Asymptotic representation theorems for poverty indices

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Abstract. We set general conditions under which the general poverty index, which summarizes all the available indices, is asymptotically represented with some empirical processes. This representation theorem offers a general key, in most directions, for the asymptotic of the bulk of poverty indices and issues in poverty analysis. Our representation results uniformly hold on a large collection of poverty indices. They enable the continuous measure of poverty with longitudinal data.

Résumé. Nous introduisons un ensemble cohérent de conditions sous lesquelles, l'Index Général de Pauvreté, qui englobe presque tous les indices usuels, admet une représentation asymptptique sous forme de somme d'un processus empirique fonctionnel et d'un processus empirique spécial. Cette représentation asymptotique offre un ensemble d'outils permettant de trouver rapidement les lois asymptotiques des mesures de pauvreté. Elle est en général utile pour l'analyse de la pauvreté. Elle est donnée de manière uniforme par rapport au temps et à la fonctionnelle de la mesure. Elle est appropriée pour l'étude de la pauvreté longitudinale.

In quantitative poverty analysis, poverty indices are the key tools as well as inequality measures. A great number of such indices have been introduced in the literature since the pioneering works of the Nobel Prize winner, Amartya Sen [11] who first derived poverty measures (see Sen [11] from an axiomatic point of view. A survey of these indices is to be found in Zheng [13]), who also discussed their properties and classified them from an axiomatic point of view.

Statistical asymptotic laws for these indices, particularly asymptotic normality, on which statistical inference on the unknown poverty index may be based, are also of great importance. Recent works which dealt with this, are available in Barrett and Donald [1], Davidson and Duclos [6] and Bishop et al. [3, 2] for instance. These results reveal themselves very powerful and showed real interest in applications. Nevertheless, the indices are studied mostly one by one. In Lo [7], a unified approach is proposed with a general form of the poverty indices, named the General Poverty Index (GPI), including almost all the proposed indices. In Lo [7] and Sall and Lo [10], a general asymptotic theory of the GPI is given, based on the so-called Hungarian approximations (see Csőrgö et al. [4] and Csőrgö and Horvàth [5]). Now there is much to do when we deal with longitudinal data. In this case, one has to move from a static approach to a time-dependent one. Moreover, the GPI and a large class of inequality measures, form classes of L-Statistics indexed by functions.

We aim at giving general tools from which, the functional and time-dependent asymptotic laws of the GPI will be derived. Precisely, we give here functional and time-dependent representation theorems of the class of poverty indices into a functional empirical process, largely described in van der Vaart and Wellner [12], and a new functional process to be especially handled in Lo [8].

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Now, let us make some notation that will permit to clarify the ideas and formulate the problems. We consider a population of individuals, each of which having a random income or expenditure Y, with distribution function G. An individual or a household is considered as poor whenever Y fulfills $Y \leq Z$, where $Z > 0$ is a specified theshold level named the poverty line.

Consider now a random sample $Y_1, Y_2, ...Y_n$ of size n of incomes, with empirical distribution function $G_n(y) = n^{-1}\# \{Y_i \leq y : 1 \leq i \leq 1\}$. The number of poor individuals within the sample is then equal to $Q_n = nG_n(Z)$. Let also be measurable functions $A(p, q, z)$, $w(t)$, and $d(t)$ of $p, q \in N$, and $z, t \in R$ and $B(q) = \sum_{i=1}^{q} w(i)$.

Let finally $Y_{1,n} \leq Y_{2,n} \leq ... \leq Y_{n,n}$ be the order statistics of the sample $Y_1, Y_2, ...Y_n$ of Y. We introduce the following

$$J_n = \frac{A(Q_n, n, Z)}{nB(Q_n)} \sum_{j=1}^{Q_n} w(\mu_1 n + \mu_2 Q_n - \mu_3 j + \mu_4) : d\left(\frac{Z - Y_{j,n}}{Z}\right), \tag{1}$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are constants, as the General Poverty Index (GPI) like in Lo et al. [7] and Sall and Lo [10].

We already showed in Sall and Lo [10] how to derive from (1) the individual poverty measures like the Sen, Kakwani, Foster-Greer-Thorbecke, Thon, Chakravarty ones and some other ones. (See Zheng [13], for a definition of such measures). We do not need to return back to this in this paper.

As said previously, our aim is to obtain functional asymptotic laws for the time-dependent GPI. We then need to define our index set in (1). Suppose that the functions A, w and d are in some classes of positive and measurable functions $C_i, i = 1, 2, 3$ with these further specifications : C_1 is a class of functions $A(p, q, z)$ with $(p, q, z) \in N^3$, C_2 of functions $w(t)$ with $t \in R$ and C_3 of functions $d(\cdot)$ continuous and defined on $[0, 1]$ onto $[0, 1]$ and bounded by one. The constants vector $\mu = (\mu_1, ..., \mu_4)^t$ lies also in some subset C_4 of N^4. We put $\lambda_0 = (A, w, u) \in \Gamma 0 = C_1 \times C_2 \times C_4$. In the longitudinal data case, we observe the same households over the time. This leads to the longitudinal observations of $Y \in C([0, T])$,

$$\{Y_1(t), ..., Y_t(t), 0 \leq t \leq T\},$$

where for each $t \in [0, T]$, $G_t(\cdot)$ stands for the distribution function of $Y(t)$. We consequently use the index

$$\lambda = (A, w, u, t) \in \Gamma = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_4 \times [0, T]$$

and

$$\phi = (A, w, d, u, t) \in \Phi = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4 \times [0, T].$$

In the time-dependent case, the poverty line $Z(t)$ may depend on the time and so does the poor headcount denoted as $Q_n(t)$. With these notations, our object study becomes

$$J_n(\phi) = \frac{A(Q_n(t), n, Z(t))}{nB(Q_n(t))} \sum_{j=1}^{Q_n(t)} w(\mu_1 n + \mu_2 Q_n(t) - \mu_3 j + \mu_4) \, d\left(\frac{Z(t) - Y_{j,n}(t)}{Z(t)}\right).$$

We are now giving our fundamental result, that is the uniform representation of $\{\sqrt{n}(J_n(\phi) - J(\phi)), \phi \in \Phi\}$ in terms of functional empirical processes, where $J(\phi)$ is the exact GPI.

In the sequel, we use almost sure limits in outer probability $(a.s.o.p)$, limits to zero in outer probability denoted $o_P^*(1)$, following the new theory in van der Vaart and Wellner [12], when dealing with non-measurable processes. We point out for once that all the limits in this paper are meant as $n \to +\infty$, unless the contrary is specified.

1. Fundamental Theorem

Define $F_0 = \{f_t : x \rightsquigarrow 1_{(x(t) \leq Z(t))}, t \in [0, T]\}$ as a subset of $\ell^\infty(C([0, T]))$, the set of real bounded and continuous functions defined on $C([0, T])$. We will consider these general assumptions.

(HP0) There exist $\beta > 0$ and $0 < \xi < 1$ such that

$$0 < \beta < \inf_{0 \leq t \leq T} G_t(Z) < \sup_{0 \leq t \leq T} G_t(Z) < \xi < 1.$$

(HP1) The familiy F_0 is a P_Y-Glivenco-Cantelli class, that is, as $n \to \infty$,

$$\sup_{t \in [0,T]} |G_{t,n}(Z(t)) - G_t(Z(t))| \to 0, \ a.s.o.p.$$

where, for any $t \in [0,T]$ and $y \in R$, $G_{t,n}(y) = n^{-1} \sum_{i=1}^{n} 1_{(Y(t) \leq y)}$.

(HP2a) There exist a class C of functions $c(s,t)$ of $(s,t) \in R^2$ and a class Π of functions $\pi(s,t)$ of $(s,t) \in R^2$, such that for any $\lambda = (A, w, \mu, t) \in \Gamma$, one can find a single function $h(p,q)$ of $(p,q) \in N^2$ such that there exist a function $c \in C$ and a function $\pi \in \Pi$, all of them independent of $t \in [0,T]$, under which we have, as $n \to +\infty$,

$$\sup_{\lambda \in \Gamma} \max_{1 \leq j \leq Q_n(t)} \left| A(n, Q_n(t)) h^{-1}(n, Q_n(t)) w(\mu_1 n + \mu_2 Q_n(t) - \mu_3 j + \mu_4) \right.$$

$$\left. -c(Q_n(t)/n, j/n) \right| = o_P^*(n^{-1/2}).$$

(HP2b) and

$$\sup_{\lambda \in \Gamma} \max_{1 \leq j \leq Q_n(t)} \left| w(j) h^{-1}(n, Q_n(t)) - \frac{1}{n} \pi(Q_n(t)/n, j/n) \right| = o_P^*(n^{-1/2})$$

(H2Pc) There exists a function $c(u,v)$ of $(u,v) \in (0,1)^2$ independent of $t \in [0,T]$, such that, as $n \to +\infty$,

$$\sup_{t \in [0,T]} \max_{1 \leq j \leq Q_n(t)} \left| A(n, Q_n(t)) h^{-1}(n, Q_n(t)) w(\mu_1 n + \mu_2 Q_n(t) - \mu_3 j + \mu_4) \right.$$

$$\left. -c(Q_n(t)/n, j/n) \right| = o_P^*(n^{-1/2}).$$

(HP3) The elements of the classes C and Π have equi-continuous partial differentials in the sense that (for example for $c \in C$) :

$$\lim_{(k,l) \to (0,0)} \sup_{(x,y) \in (0,1)^2} \sup_{c \in C} \left| \frac{\partial c}{\partial y}(x+l, y+k) - \frac{\partial c}{\partial y}(x,y) \right| = 0,$$

and

$$\lim_{(k,l) \to (0,0)} \sup_{\beta \leq x \leq \xi, y \in (0,1),} \sup_{c \in C} \left| \frac{\partial c}{\partial x}(x+l, y+k) - \frac{\partial c}{\partial x}(x,y) \right| = 0.$$

(HP4) For any $(c, \pi) \in C \times \Pi$, for fixed x, the functions $y \to \frac{\partial c}{\partial y}(x,y)$ and $y \to \frac{\partial \pi}{\partial y}(x,y)$ are monotone.

(HP5) For any $t \in [0,T]$, G_t is strictly increasing, and the functions G_t are equi-continuous in $t \in [0,1]$.

(HP6) There exist $H_0 > 0$ and $H_\infty < +\infty$ such that, for any $(t, c, \pi, d) \in [0,T] \times C \times \Pi \times C_3$,

$$H_0 < H_c(\phi) = \int_0^{+\infty} c(G_t(Z), G_t(y)) \gamma(y) dG_t(y) < H_\infty,$$

and

$$H_0 < H_\pi(\phi) = \int_0^{+\infty} \pi(G_t(Z), G_t(y)) e(y) dG_t(y) < H_\infty;$$

where $\gamma(x) = d(\frac{Z-x}{Z}) 1_{(x \leq Z)}$ and $e(x) = 1_{(x \leq Z)}$ for $x \in R$. Here and in the sequel, the functions depend, in some way, on ϕ even when we do not specify it or even when we only partially do it.

(HP7) There is a universal constant K_0, such that there exists $\delta > 0$, there exists $r > 0$ such that

$$|s - t| \leq \delta \Longrightarrow \left| \frac{1}{3} - \mathbb{E}(G_t(Y(t)) G_s(Y(s))) \right| \leq K_0 |s - t|^{1+r}, \tag{2}$$

Here is our fundamental tool Theorem.

Theorem 1. *Suppose that the hypotheses (HP1)-(HP7) hold. Put*

$$J(\phi) = H_c(\phi) / H_\pi(\phi),$$

$$g_t = H_2^{-1} g_c - H_c H_\pi^{-2} g_\pi + K e(f_t(\cdot))$$

with

$$g_c(\cdot) = c(G_t(Z), G_t(f_t(\cdot))) \gamma(f_t(\cdot)), \ g_\pi = \pi(G_t(Z), G_t(f_t(\cdot))) e(f_t(\cdot)), \tag{3}$$

$$K(\phi) = H_\pi^{-1} K_c - H_c H_\pi^{-2} K_\pi \tag{4}$$

$$K_c(\phi) = \int_0^1 \frac{\partial c}{\partial x}(G_t(Z), s)\gamma(G_t^{-1}(s))ds, \ \ K_\pi(\phi) = \int_0^1 \frac{\partial \pi}{\partial x}(G_t(Z), s)e(G_t^{-1}(s))ds, \tag{5}$$

$$\nu = H_\pi^{-1}\nu_c - H_c H_\pi^{-2}\nu_\pi,$$

where

$$\nu_{c,t}(y) = \frac{\partial c}{\partial x}(G_t(Z), G_t(f_t(y)))\gamma(f_t(y)),: \nu_{\pi,t}(y) = \frac{\partial \pi}{\partial x}(G_t(Z), G_t(f_t(y)))e(f_t(y)).$$

Then we have, uniformly in

$$\phi = (A, w, d, \mu, t) \in \Phi = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4 \times [0, T],$$

the following representation :

$$\sqrt{n}(J_n(\phi) - J(\phi)) = \alpha_{t,n}(g_t) + \beta_n(t, \nu_t) + o_P^*(1), \tag{R}$$

with

$$\alpha_{t,n}(g_t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n g_t(Y_j) - Eg(Y_j)$$

and

$$\beta_n(t, \nu_t) = \frac{1}{\sqrt{n}} \sum \{G_{t,n}(Y_j(t)) - G_t(Y_j(t))\} \nu_t(Y_j).$$

Suppose that (H2Pc) holds in place of (H2Pa) and (HP2b) and the other assumptions are true. Then the representation (R) holds with

$$K(t) = K_c(t), g_t = g_{c,t} \text{ and } v_t = \nu_{c,t} \tag{RD}$$

Remark 1. The conditions may seem hard to hold. But, for the classical poverty measures, (HP2a), (HP2b) and (HP4) hold. (HP4) also holds for the Kakwani class. The other hypothesises depend on the distribution of $\{Y(t), t \in [0, T]\}$ and the properties of the poverty lines $\{Z(t), t \in [0, T]\}$.

Remark 2. With the representation (R), $\sqrt{n}(J_n(\phi) - J(\phi))$ may be studied in many ways whenever the properties of the processes $\alpha_{t,n}(g_t)$ and $\beta_n(t, \nu_t)$ are known as well as their covariance structure. The first is nothing else than the functional process. It is largely studied in van der Vaart and Wellner [12]. The second $\beta_n(t, \nu_t)$, apparently new, is entirely described in Lo [8] as well as its correlation with the functional empirical process. Thus, Formula (R) opens various poverty study fields : using poverty indices with longitudinal data, simultaneous comparison poverty situations with several indices, statistical estimation of lack of decomposability, uniform bootstrapping of poverty indices, etc. Such studies using (R) are underway. As an example, it is used in Lo and Sall [9] to get confidence intervals of the relative change in poverty. This method enables to check whether the Millennium Development Goal (MDG) of halving poverty in some time interval is achieved with a probability confidence, say 95%. In the same time, it is showed in Lo and Sall [9] how to combine (R) with the results of $\beta_n(t, \nu_t)$ in Lo [8] to get concluding applications in specific problems.

2. PROOFS

Let us begin by giving general considerations. We introduce these notations to be used later in the proofs. First, based on the hypothesis $(H5)$, we may use the following representation. First, define the rank statistic $R_n(t) = (R_{1,n}(t), ..., R_{n,n}(t))$ based on $Y_1(t), ..., Y_n(t)$ defined by

$$\forall (i, j) \in \{1, ..., n\}^2, \ R_{j,n}(t) = i \Leftrightarrow Y_j(t) = Y_{i,n}(t).$$

Since each G_t is a one-to-one mapping function, we have

$$G_{t,n}(Y_j) = R_{j,n}(t)/n, \text{ a.s.}$$

Secondly, we remark that $U_1(t) = G_t(Y_1), U_2(t) = G_t(Y_2(t)), ...$ are independent uniform random variables whenever G_t is increasing. We will have to consider $U_{t,n}(\cdot)$ and $V_{t,n}(\cdot)$, respectively the empirical distribution and quantile functions based on $U_1(t), ..., U_n(t)$. Denote also $\alpha_{t,n}(s) = \{\sqrt{n}(U_{t,n}(s) - s), \ s \in (0, 1)\}$, the empirical process based on $Y_1(t), Y_2(t), ..., Y_n(t)$; for $n \geq 1$.

Now, by (HP1),

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \sup_{y\in\mathbb{R}} |G_{t,n}(y) - G_t(y)| = 0, \ a.s.o.p$$

By $(HP2)$, uniformly in $\phi \equiv (A, w, \mu, d, t) \in \Phi = C_1 \times C_2 \times C_3 \times C_4 \times [0,T]$,

$$J_n = \frac{A(n, Q_n(t))/h(n, Q_n(t), j)}{n \sum_{j=1}^{Q_n(t)} w(j)/h(n, Q_n(t), j)} \sum_{j=1}^{Q_n(t)} w(\mu_1 n + \mu_2 Q_n(t) - \mu_3 j + \mu_4) d(\frac{Z - Y_{j,n}(t)}{Z}).$$

$$= (J_n(c) + o_P^*(n^{-1/2}))/(J_n(\pi) + o_P^*(n^{-1/2}),\tag{6}$$

where

$$J_n(c) = \frac{1}{n} \sum_{j=1}^{n} c(G_{t,n}(Z), G_{t,n}(Y_j))\gamma(Y_j(t)),\tag{7}$$

and

$$J_n(\pi) = \frac{1}{n} \sum_{j=1}^{n} \pi(G_{t,n}(Z), G_{t,n}(Y_j))e(Y_j).\tag{8}$$

Now we have

$$J_n(c) = \int_0^1 c(G_{t,n}(Z), U_{t,n}(V_{t;n}(s)))\gamma(G^{-1}(V_{t,n}(s)))ds,$$

and

$$H_c(\phi) = \int_0^1 c(G_t(Z), s)\gamma(G^{-1}(s))ds.$$

Then

$$J_n(1) = \frac{1}{n} \sum_{j=1}^{n} c(G_t(Z), G_t(Y_j))\gamma(Y_j(t))$$

$$+\frac{1}{n} \sum_{j=1}^{n} \{c(G_{t,n}(Z), G_{t,n}(Y_j)) - c(G_t(Z), G_{t,n}(Y_j))\} \gamma(Y_j(t))$$

$$+\frac{1}{n} \sum_{j=1}^{n} \{c(G_t(Z), G_{t,n}(Y_j)) - c(G_t(Z), G_t(Y_j))\} \gamma(Y_j(t)).$$

Set,

$$D_1 = \frac{1}{n} \sum_{j=1}^{n} \{c(G_{t,n}(Z), G_{t,n}(Y_j)) - c(G_t(Z), G_{t,n}(Y_j))\} \gamma(Y_j(t)),$$

and

$$D_2 = \frac{1}{n} \sum_{j=1}^{n} \{c(G_t(Z), G_{t,n}(Y_j)) - c(G_t(Z), G_t(Y_j))\} \gamma(Y_j(t)) \equiv D_1 + D_2.$$

First, we have by using the classical representations,

$$D_1 = \int_0^1 \{c(G_{t,n}(Z), U_{t,n}(V_{t,n}(s))) - c(G_t(Z), U_{t,n}(V_{t;n}(s)))\} \gamma(G^{-1}(V_{t,n}(s))ds$$

$$= (G_{t,n}(Z) - G_t(Z)) \int_0^1 \left\{ \frac{\partial c}{\partial x}(\zeta_n(t, Z), U_{t,n}(V_{t;n}(s))\gamma(G^{-1}(V_{t,n}(s)) \right\} ds\tag{9}$$

where $\zeta_n(t, Z)$ lies between $G_t(Z)$ and $G_{t,n}(Z)$. We want now to prove that integral factor in (9) tends to $K_c(\phi)$ uniformly in ϕ. We shall use a method that will be repeated throughout the paper. First, we remark that this is performed in $s \in (V_{t,n}(s) \le G_t(Z))$. Since $V_{t,n}(s) \to s$ uniformly in $(t, s) \in (0, T) \times (0, 1)$, in outer probability, we have for any $\epsilon > 0, 0 < \xi + \epsilon < 1$, $(V_{t,n}(s) \le G(Z)) \subseteq (s \le G_t(Z) + \epsilon)$, uniformly in $(t, s) \in (0, T) \times (0, 1)$, with outer probability greater or equal to $1 - \epsilon$ (denoted $w.p.1 - \epsilon$), for large values of n. Thus for such n's, we have, uniformly in $(t, s) \in (0, T) \times (0, 1)$, $w.p.1 - \epsilon$,

$$\left| \int_0^1 \frac{\partial c}{\partial x}(\zeta_n(t, Z), U_{t,n}(V_{t,n}(s))\gamma(G_t^{-1}(V_{t,n}(s)))ds \right.$$

$$
-\int_0^1 \frac{\partial c}{\partial x}(G_t(Z),s)d(\frac{Z-G_t^{-1}(s)}{Z})e(G_t^{-1}(V_n(s)))ds\Bigg|
$$

$$
\leq \int_0^{\xi+\epsilon}\left|\frac{\partial c}{\partial x}(\zeta_n(t,Z),U_{t,n}(V_{t,n}(s)))d(\frac{Z-G_t^{-1}(V_{t,n}(s))}{Z})\right. \tag{10}
$$

$$
\left.-\frac{\partial c}{\partial x}(G_t(Z),s)d(\frac{Z-G_t^{-1}(s)}{Z})\right|e(G_t^{-1}(V_{t,n}(s)))ds
$$

This latter tends uniformly in ϕ to zero since $\frac{\partial c}{\partial x}(\cdot,\cdot)$ is uniformly continuous on $[0,\xi+\epsilon]$ in the sense described in (HP3). By letting $\epsilon\to 0$, we get the convergence of (10) to zero, uniformly in ϕ, in outer probability. Remark further that

$$
\left|\int_0^1 \frac{\partial c}{\partial x}(G_t(Z),s)d(\frac{Z-G_t^{-1}(s)}{Z})e(G_t^{-1}(V_{t,n}(s)))ds - K_c\right|
$$

$$
\leq \int_0^1\left|\frac{\partial c}{\partial x}(G_t(Z),s)d(\frac{Z-G_t^{-1}(s)}{Z})\right|\left|e(G_t^{-1}(V_{t,n}(s)))-e(G_t^{-1}(s))\right|ds. \tag{11}
$$

Now $\left|e(G_t^{-1}(V_{t,n}(s)))-e(G_t^{-1}(s))\right|$ is the indicator function of the symmetrical difference of the sets $(V_{t,n}(s)\leq G(Z))$ and $(s\leq G_t(Z))$. And we have, for large values of n, $w.p.1-\epsilon$

$$
(V_{t,n}(s)\leq G_t(Z))\Delta(s\leq G_t(Z))\subseteq \tag{12}
$$

$$
(G_t(Z)-\epsilon\leq s<G_t(Z))+(G_t(Z)\leq s\leq G_t(Z)+\epsilon).
$$

This and the uniform boundedness in ϕ, (say by M), of the functions $\frac{\partial c}{\partial x}(G_t(Z),s)d(\frac{Z-G_t^{-1}(s)}{Z})$, due its uniform continuity on $[0,G(Z)+\epsilon_0]$ in the sense of (HP3), imply that the second member in (11) is, uniformly in ϕ, less than $2M\epsilon$, $w.p.1-\epsilon$. By letting $\epsilon\to 0$, we finally get the convergence of the integral in 11, uniformly in ϕ, to K_c. This kind of arguments will be used in the sequel without further details. It follows that

$$
D_1=K_c(G_n(Z)-G(Z))/\sqrt{n}+o_p^*(n^{-1/2}), \tag{13}
$$

uniformly in ϕ. Next

$$
D_2=\frac{1}{n}\sum\{G_{t,n}(Y_j(t))-G_t(Y_j(t))\}\frac{\partial c}{\partial y}(G_t(Z),\zeta_{t,n}(j))\;\gamma(Y_j(t)); \tag{14}
$$

where $\zeta_{t,n}(j)$ lies between $G_{t,n}(Y_j(t))$ and $G_t(Y_j(t))$. Now, denoting $I_n=[G_{t,n}(Y_j(t))\wedge G_t(Y_j(t)),G_{t,n}(Y_j(t))\vee G_t(Y_j(t))]$, we have to show that

$$
\max_{1\leq j\leq n}\sup_{t\in[0,T]}\sup_{(x,y)\in I_n^2}\left|\frac{\partial c}{\partial y}(G_t(Z),x)-\frac{\partial c}{\partial y}(G_t(Z),y))\right|\gamma(Y_j(t))\to_{P^*}0, \tag{15}
$$

as $n\to+\infty$. The boundedness of $\gamma(\cdot)$, the equi-continuity of $\frac{\partial c}{\partial y}$ and (HP1) establish 15. By $(HP7)$ and Theorem 2 in Lo [8], the process $B_n^*(\cdot)$ defined by:

$$
\{B_n^*(t),0\leq t\leq T\}=\left\{\frac{1}{\sqrt{n}}\sum\{G_{t,n}(Y_j(t))-G_t(Y_j(t))\},0\leq t\leq T\right\}
$$

converges to a Gaussian process $\{G_0(t),0\leq t\leq T\}$ in $\ell^\infty([0,T])$ under the hypotheses. Thus,

$$
\sqrt{n}D_2=\frac{1}{\sqrt{n}}\sum\{G_{t,n}(Y_j(t))-G_t(Y_j(t))\}\frac{\partial c}{\partial y}(G_t(Z),G_t(Y_j(t))\;\gamma(Y_j(t))+o_P^*(1),
$$

uniformly in $\phi\in\Phi$. We conclude that

$$
\sqrt{n}(J_n(c)-H_c)=\frac{1}{\sqrt{n}}\sum g(Y_j)-Eg(Y_j)+K_1\sqrt{n}(G_{t,n}(Z)-G(Z)
$$

$$
+\frac{1}{\sqrt{n}}\sum\{G_{t,n}(Y_j(t))-G_t(Y_j(t))\}\nu_{c,t}(y)+o_P^*(1)
$$

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Now, we have to handle in the same lines to get for $J_n(\pi)$

$$\sqrt{n}(J_n(\pi) - H_\pi) = \frac{1}{\sqrt{n}} \sum g_\pi(Y_j) - Eg_\pi(Y_j) + K_\pi\sqrt{n}(G_{t,n}(Z) - G(Z)$$

$$+ \frac{1}{\sqrt{n}} \sum \{G_{t,n}(Y_j(t)) - G_t(Y_j(t))\} \nu_{\pi,t}(Y_j) + o_P^*(1).$$

Remark that $H_c(\phi)$ and $H_\pi(\phi)$ are uniformly bounded. Then we arrive at the representation (R).

Now, when $(HP2c)$ holds for $h(n, Q) = B(n, Q)$, the quotient of (6) is one. This leads to the representation (RD) only based on that of $J_n(c)$.

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