-1}{\gamma}}{A(t)} = H(x), \tag{SOCU}$$

holds. According to Theorem 2.3.3 in (de haan and Ferreira), the function H, when it is not a multiple of $D_\gamma(x)$, can be written as

$$H_{\gamma,\rho}(x) = c_1 \int_1^x s^{\gamma-1} \int_1^s u^{\gamma-1}\, du\, ds + c_2 \int_1^x s^{\gamma+\rho-1} ds,$$

where ρ is a negative number and the functions $a(t)$ and $A(t)$ satisfies for any $x > 0$,

$$\lim_{t\to\infty} \frac{a(tx)/a(t) - x^\gamma}{A(t)} = c_1 x^\gamma \frac{x^\rho - 1}{\rho}$$

and

$$\lim_{t\to\infty} A(tx)/A(t) = x^\rho.$$

According to Corollary 2.3.5 in de Haan and Ferreira, one can choose a positive function $a^*(t)$ and a function $A^*(t)$ of constant sign such that

$$\frac{\frac{U(xt)-U(t)}{a^*(t)} - \frac{x^\gamma-1}{\gamma}}{A^*(t)} \to H^*_{\gamma,\rho}(x),$$

with

$$H^*_{\gamma,\rho}(x) = \begin{cases} \frac{x^{\gamma+\rho}-1}{\gamma+\rho}, & \gamma+\rho \neq 0, \rho < 0, \\ \log x, & \gamma+\rho = 0, \rho < 0, \\ \frac{1}{\gamma} x^\gamma \log x, & \gamma \neq 0, \rho = 0, \\ (\log(x))^2/2, & \gamma = \rho = 0, \end{cases}$$

$$a^*(t) = \begin{cases} a(t)(1 - A(t)/\rho), & \rho > 0, \\ a(t)(1 - A(t)/\gamma), & \rho = 0, \gamma \neq 0, \\ a(t), & \gamma = \rho \end{cases}$$

and

$$A^*(t) = A(t)/\rho I(\rho > 0) + A(t) I(\rho = 0).$$

To see that the *SOC* does not necessary hold, consider the standard exponential distribution function. We have $U(t) = log(t)$ and

$$U(tx) - U(t) = \log x = [(x^\gamma - 1)/\gamma]_{\gamma=0}. \tag{1}$$

It is clear that the function a is necessarily constant and equal to the unity and the second order condition is here meaningless. As a consequence, the results obtained under a second order condition are partial.

2.2. Expression in terms of generalized inverse functions.

We are going to express the SOC through the generalized function $F^{-1}(1-u)$. Let

$$F \in D(G_\gamma).$$

With this parameterization, the case $\gamma = 0$ corresponds to $D(\Lambda)$, the case $-\infty < \gamma < 0$ to $D(\psi_{1/\gamma})$ and finally, the case $0 < \gamma < +\infty$ to $D(\phi_{1/\gamma})$. The second order condition will become : there exist a positive function $s(u)$ and a function $S(u)$ with constant sign such that for any $x > 0$,

$$\frac{\dfrac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \gamma^{-1}(x^\gamma - 1)}{S(u)} = H_{\gamma,\rho}(1/x) = h_{\gamma,\rho}(x). \tag{SOCF}$$

3. Representation for F$\in D(G_\gamma)$

3.1. Representations

Now we recall the classical representations of *df* attracted to some nondegenerated extremal *df*.

Theorem 1. *We have :*

1. *Karamata's representation (KARARE)*
   (a) *If $F \in D(G_\gamma)$, $\gamma > 0$, then there exist two measurable functions $p(u)$ and $b(u)$ of $u \in (0,1)$ such that $\sup(|p(u)|, |b(u)|) \to 0$ as $u \to 0$ and a positive constant c so that*

   $$G^{-1}(1-u) = \log c + \log(1+p(u)) - \gamma \log u + \left(\int_u^1 b(t)t^{-1}dt\right), \; 0 < u < 1, \tag{2}$$

   where $G^{-1}(u) = \inf\{x, G(x) \geq u\}$, $0 < u \leq 1$ is the generalized inverse of G with $G^{-1}(0) = G^{-1}(0+)$.

   (b) *If $F \in D(G_\gamma)$, $\gamma < 0$, then $y_0(G) = \sup\{x, \; G(x) < 1\} < +\infty$ and there exist two measurable functions $p(u)$ and $b(u)$ for $u \in (0,1)$ and a positive constant c as defined in (2) such that*

   $$y_0 - G^{-1}(1-u) = c(1+p(u))u^{-\gamma} \exp\left(\int_u^1 b(t)t^{-1}dt\right), 0 < u < 1. \tag{3}$$

2. *Representation of de Haan (Theorem 2.4.1 in [5]),*
   If $G \in D(G_0)$, then there exist two measurable functions $p(u)$ and $b(u)$ of $u \in (0,1)$ and a positive constant c as defined in (2) such that for

   $$s(u) = c(1+p(u)) \exp\left(\int_u^1 b(t)t^{-1}dt\right), \; 0 < u < 1, \tag{4}$$

   we have for some constant $d \in R$,

   $$G^{-1}(1-u) = d - s(u) + \int_u^1 s(t)t^{-1}dt, 0 < u < 1. \tag{5}$$

It is important to remark at once that any *df* in the extremal domain of attraction is associated with a couple of functions (p, b) used in each appropriate representation.

3.2. Preparation of second order condition.

We are now proving, under $F \in D(G_\gamma)$ that for $x > 0$,

$$\lim_{u\to\infty} \frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} = d_\gamma(x) = \gamma^{-1}(x^{-\gamma} - 1).$$

3.2.1. F $\in D(G_0)$

The representation (5) is valid for $F^{-1}(1-u)$, $u \in (0,1)$. We get for $u \in (0,1)$ and $x > 0$:

$$F^{-1}(1-xu) - F^{-1}(1-u) = s(u) - s(ux) + \int_{ux}^{u} \frac{s(t)}{t} dt.$$

For $v \in [\min(ux,u), \max(ux,u)] = A(u,x)$,

$$s(v)/s(u) = \frac{1+p(t)}{1+p(u)} \exp\left(\int_v^u \frac{b(t)}{t} dt\right).$$

By letting $pr(u,x) = \sup\{|p(t)|, 0 \leq t \leq \max(ux,u)\}$ and $br(u,x) = \sup\{|b(t)|, 0 \leq t \leq \max(ux,u)\}$, one quickly shows that, for u sufficiently small,

$$\sup_{t\in A(u,x)} |1 - (1+p(v))/(1+p(u)| \leq 2pr(u,x).$$

and

$$x^{-br(u,x)} \leq \exp\left(\int_{ux}^u \frac{b(t)}{t} dt\right) \leq x^{br(u,x)}$$

and then

$$\sup_{v\in A(u,x)} \left|1 - \exp\left(\int_v^u \frac{b(t)}{t} dt\right)\right| = O(-br(u,x)).$$
$$\leq |1+x|\,(1 \vee |x|)^{1+br(u,x)}$$

It follows that

$$\sup_{v\in A(u,x)} |1 - s(v)/s(u)| = O(\max(pr(u,x), br(u,x)) \to 0,$$

as $u \to 0$. We get, for $pbr(u,x) = pr(u,x) \times br(u,x)$,

$$\frac{F^{-1}(1-xu) - F^{-1}(1-u)}{s(u)} + \log x$$
$$= O(pbr(u,x)) + O(pbr(u,x)\log x) \to 0.$$

Then

$$\lim_{u\to 0} \frac{F^{-1}(1-xu) - F^{-1}(1-u)}{s(u)} = -\log x = \left[\gamma(x^{-1/\gamma} - 1)\right]_{\gamma=\infty}.$$

We notice that

$$\frac{F^{-1}(1-xu) - F^{-1}(1-u)}{s(u)} + \log x \tag{6}$$
$$= O(p(u,x) + b(u,x)),$$

where

$$p(u,x) = 1 - (1-p(ux))/(1-p(u)) \to 0 \ as \ u \to 0 \tag{7}$$

and

$$b(u,x) = \int_{ux}^u \frac{1}{t}\left[\frac{1-p(t)}{1+p(u)} \exp\left(\int_u^t v^{-1}b(v)\ dv\right) - 1\right] dt \to 0 \ as \ u \to 0. \tag{8}$$

3.2.2. F$\in D(G_{1/\gamma})$ $(\gamma > 0)$

We have the KARARE representation

$$F^{-1}(1-u) = c(1+p(u))u^{-\gamma}\exp\left(\int_u^1 t^{-1}b(t)dt\right), u \in (0,1).$$

Then, for $s(u) = \gamma c u^{-\gamma}\ exp(\int_u^1 t^{-1}b(t)\ dt)$, for $x > 0$

$$\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)}$$
$$= \gamma^{-1}\left\{(1+p(ux))x^{-1/\gamma}\exp\left(\int_{ux}^x t^{-1}b(t)\ dt\right) - 1 - p(u)\right\}.$$

As previously, we readily see that

$$\exp\left(\int_{ux}^x t^{-1}b(t)\ dt\right) = \exp(O(br(u,x)\log x))$$
$$= 1 + O(br(u,x)\log x) \to 1,$$

as $u \to 0$. It follows that

$$\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} \to \gamma^{-1}(x^{-\gamma} - 1). \tag{9}$$

Moreover we have

$$\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \gamma^{-1}(x^{-\gamma} - 1)$$
$$= \gamma^{-1}\left\{x^{\gamma}\left((1+p(ux))\ exp\left(\int_{ux}^x t^{-1}b(t)\ dt\right) - 1\right) - p(u)\right\}$$
$$= pb(u,x). \tag{10}$$

$$= \gamma^{-1}\left\{x^{\gamma}\left((1+p(ux))\ exp\left(\int_{ux}^x t^{-1}b(t)\ dt\right) - 1\right) - p(u)\right\} = pb(u,x). \tag{11}$$

Notice that we may also take $s(u) = F^{-1}(1-u)$.

3.2.3. F$\in D(G_\gamma), \gamma < 0$

We have $x_0 = \sup\{x, F(x) < 1\} < +\infty$ and the following representation holds :

$$x_0 - F^{-1}(1-u) = c(1+p(u))u^{-\gamma}\exp\left(\int_u^1 t^{-1}b(t)dt\right), u \in (0,1).$$

Then, for $s(u) = -\gamma c u^{-\gamma}\exp(\int_u^1 t^{-1}b(t)dt)$, $u \in (0,1)$ and $x > 0$,

$$\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} = \frac{(x_0 - F^{-1}(1-u)) - (x_0 - F^{-1}(1-ux))}{s(u)}$$
$$= -\gamma^{-1}\left\{1 + p(u) - x^{-\gamma}(1+p(ux))\overset{x}{\exp}\left(\int_{ux} t^{-1}b(t)dt\right)\right\}$$

$$= \gamma^{-1} \left\{ x^{-\gamma}(1 + p(ux)) \exp\left(\int_{ux}^{x} t^{-1} b(t) dt \right) \right\} - 1 - p(u)$$

$$\to \gamma^{-1}(x^{-\gamma} - 1),$$

as $u \to 0$. Likely to the case $0 < \gamma < \infty$,

$$\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \gamma(x^{-1/\gamma} - 1) \tag{12}$$

$$= \gamma^{-1} \left\{ x^{-\gamma} \left((1 + p(ux)) \ \exp\left(\int_{ux}^{x} t^{-1} b(t) \ dt \right) - 1 \right) - p(u) \right\} = pb(u, x). \tag{13}$$

Notice that we may also take $s(u) = x_0 - F^{-1}(1-u)$.

4. Second order condition via representations

4.1. Case by case

4.1.1. $F \in D(G_\gamma), 0 < \gamma < \infty$

The second order condition is equivalent to finding a fonction $S(u)$ of constant sign such that

$$S(u)^{-1} \left\{ \frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \gamma^{-1}(x^{-\gamma} - 1) \right\} = S(u)^{-1} pb(u, x)$$

converges to a function $h_{\gamma,\rho}$, where

$$pb(u, x) = \gamma \left\{ x^{-\gamma} \left((1 + p(ux)) \ \exp\left(\int_{ux}^{x} t^{-1} b(t) \ dt \right) - 1 \right) - p(u) \right\}$$

and $s(u)$ may be taken *as* $F^{-1}(1-u)$.

4.1.2. $F \in D(G_\gamma), -\infty < \gamma < 0$

The second order condition is equivalent to finding a fonction $S(u)$ of constant sign such that

$$S(u)^{-1} \left\{ \frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \gamma^{-1}(x^{-\gamma} - 1) \right\} = S(u)^{-1} pb(u, x)$$

converges to a function $h_{\gamma,\rho}$ where

$$pb(u, x) = \gamma^{-1} \left\{ x^{-\gamma} \left((1 + p(ux)) \ \exp\left(\int_{ux}^{x} t^{-1} b(t) dt \right) - 1 \right) - p(u) \right\}$$

and $s(u)$ may be taken *as* $x_0 - F^{-1}(1-u)$ and x_0 is the upper endpoint of F.

4.1.3. $F \in D(G_0)$

The second order condition is equivalent to finding a fonction $S(u)$ of constant sign such that

$$S(u)^{-1}\left\{\frac{F^{-1}(1-ux)-F^{-1}(1-u)}{s(u)}-\log x\right\}=S(u)^{-1}(p(u,x)+b(u,x))$$

converges to a function $h_{1/\gamma,\rho}$, where

$$p(u,x)=1-(1-p(ux))/(1-p(u))\rightarrow 0 \ as \ u\rightarrow 0$$

and

$$b(u,x)=\int_{ux}^{u}\frac{1}{t}\left(\frac{1-p(t)}{1+p(u)}\exp\left(\int_{u}^{t}v^{-1}b(v)dv\right)-1\right)dt\rightarrow 0 \ as \ u\rightarrow 0.$$

and $s(u)$ may be taken $u^{-1}\int_u^1(1-s)dF^{-1}(s)$.

5. Special cases

5.1. Statistical context.

In the statistical context, especially in the exteme value index estimation, the bulk of the work is done with

$$G^{-1}(1-u)=\log F^{-1}(1-u).$$

Let $F\in D(G_\gamma)$. The three cases $-\infty<\gamma<0$, $0<\gamma<+\infty$, $\gamma=0$ respectively imply

$$G\in D(G_{1/\gamma}),$$

$$G\in D(G_0) \text{ and } s(u,G)\rightarrow\gamma$$

and

$$G\in D(G_0) \text{ and } s(u,G)\rightarrow 0.$$

For $F\in D(G_\gamma)$, $0<\gamma<\infty$, we have a representaion like

$$G^{-1}(1-u)=c+\log(1+p(u))-\gamma\log u+\int_u^1 t^{-1}b(t)dt.$$

We take here $s(u)=\gamma$. The second order conditions becomes

$$\begin{aligned}S(u)^{-1}&\left\{\frac{G^{-1}(1-ux)-G^{-1}(1-u)}{\gamma}-\log x\right\}\\ &= S(u)^{-1}\left\{\gamma^{-1}\log\frac{1+p(ux)}{1+p(u)}+\gamma^{-1}\int_{ux}^{u}t^{-1}b(t)dt\right\}\\ &= A(u)^{-1}\ pb(u,x)\rightarrow h_{0,\rho}(x).\end{aligned}$$

Denote $dG^{-1}(1-u)/du=G^{-1}(1-u)'$ whenever if exists. Now if $G^{-1}(1-u)'$ exists for u near zero, we may take

$$b(u)=G^{-1}(1-u)'+\gamma\rightarrow 0,$$

For $F\in D(G_\gamma)$, $-\infty<\gamma<0$, we may transfer the SOC to G in a way similar as to F, with

$$\log x_0-G^{-1}(1-u)=c(1+p(u))u^{\gamma}\exp\left(\int_u^1 t^{-1}b(t)dt\right).$$

For $F \in D(G_0)$. If $s(u) = u(G^{-1}(1-u)') \to 0$, we will have

$$G^{-1}(1-s) = d - \int_u^1 t^{-1} s(t) dt.$$

We may take

$$b(u) = us'(u).$$

The second order condition becomes simpler as

$$\begin{aligned} &S(u)^{-1} b(u,x) \\ &= S(u)^{-1} \int_{ux}^{u} \frac{1}{t} \left(\frac{1-p(t)}{1+p(u)} \exp\left(\int_u^t v^{-1} b(v)\ dv \right) - 1 \right) dt \\ &\to h_{0,\rho}(x). \end{aligned}$$

Moreover, for $g(x) = dG(x)/dx$, if

$$b(u) = us'(u)/s(u) = 1 - u(G^{-1}(1-u) g(G^{-1}(1-u))^{-1} \to 0,$$

we have

$$s(u) = c \exp\left(\int_u^1 t^{-1} b(t) dt \right),$$

and the SOC becomes

$$\begin{aligned} &S(u)^{-1} b(u,x) \\ &= S(u)^{-1} \int_{ux}^{u} \frac{1}{t} \left[\exp\left(\int_u^t \left(\frac{1}{v} - \frac{1}{G^{-1}(1-\nu) g(G^{-1}(1-\nu)} \right) - 1 \right) d\nu \right] dt \\ &\to h_{0,\rho}(x). \end{aligned}$$

6. Finding the functions b and S.

6.1. Determination of the function b

In the usual cases, the function is ultimately differentiable, that is in a right neigbourhood of $x_0(F)$. It is then easy to find the function b by derivating $G^{-1}(1-u)$. In summary, for $D(G_\gamma)$, $\gamma > 0$, the function b in the representation of $G^{-1}(1-u)$ is

$$b(u) = -\left\{ \gamma + u(G^{-1}(1-u))' \right\}.$$

The function b in the representation of $F^{-1}(1-u)$ is defined by

$$-b(u) = \gamma + u(G^{-1}(1-u))'/G^{-1}(1-u))$$

For $\gamma = +\infty$, the function b is the representation of $G^{-1}(1-u)$, that is,

$$b(u) = -us'(u)/s(u),$$

where

$$s(u) = u(G^{-1}(1-u))'.$$

For $\gamma < 0$ and $y_0(G) = y_0$,

$$b(u) = -\gamma + u(G^{-1}(1-u))'/(x_0 - G^{-1}(1-u))$$

Then we apply these formulas and determine the function b for usual *df*'s. Regularity conditions in the representational approach mainly rely on the function b, while they rely of the function S for the SOC approach. It is then interesting to have both functions for usual *df*'s in tables in Subsection 6.3, following [12].

6.2. The function S for the second order condition

Functions a and A in the $(SOCU)$, as well as the functions $H_{\gamma,\rho}$ are available in the usual cases (see [12] for example). It is not the case for the $(SOCF)$ expressed in terms of the quantile functions. We then seize this opportunity to compute their analogs s and S in the this case for the usual *df*'s. The results are summarized in our tables in Subsection 6.3.

6.2.1. The Singh-Maddala Law

Let for constants a, b and c, for $x \geq 0$,

$$1 - F(x) = (1 + ax^b)^{-c},$$

the so-called Singh-Madalla *df*. This function plays a special role in income fitting distribution. It is clear that

$$F \in D(G_{bc}).$$

Put

$$\lambda = 1/bc.$$

Straightforward calculations give

$$\begin{aligned} G^{-1}(1-u) &= -b^{-1}\log a - \gamma \log u + b^{-1}\log(1-u^{1/c}) \\ &= d - \gamma \log u - \frac{1}{b}\log(1-u_0) + \int_u^{u_0} t^{-1}B(t)dt, \end{aligned}$$

where

$$a = -b^{-1}\log a,$$

$$B(u) = \gamma u^{1/c}(1-u^{1/c})^{-1}$$

and $u_0 \in]0,1[$. Put $K_0 = -\frac{1}{b}\log(1-u_0)/u_0$ and

$$b(u) = B(u)\mathbb{I}_{(0 \leq u \leq u_0)} + K_0\mathbb{I}_{(u_0 \leq u \leq 1)},$$

we get

$$G^{-1}(1-u) = d - \gamma \log u + \int_u^1 t^{-1}b(t)dt,$$

with

$$b(u) \to 0.$$

We have

$$\begin{aligned} \frac{G^{-1}(1-ux) - G^{-1}(1-u)}{\gamma} + \log x &= \frac{1}{\gamma b}(\log(1-(ux)^{1/c}) - \log(1-u^{1/c})) \\ &= \frac{1}{\gamma b}(-x^{1/c}u^{1/c} + u^{1/c} + O(u^{2/c})). \end{aligned}$$

Thus, for $S(u) = cu^{1/c}/(\gamma b)$, we get

$$\frac{\frac{G^{-1}(1-ux) - G^{-1}(1-u)}{\gamma} - \log x}{S(u)} = \frac{x^{1/c}-1}{-1/c} = h_{0,\rho}(x) = H_{0,\rho}(1/x).$$

This corresponds to a second order condition. As for $F^{-1}(1-u)$ itself, we have

$$F^{-1}(1-u) = (1/a)^{1/b}(1-u^{-1/c})^{1/b}$$

and

$$b(u) = F^{-1}(1-u)'/F^{-1}(1-u) + \frac{1}{bc} = (1 - \frac{u^{-1/c}}{1+u^{-1/c}})/bc \to 0.$$

We have for $s(u) = F^{-1}(1-u)/(bc)$,

$$\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} = bc\left(\left(\frac{1-x^{-1/c}u^{-1/c}}{1-u^{-1/c}}\right)^{1/b} - 1\right) \to \frac{x^{-1/bc}-1}{1/bc}.$$

Next

$$\begin{aligned}\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \frac{x^{-1/bc}-1}{1/bc} &= (bc)\left\{\left(\frac{1-x^{-1/c}u^{-1/c}}{1-u^{-1/c}}\right)^{1/b} - \left(x^{-1/c}\right)^{1/b}\right\} \\ &= (bc)b^{-1}\left\{\frac{1-x^{-1/c}u^{-1/c}}{1-u^{-1/c}} - \frac{x^{-1/c}(1-u^{-1/c})}{1-u^{-1/c}}\right\}\zeta(u,x)^{(1/b)-1},\end{aligned}$$

where $\zeta(u,x) \in I(x^{-1/c}, b)$, with $b = (1-x^{-1/c}u^{-1/c})/(1-u^{-1/c}) \sim x^{-1/c}$. Hence

$$\begin{aligned}\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \frac{x^{-1/bc}-1}{1/bc} &= c\left\{\frac{1-x^{-1/c}u^{-1/c}}{1-u^{-1/c}} - \frac{x^{-1/c}(1-u^{-1/c})}{1-u^{-1/c}}\right\}\zeta(u,x)^{(1/b)-1} \\ &= c\left\{\frac{1-x^{-1/c}}{1-u^{-1/c}}\right\}\zeta(u,x)^{(1/b)-1}.\end{aligned}$$

Finally for $S(u) = -(1-u^{-1/c})^{-1}$,

$$\frac{\frac{F^{-1}(1-ux)-F^{-1}(1-u)}{s(u)} - \frac{x^{-1/bc}-1}{1/bc}}{S(u)} \to h(x) = \frac{\left(x^{-1/c}-1\right)x^{-1/(bc)+1/c}}{1/c}$$

6.2.2. Burr's df

$1-F(x) = (x^{-\rho/\gamma}+1)^{1/\rho}, \rho < 0$, and

$$F^{-1}(1-u) = (u^{\rho}-1)^{-\gamma/\rho}.$$

For $s(u) = \gamma F^{-1}(1-u)$,

$$\begin{aligned}\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} &= \frac{1}{\gamma}\left\{\frac{(x^{\rho}u^{\rho}-1)^{-\gamma/\rho}}{(u^{\rho}-1)^{-\gamma/\rho}} - 1\right\} \\ &= \frac{1}{\gamma}\left\{\left(\frac{x^{\rho}u^{\rho}-1}{u^{\rho}-1}\right)^{-\gamma/\rho} - 1\right\} \\ &\to \frac{x^{-\gamma}-1}{\gamma}.\end{aligned}$$

Next

$$\begin{aligned}\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \frac{x^{-\gamma}-1}{\gamma} &= \frac{1}{\gamma}\left\{\left(\frac{x^{\rho}u^{\rho}-1}{u^{\rho}-1}\right)^{-\gamma/\rho} - x^{-\gamma}\right\} \\ &= \frac{1}{\gamma}\left\{\left(\frac{x^{\rho}u^{\rho}-1}{u^{\rho}-1}\right) - (x^{\rho})^{-\gamma/\rho}\right\} \\ &= -\frac{1}{\rho}\left\{\frac{x^{\rho}u^{\rho}-1-x^{\rho}(u^{\rho}-1)}{u^{\rho}-1}\right\}\zeta(u,x)^{-\gamma/\rho-1},\end{aligned}$$

where $\zeta(u,x) = I(a,b) = [a \wedge b, a \vee b]$, with $a = x^{\rho}$ and $b = (x^{\rho}u^{\rho}-1)/(u^{\rho}-1) \sim x^{\rho}$. Hence

$$\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \frac{x^{-\gamma}-1}{\gamma} = -\frac{1}{\rho}\left\{\frac{x^{\rho}-1}{u^{\rho}-1}\right\}\zeta(u,x)^{-\gamma/\rho-1}.$$

Hence for $S(u) = (u^{\rho}-1)^{-1}$,

$$\frac{\dfrac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \dfrac{x^{-\gamma}-1}{\gamma}}{S(u)} \to -\frac{(x^{\rho}-1)x^{-\gamma-\rho}}{\rho}.$$

6.2.3. Log Exponential law

$F(x) = 1 - exp(-e^{-x})$, that is

$$F^{-1}(1-u) = \log\log(1/u).$$

We have

$$F^{-1}(1-u)' = -1/(u\log u)$$

and let

$$s(u) = uF^{-1}(1-u)' = -1/\log u$$

so that

$$s'(u) = -1/(u(\log u)^2).$$

Finally

$$b(u) = -(\log u)^{-2} \to 0.$$

With the representation

$$F^{-1}(1-u) = c + \int_u^1 \frac{s(t)}{t}dt,$$

where s is slowly varying at zero, we get that

$$\frac{F^{-1}(1-ux)-F^{-1}(1-u)}{s(u)} \to -\log x.$$

We can use direct methods and get

$$F^{-1}(1-ux)-F^{-1}(1-u)=\log(\log(ux)/\log u).$$

We remark $v(u,x)=\log(ux)/\log u \to 1$, and that $v(u,x)-1=(\log x)/\log u$. We may use the expansion of the logarithm function and get

$$\begin{aligned} F^{-1}(1-ux)-F^{-1}(1-u) &= (v-1)-(v-1)^2+O\left((v-1)^3\right) \\ = (\log x)/\log u+((\log x)/\log u)^2/2+O((\log x)/\log u)^3). \end{aligned}$$

By putting

$$S(u)=s(u)=1/(\log u),$$

It comes that

$$\frac{\frac{F^{-1}(1-ux)-F^{-1}(1-u)}{s(u)}+\log x}{S(u)} \to (\log x)^2/2.$$

6.2.4. Normal standard

Let F be the $d.f.$ of a standard normal law. We have the simple approximation, for $M=\sqrt{2\pi}$, for $x>1$,

$$M^{-1}(x^{-1}-x^{-3})e^{-x^2/2} \leq 1-F(x) \leq M^{-1}x^{-1}e^{-x^2/2}.$$

For $s=1-F(x)$,

$$-\log M-\log x-\log(1-x^{-2})-x^2/2 \leq s \leq -\log M-\log x-x^2/2$$

$$-\log M-\log x+\frac{1}{x}+O(x^{-2})-x^2/2 \leq \log s \leq -\log M-\log x-x^2/2$$

$$\log M+\log x+x^2/2 \leq \log(1/s) \leq \log M+\log x-\frac{1}{x}+O(x^{-2})+x^2/2.$$

And as $x \to 0 \iff s \to 0$,

$$x=F^{-1}(1-s)=(2\log 1/s)^{1/2}(1+o(1)).$$

We easily see that the $o(1)$ term is at least of order $(\log 1/s)^{-1}$. This gives

$$x^2/2+\log M+\log x \leq \log(1/s) \leq x^2/2+\log M+\log x-\frac{1}{x}+O(x^{-2})$$

$$\log(1/s)+\log(1/s)\times o(1)+\log M+\log((2\log 1/s)^{1/2}(1+o(1)).$$

The left term is

$$=\log(1/s)(1+o(1)+\log M(1/2)\log 2+(1/2)\log\log(1/s)+o(1).$$

The right term is

$$\log(1/s)(1+o(1)) + \log M + (1/2)\log 2 + (1/2)\log\log(1/s) \\ -(2\log 1/s)^{-1/2}(1+o(1)) + O((\log 1/s)^{-1}) + o(1).$$

The middle term is

$$(1+o(1))x^2/2.$$

By dividing by $\log(1/s)$, we get

$$\begin{aligned} 1+&\frac{(1/2)\log 4\pi + (1/2)\log\log(1/s) + (2\log 1/s)^{-1/2}(1+o(1)) + O((\log 1/s)^{-1})}{\log(1/s)} \\ &\leq (x/(2\log(1/s))^{1/2})^2 + o(1)\times(x/(2\log(1/s))^{1/2})^2 \\ &\leq 1+\frac{(1/2)\log 4\pi + (1/2)\log\log(1/s)}{\log(1/s)}. \end{aligned}$$

Then

$$\begin{aligned} 1+&\frac{(2\log 1/s)^{-1/2}(1+o(1)) + O((\log 1/s)^{-1}) + \log(1/s)\times o(1)}{\log(1/s)} \\ &\leq (x/(2\log(1/s))^{1/2})^2 - \frac{(1/2)\log 4\pi + (1/2)\log\log(1/s)}{\log(1/s)} \\ &= (x/(2\log(1/s))^{1/2})^2 - \frac{(1/2)\log 4\pi + (1/2)\log\log(1/s)}{\log(1/s)} \\ &= 1+\frac{o(1)}{\log(1/s)}. \end{aligned}$$

Then

$$x^2 = (2\log(1/s)\left\{1 - \frac{(1/2)\log 4\pi + (1/2)\log\log(1/s) + o(1)}{\log(1/s)}\right\},$$

and

$$x = F^{-1}(1-u) = (2\log(1/s))^{1/2}\left\{1 - \frac{(1/2)\log 4\pi + (1/2)\log\log(1/s) + o(1)}{2\log(1/s)}\right\}.$$

We have

$$F^{-1}(1-s) - F^{-1}(1-xs) = A(x,s) + B(x,s) + C(x,s)$$

with

$$A(x,s) = (2\log(1/s))^{1/2} - (2\log(x/s))^{1/2} = -(2\log(1/s))^{1/2}((\frac{\log(x/s)}{\log(1/s)})^{1/2} - 1).$$

But

$$\begin{aligned} \left(\frac{\log(x/s)}{\log(1/s)}\right)^{1/2} &= \left(1+\left(\frac{\log(x/s)}{\log(1/s)}-1\right)\right)^{1/2} \\ &= 1+\frac{1}{2}\left(\frac{\log(x/s)}{\log(1/s)}-1\right) - \frac{1}{8}\left(\frac{\log(x/s)}{\log(1/s)}-1\right)^2 \\ &\quad +O\left(\left(\frac{\log(x/s)}{\log(1/s)}-1\right)^3\right) \\ &= 1+\frac{1}{2}\frac{\log x}{\log 1/s} - \frac{1}{2}\frac{(\log x)^2}{(2\log 1/s)^2} + O\left((\log 1/s)^{-3}\right). \end{aligned}$$

We get

$$A(x,s) = -\frac{\log x}{(2\log 1/s)^{1/2}} - \frac{1}{2}\frac{(\log x)^2}{(2\log 1/s)^{3/2}} + O\left((\log 1/s)^{-5/2}\right)$$

and

$$\begin{aligned} B(x,s) &= \frac{(1/2)\log 4\pi + (1/2)\log\log(x/s) + o((\log 1/s)^{-1})}{(2\log(x/s))^{1/2}} \\ &\quad - \frac{(1/2)\log 4\pi + (1/2)\log\log(1/s) + o((\log 1/s)^{-1})}{(2\log(x/s))^{1/2}}, \end{aligned}$$

$$\begin{aligned} C(x,s) &= \frac{(1/2)\log 4\pi + (1/2)\log\log(1/s) + o((\log 1/s)^{-1})}{(2\log(x/s))^{1/2}} \\ &\quad - \frac{(1/2)\log 4\pi + (1/2)\log\log(1/s) + o((\log 1/s)^{-1})}{(2\log(1/s))^{1/2}} \end{aligned}$$

and

$$B(x,s) = \frac{1}{(2\log(x/s))^{1/2}}\left(\frac{1}{2}\log(\log(x/s)/\log(1/s)\right) + \left(o((\log 1/s)^{-1})\right).$$

But

$$\begin{aligned} \log(\log(x/s)/\log(1/s)) &= \log(1 + (\log(x/s)/\log(1/s) - 1)) \\ &= (\log(x/s)/\log(1/s) - 1) \\ &\quad -\frac{1}{2}(\log(x/s)/\log(1/s) - 1)^2 \\ &\quad +O\left((\log(x/s)/\log(1/s) - 1)^3\right) \\ &= \frac{\log x}{\log 1/s} - \frac{1}{2}\frac{(\log x)^2}{(\log 1/s)^2} + O((\log 1/s)^{-3}). \end{aligned}$$

Then

$$\begin{aligned} B(x,s) &= \frac{1}{(2\log 1/s)^{1/2}}\left\{\frac{\log x}{\log 1/s} - \frac{1}{2}\frac{(\log x)^2}{(\log 1/s)^2} + O((\log 1/s)^{-3})\right\} \\ &\quad + \left(o((\log 1/s)^{-3/2})\right), \end{aligned}$$

and

$$\begin{aligned} C(x,s) &= \frac{(1/2)\log 4\pi + (1/2)\log\log(1/s) + o((\log 1/s)^{-1})}{(2\log(x/s))^{1/2}} \\ &\quad \times\left\{(2\log(1/s))^{1/2} - (2\log(x/s))^{1/2}\right\} \\ &= \frac{(1/2)\log 4\pi + (1/2)\log\log(1/s) + o((\log 1/s)^{-1})}{(2\log(1/s))^{1/2}} \\ &\quad \times\left\{-\frac{\log x}{(2\log 1/s)^{1/2}} - \frac{1}{2}\frac{(\log x)^2}{(2\log 1/s)^{3/2}} + O((\log 1/s)^{-5/2})\right\}. \end{aligned}$$

Recall

$$A(x,s)=\left\{-\frac{\log x}{(2\log 1/s)^{1/2}}-\frac{1}{2}\frac{(\log x)^2}{(2\log 1/s)^{3/2}}+O((\log 1/s)^{-5/2})\right\}.$$

We conclude that

$$\frac{(2\log(1/s))^{1/2}}{(1/2)\log 4\pi+(1/2)\log\log(1/s)}\left\{\frac{F^{-1}(1-s)-F^{-1}(1-xs)}{(2\log(1/s))^{-1/2}}+\log x\right\}$$
$$\to -\log x. \qquad (14)$$

6.2.5. Lognormal

We have

$$G^{-1}(1-s)=\exp(F^{-1}(1-s)),\ s\in(0,1),$$

where F is standard normal. This gives

$$G^{-1}(1-u)-G^{-1}(1-ux)=\exp(F^{-1}(1-s))\times(1-\exp(F^{-1}(1-xs)-F^{-1}(1-s))).$$

But

$$\begin{aligned}&\exp(F^{-1}(1-xs)-F^{-1}(1-s))\\ &\quad=\exp\left(-\frac{F^{-1}(1-s)-F^{-1}(1-xs)}{(2\log(1/s))^{1/2}}(2\log(1/s))^{1/2}\right)\\ &\quad=\exp\left(-\frac{D(x,s)}{(2\log(1/s))^{1/2}}\right)\\ &\quad=1-\frac{D(x,s)}{(2\log(1/s))^{1/2}}+\frac{1}{2}\frac{D(x,s)^2}{(2\log(1/s))}+\left(-\frac{1}{(2\log(1/s))^{3/2}}\right),\end{aligned}$$

where $D(x,s)$ is defined in (14). This yields

$$\begin{aligned}\frac{G^{-1}(1-s)-G^{-1}(1-sx)}{\exp(F^{-1}(1-s))(2\log(1/s))^{-1/2}}&=D(x,s)-\frac{1}{2}\frac{D(x,s)^2}{(2\log(1/s))^{1/2}}\\ &\quad+O\left(-\frac{1}{(2\log(1/s))^{1}}\right).\end{aligned}$$

Thus

$$\frac{(2\log(1/s))^{1/2}}{(1/2)\log 4\pi+(1/2)\log\log(1/s)}\left\{\frac{G^{-1}(1-s)-G^{-1}(1-sx)}{\exp(F^{-1}(1-s))(2\log(1/s))^{-1/2}}+\log x\right\}$$
$$\to -\frac{1}{2}(\log x)^2.$$

6.2.6. Logistic law

$$F(x)=1-2/(1+e^x),$$

that is

$$F^{-1}(1-s)=\log s^{-1}(2-s).$$

Routine computations yield

$$\begin{aligned}F^{-1}(1-s)-F^{-1}(1-xs) &= \log x+\log(2-s)/(2-xs)\\ &= \log x+\log(1+(s(x-1)/(2-xs))\\ &= \log x+(s(x-1)/(2-xs))+O(s^2).\end{aligned}$$

Thus

$$(s/2)^{-1}\left\{F^{-1}(1-s)-F^{-1}(1-xs)-\log x\right\}\rightarrow\frac{x-1}{2}.$$

6.2.7. Log-Expo

We have

$$F^{-1}(1-s)=\log\log(1/s),\ \ s\in(0,1).$$

$$\begin{aligned}F^{-1}(1-s)-F^{-1}(1-xs) &= \log((\log(x/s)/(\log(1/s)))\\ &= \log(1+(\log(x/s)/\log(1/s)-1))\\ &= (\log(x/s)/\log(1/s)-1)\\ &\quad -\frac{1}{2}\left(\log(x/s)/\log(1/s)-1\right)^2\\ &\quad +O((\log(x/s)/\log(1/s)-1)^3)\\ &= \frac{\log x}{\log(1/s)}-\frac{1}{2}\frac{(\log x)^2}{(\log(1/s))^2}\\ &\quad +O((\log(1/s))^{-3}).\end{aligned}$$

This gives

$$(\log(1/s))^2\left\{\frac{F^{-1}(1-s)-F^{-1}(1-xs)}{(\log(1/s))^{-1}}-\log x\right\}\rightarrow-\frac{1}{2}(\log x)^2$$

6.2.8. Reversed Burr's df

We have

$$F(x)=1-((-x)^{-\rho/\gamma}+1)^{1/\rho},\ x\leq 0,\ \rho<0 \text{ and } \gamma>0.$$

Then

$$F^{-1}(1-u)=-(u^\rho-1)^{-\gamma/\rho}.$$

and

$$\begin{aligned}F^{-1}(1-u)-F^{-1}(1-ux) &= ((xu)^\rho-1)^{-\gamma/\rho}-(u^\rho-1)^{-\gamma/\rho}\\ &= (u^\rho-1)^{-\gamma/\rho}\left\{\left(\frac{x^\rho u^\rho-1}{u^\rho-1}\right)^{-\gamma/\rho}-1\right\}\\ &= (u^\rho-1)^{-\gamma/\rho}\left\{x^{-\gamma}\left(\frac{1-x^{-\rho}u^{-\rho}}{1-u^{-\rho}}\right)^{-\gamma/\rho}-1\right\}.\end{aligned}$$

But

$$\begin{aligned}\left(\frac{1-x^{-\rho}u^{-\rho}}{1-u^{-\rho}}\right)^{-\gamma/\rho} &= \left(1+\left\{\frac{1-x^{-\rho}u^{-\rho}}{1-u^{-\rho}}-1\right\}\right)^{-\gamma/\rho}\\ &= \left(1+\left\{\frac{(1-x^{-\rho})u^{-\rho}}{1-u^{-\rho}}\right\}\right)^{-\gamma/\rho}\\ &= 1-\frac{\gamma}{\rho}\frac{(1-x^{-\rho})u^{-\rho}}{1-u^{-\rho}}+O(u^{-2\rho}).\end{aligned}$$

Thus

$$\frac{F^{-1}(1-u)-F^{-1}(1-ux)}{\gamma(u^{\rho}-1)^{-\gamma/\rho}}=\frac{x^{-\gamma}-1}{\gamma}-x^{-\gamma}\frac{1}{\rho}\frac{(1-x^{-\rho})u^{-\rho}}{1-u^{-\rho}}+O(x^{-\gamma}\gamma^{-1}u^{-2\rho}).$$

So

$$\left(u^{-\rho}\right)^{-1}\left\{\frac{F^{-1}(1-u)-F^{-1}(1-ux)}{\gamma(u^{\rho}-1)^{-\gamma/\rho}}-\frac{x^{-\gamma}-1}{\gamma}\right\}=-\frac{x^{-\gamma}(1-x^{-\rho})}{\rho}$$

We now summarize the results of these computations in the next subsection.

6.3. Tables of functions s and b

Name	F		b
Burr	$1-(x^{-\rho/\gamma}+1)^{1/\rho}$,	$x\geq 0,\rho>,0,\gamma<0$	u^{ρ}
Reversed Burr	$1-((-x)^{-\rho/\gamma}+1)^{1/\rho}$,	$x\leq 0,\rho<0,\gamma>0$	$u^{-\rho}$
Singh-Maddala	$1-F(x)=(1+ax^{b})^{-c}$,	$x\geq 0$	$u^{1/u}$
Log-Sm	$1-G(x)=(1+ae^{xb})^{-c}$		$\frac{-u^{1/c}}{(bc)(1-u^{1/c})}$
Exponentiel	$1-e^{-x}$,	$x\geq 0$	$(\log u)^{-1}$
Log-Expo	$1-F(x)=\exp(-e^{-x})$		$1/\log u$
Normal	$\phi(x)$		$(\log 1/u)^{-3/2}$
Lognormal	$\phi(e^{x})$		$(\log 1/u)^{-1}$
Logistic	$1-2/(1+e^{x})$,	$x\geq 0$	$(\log 1/u)^{-1}$

Name	*gamma*	$h_{\gamma,\rho}(x)$	$S(u)$
Burr	$1/\gamma$	$-\frac{(x^{\rho}-1)x^{-\gamma-\rho}}{\rho}$	$(u^{\rho}-1)^{-1}$
Reversed Burr	$-1/\gamma$	$-\frac{x^{-\gamma}(1-x^{-\rho})}{\rho}$	$u^{-\rho}$
Singh-Maddala	$1/(bc)$	$\frac{\left(x^{-1/c}-1\right)x^{-1/(bc)+1/c}}{1/c}$	$-(1-u^{1/c})^{-1}$
Log-Sm	0	$-c(x^{-1/c}-1)$	$cu^{1/c}/(\gamma b)$
Exponentiel	0	Not applicable	Not applicable
Log-Expo	0	$-(\log x)^{2}/2$	$1/\log u$
Normal	0	$-\log x$	$D(s)=\left\{\frac{(1/2)\log 4\pi+(1/2)\log\log(1/u)}{(2\log(1/u))^{1/2}}\right\}^{-1}$
Lognormal	0	$(\log x)^{2}/2$	$D(s)$
Logistic	0	$x-1$	$u/2$

7. Statistical applications

Pratically, the normality results on statistics based on the extremes are applied for ultimately differentiable distribution functions (at $+\infty$). They usually depend of the functions $b(.)$ for in the representation scheme and, on S in the second order condition one. This means that we may move from one approach to the other. Let us illustrate this with two examples.

7.1. Large quantiles process

Let $X_1, X_2, ...$ be a sequence of real and independant random variables indentically distributed and associated to the distribution function $F(x) = P(X_i \leq x), x \in R$. We suppose that these random variables are represented as $X_i = F^{-1}(U_i)$, $i = 1, 2, ...$, where $U_1, U_2, ...$ are standard uniform independant random variables. For each n, $U_{1,n} < ... < U_{n,n}$ denote the order statistics based on $U_1, ..., U_n$. Finally let $\alpha > 0$ and $a > 0$ and

$$k \to \infty;\ k/n \to 0 \ \textit{and} \ \log\log n/k \to 0 \text{ as } n \to \infty.$$

Consider this large quantile proccess (see Drees [3])

$$A_n(s, \alpha) = X_{n-[k/s^{\alpha}]+1,n} - F^{-1}(1 - [k/s^{\alpha}]/n).$$

We suppose that F is in the extremal domain. We use first the representation scheme.

7.1.1. Representation approach

Consider the function p and b defined in Theorem 1. For any $\lambda > 1$, put the convention

$$d_n(h, a, \alpha) = \sup\{|h(t)|, |t| \leq [a^{-\alpha}]\lambda k/n.$$

We may then define the regularity condition,

$$\sqrt{k}(d_n(p, a, \alpha) \vee d_n(b, a, \alpha)) \to 0 \qquad \text{(RCREP)}$$

under which we may find a uniform Gaussian approximation of $A_n(s, \alpha)$. Put for convenience

$$k(s, \alpha) = [k/s^{\alpha}],\ l(n, \alpha) = k(s, \alpha)/n, \quad k'(s, \alpha) = k(s, \alpha)/k,$$

and

$$U_{[k/s^{\alpha}],n} = U_{k(s,\alpha),n}.$$

For

$$a(k/n) = c(1 + p(k/n))(k/n)^{-\gamma} \exp\left(\int_{k/n}^{1} b(t)dt\right),$$

we have

$$A_n(1) = X_{n-[k/s^{\alpha}]+1,n}/a(k/n)$$
$$= (1 + O(d_n(p, \alpha, \lambda))(1 + O(d_n(p, \alpha, \lambda)) \times (U_{k(s,\alpha),n}/l(n, s, \alpha)^{-\gamma} k'(s, \alpha)^{-\gamma}.$$

We also have

$$\begin{aligned} A_n(2) &= F^{-1}(1 - l(n, s, \alpha))/a(k/n) \\ &= (1 + O(d_n(p, \alpha, \lambda))(1 + O(d_n(b, \alpha, \lambda))k'(s, \alpha)^{-\gamma}. \end{aligned}$$

It follows, since $k'(s,\alpha)^{-\gamma} = s^{\alpha\gamma}(1+O(k^{-1}))$ uniformly in $s \in (a,1)$, that

$$\begin{aligned}A_n(1) - A_n(2) &= s^{\alpha\gamma}\left\{(l(n,s,\alpha)^{-1}U_{k(s,\alpha),n})^{-\gamma} - 1\right\}\\ &\quad +O(d_n(p,\alpha,\lambda) \vee d_n(b,\alpha,\lambda)) + O(k^{-1});\end{aligned}$$

and by [8]

$$\begin{aligned}\sqrt{k}\left\{A_n(1) - A_n(2)\right\} &= -\gamma s^{\alpha\gamma}\sqrt{k}\left\{(l(n,s,\alpha)^{-1}U_{k(s,\alpha),n}) - 1\right\}\\ &\quad +O(\sqrt{k}d_n(p,\alpha,\lambda)) + O(\sqrt{k}d_n(b,\alpha,\lambda)))\\ &= -\gamma s^{\alpha\gamma}W_n(1,s^{\alpha}) + o_p(a,\alpha,s)\end{aligned}$$

whenever (RCREP) is valid. We then obtain the limiting law of the process of large quantiles under this condition. When F is differentiable in the neighborhood of $+\infty$, we may take $p=0$ and (RCREP) becomes

$$\sqrt{k}d_n(b,a,\alpha) \to 0.$$

Under this (RCREP), the large quantile process behaves as the Gaussian stochastic process $-\gamma s^{\alpha\gamma}(s,\alpha)W_n(1,s^{\alpha})$.

7.1.2. Second order condition approach

There exist functions $a(\cdot)$ and $S(\cdot)$ ($a(\cdots)$ is not necessarily the same as the previous function $s(\cdots)$), such that the SOC holds. But for statistical purposes, it is more convenient to use the continuous second order condition, that is for $u_n \to 0$, for $x_n \to x > 0$,

$$\lim_{n\to\infty} S(u_n)\left\{\frac{F^{-1}(1-u_n) - F^{-1}(1-u_n x_n)}{a(u_n)} - \frac{x_n^{-\gamma}-1}{\gamma}\right\} = h_\gamma(x).$$

A simple argument based on compactness yields for $u_n \to 0$ and for $0<a<b$,

$$\lim_{n\to\infty}\sup_{a\leq x\leq b}\left|S(u_n)\left\{\frac{F^{-1}(1-u_n) - F^{-1}(1-u_n x)}{a(u_n)} - \frac{x^{-\gamma}-1}{\gamma}\right\} - h_\gamma(x)\right| = 0.$$

Put $x(n,s,\alpha) = l(n,s,\alpha)^{-1}U_{k(s,\alpha),n} \to 1$. For $s \geq a$, we may see that $l(n,s,\alpha)^{-1}U_{k(s,\alpha),n} = 1 + k^{-1/2}(W_n(s^{\alpha}) + o_P(1))$, uniformly in $s \in (a,1)$, where W_n is a standard Wiener process (see Lemma 1 in [8]). Then we may apply the CSOC as follows :

$$\begin{aligned}\sup_{a\leq s\leq b} S(l(n,s,\alpha))^{-1}&\left|\left\{\frac{F^{-1}(1-l(n,s,\alpha)) - F^{-1}(1-U_{k(s,\alpha),n})}{a(l(n,s,\alpha))}\right.\right.\\ &\left.\left. - \frac{(l(n,s,\alpha)^{-1}U_{k(s,\alpha),n})^{-\gamma}-1}{\gamma}\right\} - h_\gamma(x(n,s,\alpha))\right| = 0\end{aligned}$$

This gives, uniformly in $s \in (a,1)$,

$$\begin{aligned}\frac{F^{-1}(1-l(n,s,\alpha)) - F^{-1}(1-U_{k(s,\alpha),n})}{a(l(n,s,\alpha))} &= \frac{(l(n,s,\alpha)^{-1}U_{k(s,\alpha),n})^{-\gamma}-1}{\gamma}\\ &\quad -(h_\gamma(x(n,s)) + o_P(1))S(l(n,s,\alpha)).\end{aligned}$$

Then

$$\frac{\sqrt{k}\left\{F^{-1}(1-l(n,s))-F^{-1}(1-U_{k(s,\alpha),n})\right\}}{a(l(n,s,\alpha))} = \frac{\sqrt{k}(l(n,s)^{-1}U_{k(s,\alpha),n})^{-\gamma}-1}{\gamma} -(h_\gamma(1)+o_P(1))S(l(n,s))\sqrt{k}.$$

We will apply Lemma 1 in ([8]). Since $S(l(n,s,\alpha)) = O(S(l(n,s))$ and $a(l(n,s,\alpha)) \sim s^{\alpha\gamma}a(l(n,1,1))$, we also get

$$\frac{\sqrt{k}\left\{F^{-1}(1-l(n,s))-F^{-1}(1-U_{k(s,\alpha),n})\right\}}{a(l(n,s,\alpha))} =$$

$$-W_n(1,s^\alpha)-(h_\gamma(1)+o_P(1))S(l(n,s,\alpha))\sqrt{k}.$$

$$\frac{\sqrt{k}\left\{F^{-1}(1-l(n,s))-F^{-1}(1-U_{k(s,\alpha),n})\right\}}{a(l(n,1,1))} =$$

$$-s^{\alpha\gamma}W_n(1,s^\alpha)+(h_\gamma(1)+o_P(1))\times S(k/n))\sqrt{k}.$$

We get the regularity condition

$$\sqrt{k}S(k/n)\rightarrow 0. \tag{RCSOC}$$

Conclusion 1. In both cases, we conclude that the large quantile process behaves as the Gaussian process $-s^{\alpha\gamma}W_n(1,s^\alpha)$ when appropriately normalized under conditions based on b or on S.

By comparing (RCREP) and (RCSOC), we see that the present normality result in the representation scheme uses the function b while the Second order one relies on S. In fact, almost all the normality results in both cases rely either on b in the Representation scheme or on S in the Second order model. We also see that the second order scheme seems to use a shorter way. But, as a compensation, the function S, as we may see it here, is more complicated to get. Indeed for differentiable distribution functions, the function b, is easiliy obtained.

7.2. Functional Hill process

7.2.1. Representation approach

Now consider the functional Hill process

$$T_n(f)=\sum_{j=1}^{j=k} f(j)\left(\log X_{n-j+1,n}-\log X_{n-j,n}\right),$$

where f is some positive and bounded function and $k=k(n)$ is a sequence if positive integer such that $1\leq 1\leq k\leq n$ and $k/n\rightarrow 0$ as $n\rightarrow +\infty$. We are going to study the process under the hypothesis $F\in D(\psi_{1/\gamma})=D(G_{-\gamma})$, $\gamma>0$. Now using the same representation $X_i=F^{-1}(U_i)$,

$i = 1, 2, ...$ We get

$$\begin{aligned}
T_n(f) &= \sum_{j=1}^{k} f(j)(\log X_{n-j+1,n} - \log X_{n-j,n}) \\
&= \sum_{j=1}^{k} f(j)(-(y_0 - \log X_{n-j+1,n}) + (y_0 - \log X_{n-j,n})) \\
&= \sum_{j=1}^{k} f(j) \left\{ c(1 + p(U_{j+1,n}))(U_{j+1,n})^{\gamma} \exp\left(\int_{U_{j+1,n}}^{1} \frac{b(t)}{t} dt\right) \right. \\
&\qquad \left. - c(1 + p(U_{j,n}))(U_{j,n})^{\gamma} \exp\left(\int_{U_{j,n}}^{1} \frac{b(t)}{t} dt\right) \right\} \\
&= \sum_{j=1}^{k} f(j) \left\{ c(1 + p(U_{j,n}))(U_{j,n})^{1/\gamma} \exp\left(\int_{U_{j,n}}^{1} \frac{b(t)}{t} dt\right) \right\} \\
&\qquad \times \left\{ \frac{1 + p(U_{j+1,n})}{1 + p(U_{j,n})} \left(\frac{U_{j+1,n}}{U_{j,n}}\right)^{\gamma} \exp\left(\int_{U_{j+1,n}}^{U_{j,n}} \frac{b(t)}{t} dt\right) - 1 \right\}.
\end{aligned}$$

But

$$\left(\frac{U_{j+1,n}}{U_{j,n}}\right)^{\gamma} = \exp\left(\frac{\gamma}{j} \log\left(\frac{U_{j+1,n}}{U_{j,n}}\right)^{j}\right) = \exp\left(\frac{\gamma}{j} E_j\right) = \exp(F_j)$$

where, by the Malmquist representation (see [13], p. 336), the $E_j's$ are independent standard exponential random variables. Let also

$$p_n = \sup\{|p(u)|, 0 \leq u \leq U_{k+1,n}\} \to_P 0,$$

$$b_n = \sup\{|b(u)|, 0 \leq u \leq U_{k+1,n}\} \to_P 0,$$

as $n \to +\infty$, and $c_n = a_n \vee (b_n \log k)$. Then

$$\left\{ \frac{1 + p(U_{j+1,n})}{1 + p(U_{j,n})} \left(\frac{U_{j+1,n}}{U_{j,n}}\right)^{\gamma} \exp\left(\int_{U_{j+1,n}}^{U_{j,n}} \frac{b(t)}{t} dt\right) - 1 \right\}$$

$$\begin{aligned}
&= \exp(F_j)(1 + O(p_n)) \exp(O(b_n)E_j/j) - 1 \\
&= F_j(1 + O(p_n))(1 + O(b_n \log k)) \\
&= F_j(1 + O(c_n)) - 1
\end{aligned}$$

Let also

$$\begin{aligned}
s_n &= y_0 - G^{-1}(1 - U_{k,n}) = y_0 - \log X_{n-k+1,n} \\
&= c(U_{k+1,n})^{1/\gamma} \left(1 + \exp\left(\int_{U_{j,n}}^{1} \frac{b(t)}{t} dt\right)\right).
\end{aligned}$$

This gives

$$= T_n(f)/s_n$$

$$= \sum_{j=1}^{k} f(j) \left\{ \frac{1+p(U_{j,n})}{1+p(U_{k,n})} \left(\frac{U_{j,n}}{U_{k,n}} \right)^{\gamma} \exp\left(\int_{U_{j,n}}^{U_{k,n}} \frac{b(t)}{t} dt \right) \right\}$$
$$\times \left\{ \exp(F_j)(1+O(c_n)) - 1 \right\}.$$

Let us remark that

$$\log\left(\frac{U_{k,n}}{U_{j,n}}\right) = \log\left(\prod_{h=j}^{k-1} \frac{U_{h+1,n}}{U_{h,n}}\right) = \sum_{h=j}^{k-1} \frac{1}{h} \log\left(\frac{U_{h+1,n}}{U_{h,n}}\right)^{h}$$
$$= \sum_{h=j}^{k-1} \frac{1}{h} E_h = O(\log k)$$

and

$$\left(\frac{U_{j,n}}{U_{k,n}}\right)^{\gamma} = \exp\left(-\gamma \sum_{h=j}^{k-1} \frac{1}{h} E_h\right) = \exp\left(-\sum_{h=j}^{k-1} F_h\right) = F_j^*,$$

where $F_k^* = 0$. Then

$$T_n(f)/s_n = \sum_{j=1}^{k} f(j) \left\{ (1+O(p_n))(1+O(b_n \log k)) \times \exp\left(-\sum_{h=j}^{k-1} F_h\right) \right\}$$
$$\times \left\{ \exp(F_j) - 1 + O(c_n) \exp(F_j) \right\}$$
$$= \sum_{j=1}^{k} f(j) F_j^* \left(\exp(F_j) - 1\right) + O(c_n F_j^* (\exp(F_j) - 1))$$
$$+ O(c_n^2 F_j^* (\exp(F_j) - 1)) + O(c_n \exp(F_j))$$

We conclude that $T_n(f)/s_n$ behaves as that of $\sum_{j=1}^{k} f(j) F_j^* \left(\exp(F_j) - 1\right)$ under regularity conditions based on the functions p and b.

7.2.2. Second order condition approach

Let use the continuous second order condition:

$$S^{-1}(u_n) \left\{ \frac{G^{-1}(1 - x_n u_n) - G^{-1}(1-u_n)}{s(u_n)} - \frac{x_n^{-\gamma} - 1}{\gamma} \right\} = h_{\gamma}(x) + o(1),$$

where $\mathrm{x}_n \to 0$ et $u_n \to 0$ as $n \to \infty$ and $u_n = \gamma \left\{ y_0(G) - G^{-1}(1-u_n) \right\}$. We get, for $G(x) = F(e^x)$, $x \in R$,

$$T_n(f) = \sum_{j=1}^{k} f(j)(\log X_{n-j+1,n} - \log X_{n-j,n})$$

$$= \sum_{j=1}^{k} f(j) \left\{ G^{-1}(1 - U_{j,n}) - G^{-1}(1 - U_{j+1,n}) \right\}.$$

Let $u_n(j) = U_{j,n}$ et $x_n(j) = U_{j+1,n}/U_{j,n} = \exp(F_j)$. Then

$$\frac{G^{-1}(1-U_{j,n}) - G^{-1}(1-U_{j+1,n})}{s(u_n(j))} = -\frac{G^{-1}(1-x_n(j)U_{j,n}) - G^{-1}(1-U_{j,n})}{s(u_n(j))}$$

$$-\left\{\frac{G^{-1}(1-x_n(j)U_{j,n}) - G^{-1}(1-U_{j,n})}{s(u_n(j))} - \frac{\exp\left(-\frac{\gamma}{j}E_j\right)-1}{\gamma}\right\}$$

$$-\frac{\exp\left(-\frac{\gamma}{j}E_j\right)-1}{\gamma}$$

$$-S(u_n(j))h_\gamma(\exp(F_j)) - \frac{E_j^{-\gamma/j}-1}{\gamma} + o_p(A(u_n(j)))$$

Let us use

$$\frac{T_n(f)}{s(u_n(k))} = \sum_{j=1}^{k} f(j)\frac{s(u_n(j))}{s(u_n(k))}\frac{G^{-1}(1-U_{j,n}) - G^{-1}(1-U_{j+1,n})}{s(u_n(j))}$$

$$= \sum_{j=1}^{k} f(j)\frac{s(u_n(j))}{s(u_n(k))} \times$$

$$\left\{-S(u_n(j))h_\gamma(\exp(F_j)) - \frac{\exp\left(-\frac{\gamma}{j}E_j\right)-1}{\gamma} + o_p(S(u_n(j)))\right\}.$$

Let us apply

$$\frac{s(u_n(j))}{s(u_n(k))} = \left\{(1+O(p_n))(1+O(b_n\log k)) \times \exp\left(-\sum_{h=j}^{k-1} F_h\right)\right\}.$$

We arrive at

$$\frac{\gamma T_n(f)}{s(u_n(k))} = \sum_{j=1}^{k} f(j)(1+O(c_n))F_j^*\left\{-S(u_n(j))h_\gamma(\exp(F_j))\right.$$

$$\left. -(\exp(F_j)-1) + o_p(A(u_n(j)))\right\}$$

$$= \sum_{j=1}^{k} f(j)F_j^*(\exp(F_j)-1) + \sum_{j=1}^{k} f(j)O(c_nF_j^*(\exp(F_j)-1))$$

$$+\sum_{j=1}^{k} f(j)(1+O(c_n))F_j^*\left\{-S(u_n(j))h_\gamma(\exp(F_j)) + o_p(S(u_n(j)))\right\}.$$

Conclusion 2. In both cases, we see that when properly normalized, $T_n(f)$ behaves as $\sum_{j=1}^{k} f(j)F_j^*(\exp(F_j)-1)$ under regularity conditions based on p, b or S.
As for the first example, the SOC approach seems shorter. But here this latter approach still needs the first one.

8. Conclusion

As a general conclusion, we say :

1. The representation approach is more general.
2. The second order condition seems to be shorter and more unified.
3. The computation of b is less complicated than that of S.
4. The representation approach is still used within the second order approach.
5. The two approaches may be simultanuously used.

We conclude that the two approaches are equivalent and we have proposed for both cases the computation of b and S for usual distribution functions.

Acknowledgement. The paper was finalized while the first author was visiting MAPMO, University of Orléans, France, in 2009 and 2011. He expresses his warm thanks to responsibles of MAPMO for kind hospitality. The second author was partially granted by the project AIRE-SUD of the Institut Regional de Development (IRD) at the Université Gaston Berger de Saint-Louis.

References

[1] Csörgő, M., Csörgő, S., Horvàth, L. and Mason, M., 1986. Weighted empirical and quantile processes. *Ann. Probab.* **14**, 31-85.

[2] Csörgő, S., Deheuvels, P. and Mason, D. M., 1985. Kernel estimates of the tail index of a distribution. *Ann. Statist.* **13**, 1050-1077.

[3] Drees, H., 1995. Refined Pickands Estimators for the extrem value index. *Ann. Statist.* **23** (6), 2059-2080.

[4] Dekkers, A.L.M, Einmahl, J.H.J., and de Haan L., 1989. A moment estimator for the index of an extreme value distribution. *Ann. Statist.* **17** (4),1833-1855.

[5] de Haan, L., 1970. On regular variation and its application to the weak convergence of sample extremes. Mathematical Centre Tracts, 32, Amsterdam. (MR0286156)

[6] de Haan, L. and Feireira A. (2006). Extreme value theory: An introduction. Springer. (MR2234156)

[7] de Haan, L. and Stadtmüller, U.(1996). Generalized regular variation of second order. *J. Australian Math. Soc.* Ser A, **61**,381-395.

[8] Fall, A.M., and Lo, G.S., 2011. On the Pickands process. To appear in C. R. Math. Acad. Sci. Soc. R. Can. Available at : http://arxiv.org/abs/1111.4469

[9] Lo, G.S., 1989. A note on the asymptotic normality of sums of extreme values. *J. Statist. Plan. and Inf.* **22** (1), 89-94.

[10] Hall, P., 1982. On simple estimates of an exponent of regular variation. *J. Roy. Statist. Soc.* Ser. B, **44**, 37-42.

[11] Hall, P., 1978. Representation and limit theorem for extreme value distributions. *J. Appl. Probab.* **15**, 639-644.

[12] Segers, J., 2002. Generalized Pickands Estimators for the Extreme Value Index. *J. Statist. Plann. Inf.*, **128** (2), 381-396.

[13] Shorack, G.R. and Wellner J.A., 1986. *Empirical Processes with Applications to Statistics. wiley-Interscience.* New-York.