Optimal Portfolios Under Dynamic Shortfall Constraints

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Abstract. Value-at-Risk (VaR), a downside risk measure, has emerged as the industry standard with regulatory authorities enforcing its use in risk measurement and management. Despite its widespread acceptance, VaR is not coherent. Tail Conditional Expectation (TCE), on the other hand, for an underlying continuous distribution, is a coherent risk measures. Our focus in this paper is the dynamic portfolio and consumption choice of a trader subject to a risk limit specified in terms of TCE.

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1 Introduction

Value-at-Risk (VaR), a downside risk measure, has emerged as the industry standard with regulatory authorities enforcing its use in risk measurement and management (Jorion [6]). Despite its widespread acceptance, VaR is not coherent (Artzner et al. [2]). Tail Conditional Expectation (TCE), on the other hand, for an underlying continuous distribution, is a coherent risk measures (Rockafellar and Uryasev [10]).

Our focus in this paper is the dynamic portfolio[4] and consumption choice of a trader subject to a risk limit

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[4]By dynamic portfolio strategy we mean portfolio re-balancing as well as re-calculation of TCE at short intervals of time within the investment horizon. This is in contrast to the static (one-period) model of Markowitz whereby the portfolio once chosen, is never revised.

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specified in terms of TCE. In the existing literature, investment and consumption strategies are often studied in separate problems. Here, we consider both in the same problem formulation. We apply the TCE constraint while maximizing the agent's utility over consumption throughout the investment horizon, and over terminal wealth. This problem has not yet received adequate attention in the existing literature. We show through numerical simulations by applying an algorithm similar to that in Yiu [11] that the introduction of a TCE constraint reduces investment in risky assets and increases consumption (cf. Cuoco et al. [3]). Putschögl and Sass [9] use Expected Shortfall instead of TCE.

2 The Model

We consider a standard Black-Scholes type market (Korn [7]) consisting of one risk-free bond and n risky stocks and a finite time horizon [0,T]. Uncertainty in the financial market is modeled by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, equipped with a filtration that is a non-decreasing family $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ of sub-σ-fields of $\mathcal{F}$. It is assumed throughout this paper that all stated processes are well defined without giving any regularity or technical conditions ensuring this. The interested reader may see [1] for such details. The risk-free rate $r = r_t$ of the risk-free asset (bond) S^0 is supposed to evolve according to $dS_t^0 = rS_t^0 dt,\ S_0^0 = s^0$. For the risky assets (stocks), for which the prices will be denoted by $S_t = (S_t^1, \ldots, S_t^n)$ for some $n \in \mathbb{N}$, the basic evolution model is that of a

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log-normal diffusion process:

$$\frac{dS_t^i}{S_t^i} = \mu^i dt + \sum_{j=1}^{k} \sigma^{ij} dW_t^j \quad \forall\, t \in [0,T], \qquad S_0^i = s^i,\ i = 1, \ldots, n, \tag{1}$$

where, for some $k \in \mathbb{N}$, $W_t = [W_t^1, \ldots, W_t^k]'$, with the symbol ($'$) standing for transpose, is a k-dimensional standard Wiener process, i.e., a vector of k independent one-dimensional Wiener processes. The n-vector $\mu = (\mu^1, \ldots, \mu^n)'$, contains the expected instantaneous rates of return and the $n \times k$-matrix $\sigma = \sigma^{ij}$, $(i = 1, \ldots, n,\ j = 1, \ldots, k)$ measures the instantaneous sensitivities of the risky asset prices with respect to exogenous shocks so that the $(n \times n)$-matrix $\sigma\sigma'$ contains the variance and co-variance rates of instantaneous rates of return. An agent invests according to an investment strategy that can be described by the $(n+1)$-dimensional, $\mathcal{F}_t$-predictable process $\theta_t = (\theta_t^0, \theta_t^1, \ldots, \theta_t^n)$, with θ_t^i $(i = 1, \ldots, n)$ denoting the fraction of wealth invested in the risky asset i at time t, whereby the remaining fraction $1 - \sum_{j=1}^{n} \theta_t^i$ of the agent's wealth is invested in the risk-free asset. Let c_t be the instantaneous consumption rate. The corresponding portfolio value process reads

$$dV_t^{\theta,c} = V_t^{\theta,c}\left[\left(1 - \sum_{i=1}^{n} \theta_t^i\right)\frac{dS_t^0}{S_t^0} + \sum_{i=1}^{n} \theta_t^i \frac{dS_t^i}{S_t^i}\right] - c_t dt, \qquad V_0^{\theta,c} = v_0. \tag{2}$$

To have a better exposition, we adopt a matrix expression: denote $\sigma = [\sigma^{ij}]$, $\theta_t = [\theta_t^1, \ldots, \theta_t^n]'$, $\mu = [\mu^1, \ldots, \mu^n]'$, $1_n = [1, \ldots, 1]'$ and $W_t = [W_t^1, \ldots, W_t^k]'$, so that σ is an $n \times k$ matrix, $\mu - r1_n$ and θ_t are n-dimensional column vectors and W_t is a k-dimensional column vector. Hence equation (2) can be rewritten as

$$dV_t^{\theta,c} = V_t^{\theta,c}\left[(r + \theta_t'(\mu - r1_n))\, dt + \theta_t' \sigma_t dW_t\right] - c_t dt, \qquad V_0^{\theta,c} = v_0. \tag{3}$$

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We have adopted an incomplete market asset pricing setting of He and Pearson [5]. To eliminate redundant assets, we assume that σ is of full row rank, that is, $\sigma\sigma'$ is an invertible matrix.

3 Tail Conditional Expectation

Tail Conditional Expectation is closely related to the Value-at-Risk concept, but overcomes some of the conceptual deficiencies of Value-at-Risk (Rockafellar and Uryasev [10]). In particular, it is a coherent risk measure for continuous distributions (Artzner et al. [2]). Given some probability level $\alpha \in (0,1)$, a time t wealth benchmark Υ_t and horizon Δt , the Value-at-Risk (VaR_t^α) of time t wealth V_t at the confidence level $1-\alpha$ is given by the smallest number L such that the probability that the loss $G_{t+\Delta t} := \Upsilon_{t+\Delta t} - V_{t+\Delta t}^{\theta,c}$ exceeds L is no larger than α.

$$VaR_t^\alpha = \inf\{L \geq 0 : \mathbf{P}(G_{t+\Delta t} \geq L|\mathcal{F}_t) \leq \alpha\} := (Q_t^\alpha)^-,$$

$$Q_t^\alpha = \sup\left\{L \in \mathbb{R} : \mathbf{P}((V_{t+\Delta t}^{\theta,c} - \Upsilon_{t+\Delta t}) \leq L|\mathcal{F}_t) \leq \alpha\right\},$$

where Q_t^α is the quantile of the projected wealth surplus at the horizon $t+\Delta t$ and $x^- = \max\{0,-x\}$.

Definition 1. ***(Tail Conditional Expectation)***
Consider the loss distribution $G_{t+\Delta t} := \Upsilon_{t+\Delta t} - V_{t+\Delta t}^{\theta,c}$ *represented by a continuous distribution function* $F_{G_{t+\Delta t}}$

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with $\int_{\mathbb{R}} |G_{t+\Delta t}| dF(G_{t+\Delta t}) < \infty$. *Then the* TCE_t^α *at confidence level* $1-\alpha$ *is defined as*

$$TCE_t^\alpha = \mathbb{E}_t \left\{ G_{t+\Delta t} \geq VaR_t^\alpha | \mathcal{F}_t \right\} =$$

$$\frac{1}{\alpha} \mathbb{E}_t \left\{ G_{t+\Delta t} I(G_{t+\Delta t} \geq -Q_t^\alpha) | \mathcal{F}_t \right\}^+,$$

where I(A) is the indicator function of the set A and $x^+ = \max\{0, x\}$.

In other words, the Tail Conditional Expectation of wealth V_t at time t is the conditional expected value of the loss exceeding $(Q_t^\alpha)^-$. Given the log-normal distribution of asset returns (μ, σ constant), the TCE_t^α can be explicitly computed as can be seen in the following proposition. We refer to [1] for the proof.

Proposition 1. *(**Computation of Tail Conditional Expectation**)*
We have

$$TCE_t^\alpha = \frac{1}{\alpha} \left(\alpha \Upsilon_{t+\Delta t} - V_t^{\theta,c} \left[\exp\left((\theta_t'(\mu - r 1_n) + r - \frac{c_t}{V_t^{\theta,c}}) \Delta t \right) \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t' \sigma\| \sqrt{\Delta t} \right) \right] \right), \tag{4}$$

where $\Phi(\cdot)$ *and* $\Phi^{-1}(\cdot)$ *denote the normal distribution and the inverse distribution functions.*

4 Statement of the Problem

We seek the optimal asset and consumption allocation that maximizes (over all admissible $\{\theta_t, c_t\}$) the expected utility of discounted terminal wealth at time T and consumption over the entire horizon $[0, T]$, for a risk averse investor who limits his risk by imposing an upper bound

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on the TCE. In mathematical terms the final stochastic optimal control problem with TCE constraint is

$$\max_{\{\theta,\ c\}\in A(v_0)} \mathbb{E}_{0,v_0}\left\{\int_0^T e^{-\rho s}U^1(c_s)ds + e^{-\rho T}U^2(V_T)\right\}, \quad (5)$$

subject to the wealth dynamics given in (3) and the TCE constraint

$$TCE_t^\alpha \le \varepsilon(V_t,t),\ \forall\ t\in[0,T-\Delta t), \quad (6)$$

for fixed α and $\Delta t > 0$, where $TCE_t^\alpha = TCE_t^\alpha(V_t,\theta_t,c_t)$ is given in (4). Here $\mathbb{E}_{t,v}$ denotes the expectation operator at time t, given $V_t^{\theta,c} = v$ (and given the chosen consumption and investment strategies), U^1 and U^2 are twice differentiable, increasing, concave utility functions, $\varepsilon(v,t)$ is an upper bound on TCE and $\rho > 0$ is the rate at which consumption and terminal wealth are discounted. We let $x^{1-\gamma}/(1-\gamma)$, where $\gamma\in(0,\infty)\backslash\{1\}$. This falls in the category of power utility functions, also known as Constant Relative Risk Aversion (CRRA) utility functions.

5 Optimality Conditions

In applying the dynamic programming approach we solve the Hamilton-Jacobi-Bellman (HJB) equation associated with the utility maximization problem (5). From Fleming and Rishel [4] we have that the corresponding HJB equation is given by

$$\rho J(v,t) = \max_{c_t\ge 0,\ \theta_t\in\mathbb{R}^n}\{U(c_t) + J_t(v,t) + J_v(v,t) \quad (7)$$

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$$\times \left(v[\theta_t'(\mu - r1_n) + r] - c_t\right) + \frac{1}{2} J_{vv}(v,t) v^2 \theta_t' \sigma\sigma' \theta_t \Bigg\},$$

subject to the terminal condition $J(v,T) = U(v)$, where J, the value function is given by

$$J(v,t) = \sup_{\{\theta,\ c\} \in A(v)} \mathbb{E}_{t,v} \left\{ \int_t^T e^{-\rho s} U(c_s) ds + e^{-\rho T} U(V_t^{\theta,c}) \right\}, \tag{8}$$

where subscripts on J denote partial derivatives and $V_t^{\theta,c} = v$, the wealth realization at time t.
In solving the HJB equation (7), the static optimization problem

$$\max_{c_t \geq 0,\ \theta_t \in \mathbb{R}^n} \left\{ U(c_t) + J_v(v,t) \left(v[\theta_t'(\mu - r1_n) + r] - c_t\right) + \frac{1}{2} J_{vv}(v,t) v^2 \theta_t' \sigma\sigma' \theta_t \right\}, \tag{9}$$

subject to the TCE constraint (6) can be tackled separately to reduce the HJB equation (7) to a nonlinear partial differential equation of J only.
We introduce the Lagrange function

$$\mathcal{L}(\theta, c, \lambda) = \mathcal{L}(\theta(v,t), c(v,t), \lambda(v,t))$$

as

$$\mathcal{L}(\theta, c, \lambda) = J_v(v,t) \left(v \left[\theta'(\mu - r1_n) + r - c\right]\right) \tag{10}$$

$$+ \frac{1}{2} v^2 \theta' \sigma\sigma' \theta J_{vv}(v,t) + U(c) - \lambda(v,t) \left(\alpha TCE_t^\alpha(v,\theta,c) - \varepsilon_1\right), \tag{11}$$

where λ is the Lagrange multiplier, $\varepsilon_1 = \varepsilon \cdot \alpha$ and TCE_t^α is given in (4). The first-order necessary conditions with

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respect to θ, c and λ respectively of the static optimization problem (11) are given by

$$\nabla_\theta \mathcal{L} = vJ_v(\mu - r1_n) + \frac{1}{2}J_{vv}v^2\sigma\sigma'\theta + \lambda v\left[\, (\mu - r1_n)\Delta t \right. \quad (12)$$

$$\exp\left((\theta'(\mu - r1_n) + r - \frac{c}{v})\Delta t\right) \cdot \Phi\left(\Phi^{-1}(\alpha) - \|\theta'\sigma\|\sqrt{\Delta t}\right)$$

$$- \exp\left((\theta'(\mu - r1_n) + r - \frac{c}{v})\Delta t\right) \cdot \frac{\sqrt{\Delta t}}{2}\frac{\sigma\sigma'\theta}{\|\theta'\sigma\|} \cdot$$

$$\left. \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(\Phi^{-1}(\alpha) - \|\theta'\sigma\|\sqrt{\Delta t})^2\right]\, \right] = 0,$$

$$\frac{\partial \mathcal{L}}{\partial c} = U_c(c) + \lambda(v,t)\Delta t \cdot \Phi\left(\Phi^{-1}(\alpha) - \|\theta'\sigma\|\sqrt{\Delta t}\right)$$
$$\cdot\left[\exp\left((\theta'(\mu - r1_n) + r - \frac{c}{v})\Delta t\right) \cdot \Phi\left(\Phi^{-1}(\alpha) - \|\theta'\sigma\|\sqrt{\Delta t}\right)\right]$$
$$= J_v(v,t), \quad (13)$$

where U_c is the first-order derivative of U with respect to c and

$$\frac{\partial \mathcal{L}}{\partial \lambda} = H(v,t) := -\alpha\Upsilon_{t+\Delta t} + v\left[\, \exp\left((\theta'(\mu - r1_n) + r - \frac{c}{v})\Delta t\right) \right. \quad (14)$$

$$\left. \cdot\Phi\left(\Phi^{-1}(\alpha) - \|\theta'\sigma\|\sqrt{\Delta t}\right)\, \right] + \varepsilon_1 = 0,$$

while the complimentary slackness condition is given as $\lambda(v,t)H(v,t) = 0$ and $\lambda(v,t) \geq 0$. Simultaneous solution

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of these first-order conditions yields the optimal solutions θ^{opt}, c^{opt} and λ^{opt}. Substituting these into (7) gives the partial differential equation

$$-\rho J(v,t) + \frac{(c^{opt}(v,t))^{1-\gamma}}{1-\gamma} + J_t(v,t) + \tag{15}$$

$$J_v(v,t)\left(v[(\theta^{opt}(v,t))'(\mu - r1_n) + r]\right.$$

$$\left.-c^{opt}(v,t)\right) + \frac{1}{2}J_{vv}(v,t)v^2(\theta^{opt}(v,t))'\sigma\sigma'\theta^{opt}(v,t) = 0,$$

with terminal condition $J(v,T) = \frac{v^{1-\gamma}}{1-\gamma}$, which can then be solved for the optimal value function $J^{opt}(v,t)$. Because of the non-linearity in θ^{opt} and c^{opt}, the first-order conditions together with the HJB equation are a non-linear system so the partial differential equation (15) has no analytic solution and numerical methods such as Newton's method or Sequential Quadratic Programming (SQP) (see, e.g., Nocedal and Wright [8]) are required to solve for $\theta^{opt}(v,t)$, $c^{opt}(v,t)$, $\lambda^{opt}(v,t)$ and $J^{opt}(v,t)$ iteratively.

6 Numerical Results

We use an iterative algorithm similar to that of Yiu [11]which yields a $\mathcal{C}^{2,1}$ approximation $\widehat{J}$ of the exact solution J. The pair $(\widehat{\theta}_t, \widehat{c}_t)$ is the investment strategy related to $\widehat{J}$. We refer to [1] for details of the algorithm. We have implemented the algorithm to illustrate the optimal portfolio of the preceding section with examples and

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Parameter	**Value**
Stock (S^1)	$\mu = 4\%$, $\sigma^{11} = 5\%$, $\sigma^{12} = 5\%$
Stock (S^2)	$\mu = 6\%$, $\sigma^{21} = 5\%$, $\sigma^{22} = 20\%$
Bond (S^0)	$r = 3\%$
Investment horizon	$t \in [0, 1]$
Initial wealth	$v_0 = 1$
Shortfall probability	$\alpha = 1\%$
Value-at-Risk horizon	$\Delta t = 1/48 \approx 7$ days
Utility function	$U(x) = x^{1-\gamma}/\ (1-\gamma), \quad \gamma = 0.9$
Wealth benchmark	$\Upsilon_t = V_t e^{r\Delta t}$ (money market)
Bound	$\varepsilon = 1$

Table 1: Parameters for the consumption and investment portfolio optimization problem.

assume $n = 2$. That is, the market is composed of two risky stocks and a risk-free bond. Table 1 shows the parameters for the portfolio optimization problem and the underlying Black-Scholes model of the financial market.

We consider the Tail Conditional Expectation of the wealth surplus $V_t - \Upsilon_{t+\Delta t}$ with respect to the benchmark $\Upsilon_{t+\Delta t}$ such that it satisfies $TCE_t^\alpha(V_{t+\Delta t} - \Upsilon_{t+\Delta t}) \leq \varepsilon$, where ε comes from Table 1. That is, the TCE is re-evaluated at each discrete time step (TCE horizon) Δt and kept below the upper bound ε, by making use of conditioning information. Here, the shortfall benchmark is taken to be the investment in the risk-free bond $\Upsilon_{t+\Delta t} = V_t e^{r\Delta t}$. Figure 1 shows in the right panel the amount of wealth invested in the risky assets with and without the TCE constraint, plotted against the possible wealth realization at different times. The left panel shows the value function. The shortfall benchmark is the investment in the risk-free bond.

As can be observed from the image, as the wealth level increases, so does the investment in risky assets. This results from the property of constant relative risk aver-

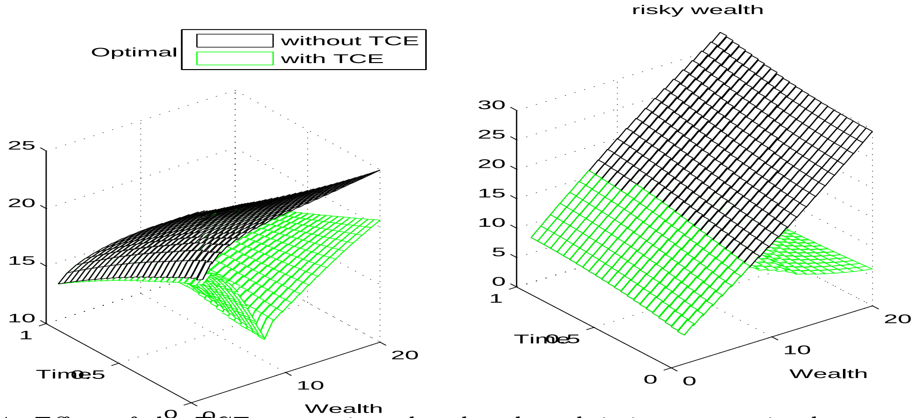

Figure 1: Effect of the TCE constraint when benchmark is investment in the bond.

sion of the utility function. A good control over the investment in the risky assets has been achieved and the proportions invested in the risky assets are reduced in order to fulfill the TCE constraint. In particular, when the constraint is not active, the optimal portfolio follows the unconstrained solution; as the portfolio value increases, the TCE constraint becomes active and allocates less to the risky assets. The local minimum (around wealth level 10) observed in the left panel of Figure 1 comes as a result of a sudden increase in the consumption rate once the constraint becomes active. The value function of the constrained problem is identical to that of the unconstrained one when the Lagrange multipliers are null, whereas it is inferior when the constraint is active.

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