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A goodness-of-fit test based on Kendall's process: Durante's bivariate copula models

N'dri Hubert Bian [1,*], **Ouagnina Hili** [2,1] **and Gueï Cyrille Okou** [3]

[1] UMRI Mathématiques et Nouvelles Technologies de l'Information. Institut National Polytechnique Félix Houphouët-Boigny (*INP-HB*), Yamoussoukro, Côte d'Ivoire.
[3] Université Jean-Lorougnon-Guédé, Daloa, Côte d'Ivoire

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Abstract. The proposed goodness-of-fit testing procedures for copula models are fairly recent. The new test statistics or omnibus tests are functional of an empirical process motivated by the theoretical and empirical versions of Kendall's or Spearman's dependence function. In this paper, we propose a fitting procedure for a symmetric and flexible copula model with a non-zero singular component using the Kendall process. The conditions under which this empirical process weakly converges are satisfied. Using a parametric bootstrap method that allows to compute approximate p-values, it is empirically shown that tests based on the Cramér-von Mises distance keeps the prescribed value for the nominal level under the null hypothesis. Simulation studies that demonstrate the power of the fit test are presented.

Key words: copula; Cramér-von Mises statistic; goodness-of-fit test; Kendall's dependence function; parametric bootstrap.
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*Corresponding author N'dri Hubert Bian : hubertbn@yahoo.fr
Ouagnina Hili: o_hili@yahoo.fr
Gueï Cyrille Okou : okou.guei.cyrille@gmail.com

Résumé. Les procédures proposées pour tester l'adéquation des modèles de copule sont assez récentes. Les nouvelles statistiques de test ou encore des tests omnibus sont des fonctionnelles d'un processus empirique motivé par les versions théorique et empirique de la fonction de dépendance de Kendall ou de Spearman. Dans cet article, nous proposons une procédure d'ajustement pour un modèle de copule symétrique et flexible ayant une composante singulière non nulle. Les conditions sous lesquelles, ce processus empirique converge faiblement sont satisfaites. En utilisant une méthode de bootstrap paramétrique permettant le calcul des p-valeurs approximatives, il est montré empiriquement que les tests basés sur la distance de Cramér-von Mises conservent la valeur prescrite pour le niveau nominal sous l'hypothèse nulle. Des études de simulation qui prouvent la puissance du test sont présentées.

The authors.

N'dri Hubert Bian, *Ph.D.* Student, is preparing a *PhD* Thesis under the second author, Institute National Polytechnique Félix Houphouët-Boigny (*INP-HB*), Yamoussoukro, Côte d'Ivoire.

Ouagnina Hili, *Ph.D.*, Full Professor of Statistic, Institute National Polytechnique Félix Houphouët-Boigny (*INP-HB*), Yamoussoukro, Côte d'Ivoire.

Gueï Cyrille Okou, *Ph.D.*, Master Assistant, Université Jean-Lorougnon-Guédé, Daloa, Côte d'Ivoire

1. Introduction

In recent years, copulas have been of great interest to statisticians as a promising and flexible tool for understanding the dependence between random variables and for modeling non-Gaussian multivariate data. Copula models are frequently used in various fields such as finance Cherubin *and al.* (2004) , actuarial sciences Frees and Valdez (1998), geostatistic Bacigál and Komorníková (2006) and hydrology Genest and Favre (2007).

A copula is a dependence function, a mathematical expression that allows the modeling of the dependence structure between stochastic variables. As such, copulas can be used to construct multivariate distribution functions. The main advantage of the copula approach is that it is capable of studying and estimating separate margins and dependence structure. Then these parts can be put together to construct a multivariate distribution function.

One notes thus in the statistical literature, see for example, Joe (1997), Nelsen (2006) and Okhrin *and al.* (2013), the introduction and study of several families of copula, motivated by the practical necessity of more complex models.

Here we investigate a class of copulas that can be used as a possible option to all class that allow to conveniently describe the dependence in the tail of the joint distribution. More precisely, we rely on a generalization of the bivariate Cuadras-Augé copula, which has been proposed in (Cuadras and Augé (1981), section 2.3), then, discussed in detail in Durante (2006) and Durante (2007) for the bivariate case, and in Durante *and al.* (2007) for the multivariate case.

Following the copula modeling step, it seems natural to want to test the goodness-of-fit of the model based on the data set at our disposal. This problem has also aroused the interest of researchers. The first step in this direction was taken by Genest and Rivest (1993). These authors were based on the integral probability transformation that can be associated with a copula. This procedure was later formalized by Wang and Wells (2000) and Genest *and al.* (2006), who obtained the asymptotic behavior of goodness-of-fit test statistics and implemented a re-sampling method, namely the parametric bootstrap, which, unlike the method proposed by Wang and Wells (2000), is valid. Another significant contribution is that of Genest *and al.* (2009), whose tests described are based on the empirical copula. Goodness-of-fit procedures have been proposed by Fermanian (2005) and Mesfioui *and al.* (2009). These methods consider univariate marginal distributions as infinite-dimensional nuisance parameters, and the observations are replaced by the statistics that are maximally invariant in this context, that is, ranks.

Using ranks instead of the observations themselves can have a significant impact on the limiting distribution of a test statistic. For example, a fit statistic suggested by Breymann *and al.* (2003), which is distributed as a chi-square variable in the case of known margins, deviates from this supposed distribution when using its rank-based version. As remarked by the authors themselves, this had an significant effect on the observed nominal level of the test.

Newly, Durante copula class have been used to model the spatial behavior of some hydrological random variables (Durante and Salvadori (2010), Salvadori *and al.* (2011)). In addition, different estimators of the univariate real value function "generator" have been proposed both in a parametric and a semi-parametric framework (Durante and Okhrin (2014)). Moreover, this class of copulas has been chosen as a suitable linking copula for the one-factor copula model to construct "*FDG*" copulas (see Mazo *and al.* (2016)). However, so far, their practical use has been limited, since standard statistical techniques for adjusting these copula have not been investigated in depth until special cases (mostly related to those Durante copulas that are also extreme value copulas). However, we can note the study made by Mesfioui *and al.* (2009) which allows us to verify the goodness-of-fit based on the Spearman process of some families of copula, for example the Durante copula. Their comparative analysis showed that the test based on the Kendall process is the best in all the situations considered. This study did not use the bootstrap version proposed by Genest and Rémillard (2008) and is also specifically focused on the Cuadras-Augé copula.

The purpose of this paper is to propose a goodness-of-fit test to check the parametric model hypothesis for models belonging to the Durante copula family. Since these first-order partial derivatives are not continuous, the fit test is developed around a sample version of the Kendall dependence function using the Cramér-von Mises test statistic. In particular, we will verify that the empirical kendall distribution of the observations of the Durante copula model is adequate with its theoretical distribution. It will be possible by estimating the p-value through a parametric bootstrap method. In addition, the asymptotic tools used in this paper are typical of the type of arguments that are needed to characterize the behavior of a large sample of goodness-of-fit statistics in semi-parametric models.

The paper is organized as follows. Second section presents Durante's bivariate copula model and also describes the proposed test statistic. In section 3, an empirical process based on the functional estimation of the Kendall distribution function is defined and the weak convergence of an appropriate standard version is established. In the fourth section, the assumptions that led to this asymptotic result are tested for a few copula of the Durante class. In section 5, a parametric bootstrap is proposed for the calculation of p-values and in the section 6 the power of the test is illustrated through simulations. Finally, the last section is dedicated to the appendix.

2. Methodology

Several statistical inference procedures involving two random variables with continuous marginal distributions are successfully solved by considering the dependence structure and marginal behaviors separately. This idea resides in the theorem of Sklar (1959), which allows to write a bivariate distribution function according to its margins F and G, namely

$$H(x, y) = C\{F(x), G(y)\}.$$

Function C, is called the copula of H, unique if its margins F and G are continuous. It is a bivariate cumulative distribution function whose univariate margins are uniform over the interval $[0, 1]$. This formula allows the development of statistical techniques to select a dependence function, that is, a copula, that fits correctly to a set of bivariate observations. In other words, we have to choose between the null and alternative hypotheses belonging to a given parametric family or not, where

$$\mathcal{H}_0 : C \in \{C_\theta; \theta \in \Theta\} \text{ versus } \mathcal{H}_1 : C \notin \{C_\theta; \theta \in \Theta\}.$$

Consider here Durante's bivariate copula model, where the unknown copula associated belongs to a class

$$\{C_\theta; \theta \in \Theta\},$$

Θ is an open subset of $\mathbb{R}$. Thus, the choice of the best copula to model is based on the Kendall function of the Durante bivariate copula. We then define the cramér-von Mises statistic, one of omnibus statistics with good power properties. Conditions which ensure the weak convergence of the goodness-of-fit process are given and verified for a some number of copulas. Hence, the limiting distribution, under the null hypothesis, of the goodness-of-fit process depend on the unknown parameter value estimated θ by Kendall's inversion. Through an algorithm based on the parametric bootstrap method, approximate p-values are determined, see Genest *and al.* (2009).

2.1. Durante's bivariate copula model and its Kendall function

2.1.1. Durante's bivariate copula model

In the case of bivariate copula, there are several families of copula. We are interested here to a particular class of copula, which is an alternative to all the models that describe dependence in the joint distribution tail. Indeed, following the seminal ideas of Marshall (1996), Durante (2006) established the copula verifying conditions for this class. Durante's class of copulas can also be interpreted as models with shocks, having a non vanishing singular part and an absolute continuous part.

In general, the expression of the Durante's copula is identified by the form below:

$$C_\theta(u, v) = \min(u, v)\ f_\theta(\max(u, v)), \tag{1}$$

with $f_\theta : [0, 1] \to [0, 1]$, called generator of C_θ, is a derivable function satisfying the following conditions:

(i) $f_\theta(1) = 1$;

(ii) f_θ is strictly increasing;

(iii) the function $t \mapsto f_\theta(t)/t$ is strictly decreasing on $(0; 1]$.

Some properties of Durante's copula are defined as follows.

Property. Let C_θ a Durante bivariate copula with generator f_θ. Then it has the following properties (see Durante (2007)):

1. C_θ is symmetric or exchangeable, that is $C_\theta(u, v) = C_\theta(v, u)$ for all u, v in $[0, 1]$.
2. If f_θ is concave and non-decreasing with $f_\theta(1) = 1$, then it is a generator of C.

3. From condition (iii) above, $t \leq f_\theta(t) \leq 1$ for all $t \in [0,1]$ and it follows that, for every copula C_θ, $\Pi \preceq C_\theta \preceq M$, with the independence copula $\Pi(u,v) = uv$ when $f_\theta(t) = t$ and the Fréchet-Hoeffing upper bound $M(u,v) = \min(u,v)$ when $f_\theta \equiv 1$. Therefore, every copula C_θ is positively quadrant dependent.

We note that this family of copulas with absolutely continuous and singular components are also identified under the name semi-linear copulas, an expression used in Durante *and al.* (2008), and proved by the fact that these copulas are linear along suitable segments of their domains. Thus, if $(X;Y)$ is an exchangeable random pair with copula C_θ, then $\mathbb{P}(X = Y) > 0$. In practice, for identically distributed random variable, this involves that large values of one variable may correspond (with non-zero probability) to the same large values in the other variable (see Mai and Scherer (2014)).

2.1.2. Kendall function of Durante's copula

For Durante copula models, the goodness-of-ft test is based on the Kendall dependence function since first-order partial derivatives are not continuous. The Kendall function is usually defined as follows:

$$D_\theta(t) = \mathbb{P}\{H(x,y) \leq t\} = \mathbb{P}\{C_\theta(F(x), F(y)) \leq t\} = \mathbb{P}\{C_\theta(U,V) \leq t\},$$

where $(U,V) \sim C_\theta$ with $U = F(x)$ and $V = F(y)$. However, for a copula of type (1), the associated Kendall distribution function is given by the univariate function following:

$$D_\theta(t) = t - t\log(t) + t\log\left\{\frac{g_\theta^{-1}(t)}{f_\theta(g_\theta^{-1}(t))}\right\}, \tag{2}$$

where $g_\theta(t) = tf_\theta(t)$, and f_θ a continuous and strictly increasing function on $[0,1]$ (see, for example, Mesfioui *and al.* (2009)).

In general, the relationship between Kendall's tau tau and a copula can be written as:

$$\tau_\theta = 4\int_0^1\int_0^1 C_\theta(u,v)\; dC_\theta(u,v) - 1 = 1 - 4\int_0^1\int_0^1 \frac{\partial}{\partial u}C_\theta(u,v)\frac{\partial}{\partial v}C_\theta(u,v)\; du\; dv.$$

For Durante bivariate copulas, this can be written in terms of the copula generator $f_\theta(t)$,

$$\tau_\theta = 4\int_0^1 tf_\theta^2(t)\; dt - 1,$$

where the function $\theta \mapsto \tau_\theta$ is continuous on $\Theta \subset \mathbb{R}_+^*$ (see 7).

2.2. Proposed goodness-of-fit test statistic

In their work, Genest *and al.* (2006) had the idea of measuring a distance between the empirical repartition function of the integral probability transformation, that is the empirical Kendall function D_n and its theoretical version D_θ under the null hypothesis. Thus, one of the test statistics considered by Genest *and al.* (2006), generally more powerful, is based on the Cramér-von Mises functional. The possible choices of Cramér-von Mises functional are presented in the different forms follows:

$$\Phi_n = n \int_0^1 \left(D_n(t) - D_{\hat{\theta}_n}(t)\right)^2 \, dD_{\hat{\theta}_n}(t) \tag{3}$$

and

$$\mathcal{W}_n = n \int_0^1 \left(D_n(t) - D_{\hat{\theta}_n}(t)\right)^2 \, dt.$$

In these expressions, $\hat{\theta}_n$ is a convergent estimator for θ. Cramér-von Mise test statistic is expected to be consistent, in the sense that divergences of very different forms can give the same value. These statistics only differ in their integration measure. If the value of the test statistic Φ_n or $\mathcal{W}_n$ is greater than the critical point, then this usually leads to the rejection of the null hypothesis $\mathcal{H}_0$. But, it should be noted that the use of other functional distances is possible, for example Kolmogorov-Smirnov type statistics.

However, Genest and Rémillard (2008) shown that the latter is generally less powerful than the Cramér-von Mises test statistic and also more difficult to compute, especially its bootstrap version.

An approximate p-value can be deduced from the limit distribution of Φ_n, which depends on the asymptotic behavior of the process. Since the asymptotic distribution of Φ_n depends on the copula and the unknown parameter, the approximate p-values of this statistic are determined by simulation.

3. asymptotic results of the goodness-of-fit test

3.1. Weak process convergence

Let $(X_1, Y_1), \cdots, (X_n, Y_n)$, be a random sample, composed of n independent copies of a population with continuous marginal distributions F and G. As in the context of copula inference and the case considered here, one generally desires to keep the marginal distributions unknown, it is necessary to rely on estimates of non-observable variables $\xi_i, 1 \leq i \leq n$. These estimates are provided by replacing F and G by their empirical versions defined as follows:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{(X_i \le x)} \text{ and } G_n(y) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{(Y_i \le y)}.$$

Using the fact that one can write $\xi = C(U, V)$, Genest and Rivest (1993) and Barbe *and al.*(1996) show that a consistent estimate of the Kendall function D_θ is given by the empirical distribution D_n of a resized version of the pseudo-observations of $(\xi_{i,n})_{1\le i\le n}$ where $\xi_{i,n} = C_n(U_i, V_i)$ and

$$C_n(w) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{(U_j \le u, V_j \le v)},$$

$w = (u, v) \in [0, 1]^2$.

$C_n(.)$ is not a copula, its is the empirical distribution of the normalized rank, because of $U_j = R_{j,n}/n$ and $V_j = S_{j,n}/n$. Here, $R_{j,n}$ represents the rank of X_j among $X_1, \cdots, X_n$ and $S_{j,n}$ represents the rank of Y_j among $Y_1, \cdots, Y_n$.

The empirical distribution D_n of the Kendall function is defined by the following expression:

$$D_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{(\xi_{i,n} \le t)}, \quad t \in [0.1].$$

It is a "efficient estimator" of the Kendall dependence function D_θ associated with a member C_θ of the given Durante parametric bivariate copula family.

More specifically, it is proved by Barbe *and al.*(1996), and Ghoudi and Rémillard (1998) that the known parametric Kendall process

$$\delta_{n,\theta} = \sqrt{n}(D_n - D_\theta)$$

converges weakly to a Gaussian process centered under $\mathcal{H}_0$. This suggests to define the goodness-of-fit process Δ_n:

$$\Delta_n = \sqrt{n}(D_n - D_{\hat{\theta}_n}),$$

where $\hat{\theta}_n$ is an estimator of θ, based on the inversion of the Kendall tau.

We provide below the conditions of regularity for the existence of the Kendall process.

Hypothesis $\mathcal{A}_1$. For all $\theta \in \Theta$, the distribution function $D_\theta(t)$ of $C_\theta(u,v)$ admits a density $d_\theta(t) = \partial D_\theta(t)/\partial t$ which is continuous on $\Theta \times (0,1]$, its generator

$$f_\theta(t) = o\left\{\log^{-2}\left(\frac{1}{t}\right)\right\}$$

as $t \to 0$ and such that

$$d_\theta(t) = o\left\{t^{-1/2}\log^{-1/2-\epsilon}\left(\frac{1}{t}\right)\right\},$$

for some $\epsilon > 0$ as $t \to 0$.

Verification of Hypothesis $\mathcal{A}_1$. Consider the dependence function of Kendall defined in (2) and its associated density is:

$$d_\theta(t) = -\log(t) + \log\left\{\frac{g_\theta^{-1}(t)}{f_\theta(g_\theta^{-1}(t))}\right\} + \frac{t(g_\theta^{-1})'(t)}{g_\theta^{-1}(t)} - \frac{t(g_\theta^{-1})'(t)f_\theta'(g_\theta^{-1}(t))}{f_\theta(g_\theta^{-1}(t))}. \tag{4}$$

To check the hypothesis $\mathcal{A}_1$, we have for all $t \in [0,1]$, $t \leq f_\theta(t) \leq 1$ and f_θ is a bijective function on $[0.1]$. It is assumed that $f_\theta^{'}(t)$ exists and is continuous.

Remark that

$$\begin{aligned} d_\theta(t) \times t^{1/2}\log^{1/2+\epsilon}(1/t) &= -t^{1/2}\log(t) \times \log^{1/2+\epsilon}(1/t) + t^{1/2}\log^{1/2+\epsilon}(1/t) \times \log\left(\frac{g_\theta^{-1}(t)}{f_\theta(g_\theta^{-1}(t))}\right) \\ &+ \frac{t(g_\theta^{-1})'(t)}{g_\theta^{-1}(t)} \times t^{1/2}\log^{1/2+\epsilon}(1/t) - \frac{t(g_\theta^{-1})'(t)f_\theta'(g_\theta^{-1}(t))}{f_\theta(g_\theta^{-1}(t))} \times t^{1/2}\log^{1/2+\epsilon}(1/t). \end{aligned}$$

Now, we will study the limit of every term.

First we note that

$$\begin{aligned} \lim_{t^+\to 0}\left\{-t^{1/2}\log(t) \times \log^{1/2+\epsilon}(1/t)\right\} &= \lim_{t^+\to 0}\left\{-t^{1/2}\log(t) \times \log^2(1/t) \times \log^{1/2+\epsilon-2}(1/t)\right\} \\ &= \lim_{t^+\to 0}\left\{-t^{1/6}\log(t) \times t^{1/6}\log(t) \times t^{1/6}\log(t) \times \log^{\epsilon-3/2}(1/t)\right\} \\ &= 0, \end{aligned}$$

for any $0 < \epsilon < 3/2$, $\lim_{t^+\to 0}\left\{\log^{\epsilon-3/2}(1/t)\right\} = 0$ and $\lim_{t^+\to 0}\left\{t^{1/6}\log(t)\right\} = 0$.

Next, one show that

$$t^{1/2}\log^{1/2+\epsilon}(1/t)\times\log\left(\frac{g_\theta^{-1}(t)}{f_\theta(g_\theta^{-1}(t))}\right) = \left\{t^{1/8}\log(t)\times t^{1/8}\log(t)\times\log^{1/2+\epsilon-2}(1/t)\right\} \times\left\{t^{1/4}\log\left(\frac{g^{-1}(t)}{f(g^{-1}(t))}\right)\right\}$$

and it follows that

$$\lim_{t^+\to 0}\left\{t^{1/2}\log^{1/2+\epsilon}(1/t)\times\log\left(\frac{g_\theta^{-1}(t)}{f_\theta(g_\theta^{-1}(t))}\right)\right\}=0.$$

Because of $\lim_{t^+\to 0}\left\{t^{1/8}\log(t)\right\}=0$, $\lim_{t^+\to 0}\left\{\log^{1/2+\epsilon-2}(1/t)\right\}=0$ for all $0<\epsilon<3/2$ and since the limits of the functions f_θ and g_θ^{-1} are assumed to be finite, then we have

$$\lim_{t^+\to 0}\left\{t^{1/4}\log\left(\frac{g_\theta^{-1}(t)}{f_\theta(g_\theta^{-1}(t))}\right)\right\}=0.$$

Moreover, by adopting the convention used by Durante *and al.* (2008) in the case of semilinear copula definition, this is $\left\{\frac{0}{0}=0\right\}$, we have

$$\begin{aligned}\lim_{t^+\to 0}\left(\frac{t(g_\theta^{-1})'(t)}{g_\theta^{-1}(t)}\times t^{1/2}\log^{1/2+\epsilon}(1/t)\right) &= \lim_{t^+\to 0}\left\{\frac{1}{g'_\theta(g_\theta^{-1}(t))\times g_\theta^{-1}(t)}\right.\\ &\quad\left.\times\ t^{3/2}\log^2(1/t)\times\log^{1/2+\epsilon-2}(1/t)\right\}\\ &= \lim_{t^+\to 0}\left\{\frac{t^{5/4}}{g'_\theta(g_\theta^{-1}(t))g_\theta^{-1}(t)}\times t^{1/4}\log^2(t)\right.\\ &\quad\left.\times\ \log^{1/2+\epsilon-2}(1/t)\right\}\\ &= 0,\quad \epsilon\in(0,3/2]\end{aligned}$$

for $\lim_{t^+\to 0}\left\{\frac{t^{5/4}}{g'_\theta(g_\theta^{-1}(t))g_\theta^{-1}(t)}\right\}=0$ and $\lim_{t^+\to 0}\left\{t^{1/4}\log^2(t)\right\}=0$.

Finally, it follows that f_θ is differentiable almost everywhere on $[0,1]$ and the left and right derivatives of f_θ exist for every $t\in[0,1]$ and assume finite values, we have:

$$\begin{aligned}
\lim_{t^+\to 0}\left\{\frac{t(g_\theta^{-1})'(t)f_\theta'(g_\theta^{-1}(t))}{f_\theta(g_\theta^{-1}(t))}\times t^{1/2}\log^{1/2+\epsilon}(1/t)\right\} &= \lim_{t^+\to 0}\left\{\frac{t^{5/4}}{f_\theta(g_\theta^{-1}(t))}\times (g_\theta^{-1})'(t)\times f_\theta'(g_\theta^{-1}(t))\right\}\\
&\times \lim_{t^+\to 0}\left\{t^{1/4}\log^2(1/t)\times \log^{1/2+\epsilon-2}(1/t)\right\}\\
&= 0,
\end{aligned}$$

because of

$$\lim_{t\to 0}\left\{\frac{t^{5/4}}{f_\theta(g_\theta^{-1}(t))}\times (g_\theta^{-1})'(t)\times f_\theta'(g_\theta^{-1}(t))\right\} = 0,$$
$$\lim_{t^+\to 0}\left\{t^{1/2}\log^2(t)\right\} = 0$$

and

$$\lim_{t^+\to 0}\left\{\log^{1/2+\epsilon-2}(1/t)\right\} = 0.$$

Hypothesis $\mathcal{A}_2$. For all $\theta\in\Theta$, the density $d_\theta(t) = \partial D_\theta(t)/\partial t$ is bounded on any compact subset of $(0,1]$ and for all continuous $h:[0,1]^2\to[0,1]$, the mapping

$$t\mapsto \mu_\theta(t,h) = d_\theta(t)\mathbb{E}\{h(X,Y)|H(X,Y)=t\}$$

is continuous on $(0,1]$, with $\mu_\theta(1,h) = d_\theta(1)\ h(1,1)$.

Verification of Hypothesis $\mathcal{A}_2$. Let us consider the equation (1). Its conditional distribution function is given by

$$c_x(y) = \mathbb{P}(Y\le y|X=x) = \frac{\partial}{\partial x}C_\theta(x,y)$$
$$c_x(y) = \frac{\partial}{\partial x}C_\theta(x,y) = \begin{cases} yf_\theta'(x) & ,\ 0\le y<x\\ f_\theta(y) & ,\ x\le y\le 1\end{cases}$$
$$c_\theta(x,y) = \frac{\partial^2}{\partial x\partial y}C_\theta(x,y) = \begin{cases} f_\theta'(x) & ,\ 0\le y<x\\ f_\theta'(y) & ,\ x\le y\le 1\end{cases}$$

Therefore, associated density at the copula is

$$c_\theta(x,y) = f_\theta'(x\vee y),\qquad x\vee y = \max(x,y). \tag{5}$$

As defined in (4), associated density $d_\theta(t)$ with the Kendall dependence function of the Durante copula family is continuous over any compact subset of $]0;1]$. Moreover, for this class of copula, the independent copula ($\Pi(x,y) = xy$) is the only absolutely continuous copula, see Proposition 3 in Durante (2006). Then the conditional density of $(X|XY = t)$ is given by:

$$q(x|t) = \frac{1}{xd_\theta(t)} f'_\theta\left(x \vee \frac{t}{x}\right) \mathbf{1}_{(x \geq t)},$$

(see Genest *and al.* (2002) and Mesfioui *and al.* (2009)).

The probability mass of the copula of type (1) spread over the diagonal section $C_\theta(x,x)$ between 0 and x, enables us to have the following equality:

$$\begin{aligned} S(x) = C_\theta(x,x) - \int_{[0,x]^2} f'_\theta(x \vee y)\, dx\, dy &= xf_\theta(x) - 2\int_0^x \int_0^m f'_\theta(m)\, dm\, dt \\ &= -xf_\theta(x) + 2\int_0^x f_\theta(m)\, dm. \end{aligned}$$

The density of S is given by

$$\frac{\partial S(x)}{\partial x} = -f_\theta(x) - xf'_\theta(x) + 2f_\theta(x) = f_\theta(x) - xf'_\theta(x)$$

For $x = \sqrt{t}$, on a $p(\sqrt{t}) = \frac{\partial S(x)}{\partial x} \times \frac{\partial x}{\partial t} = \frac{1}{2}\left\{t^{-1/2} f_\theta(\sqrt{t}) - f'_\theta(\sqrt{t})\right\}$, hence,

$$\begin{aligned} \mu_\theta(t,h) &= d_\theta(t)\mathbb{E}\left\{h\left(X, \frac{t}{X}\right) |H(X,Y) = t\right\} \\ &= \int_t^1 h\left(x, \frac{t}{x}\right) \frac{1}{x} f'_\theta\left(x \vee \frac{t}{x}\right) dx + p(\sqrt{t})h(\sqrt{t}, \sqrt{t}), \end{aligned}$$

is continuous since f'_θ is assumed continuous. This shows that Hypothesis $\mathcal{A}_2$ is satisfied.

As observed by Barbe *and al.*(1996) and Ghoudi and Rémillard (1998), these two hypotheses are already enough to show the weak convergence of Kendall's process $\delta_{n,\theta}$. To ensure the convergence of the empirical process Δ_n, one needs two additional hypotheses that deals with the extra term that appears from the fact that θ is unknown.

Hypothesis $\mathcal{A}_3$. For every given $\theta \in \Theta$, $\dot{D}_\theta = \partial D_\theta / \partial \theta$ exists and is continuous on $[0,1]$. Moreover, as $\alpha \to 0$

$$\sup_{|\theta^*-\theta|<\alpha} \sup_{t\in[0,1]} \left| \dot{D}_{\theta^*}(t) - \dot{D}_\theta(t) \right| \to 0.$$

Hypothesis $\mathcal{A}_4$. The sequence $\Lambda_n = \sqrt{n}\{\hat{\theta}_n - \theta\}$ and $\delta_{n,\theta} = \sqrt{n}(D_n - D_\theta)$ jointly converges in law to Λ and δ_θ.

The main theoretical result of the article is now stated. Theorem 1 below identifies the weak limit Δ_θ of the process Δ_n. The conditions are sufficient to ensure that the Cramér-von Mises test statistic Φ_n is a continuous functional.

Theorem 1. *Under hypotheses $\mathcal{A}_1$ - $\mathcal{A}_4$, the empirical process $\Delta_n = \sqrt{n}(D_n - D_{\hat{\theta}_n})$ converges in the Skorohod space of càdlàg functions (right continuous and having left-hand limits) in $\mathfrak{D}[0,1]$ to a continuous and centered Gaussian process whose representation is $\Delta_\theta(t) = \delta_\theta(t) - \Lambda_\theta \dot{D}_\theta(t)$ on $[0,1]$, where Λ_θ and δ_θ are respectively the limit law of Λ_n and centered Gaussian process of $\delta_{n,\theta}$.*

Proof . Let $\Delta_n = \delta_{n,\theta} - \mathrm{B}_n$, where $\mathrm{B}_n(t) = \sqrt{n}(D_{\hat{\theta}_n}(t) - D_\theta(t))$ for $t \in [0,1]$.

Proceeding in the same manner as Genest *and al.* (2006), the proof is as follows:

First, since the hypotheses $\mathcal{A}_1$ and $\mathcal{A}_2$ holds for the Durante's copula, $\delta_{n,\theta} \rightsquigarrow \delta_\theta$ in $\mathfrak{D}[0,1]$, where δ_θ is the continuous, centered Gaussian process (see Barbe *and al.*(1996)). It remains to prove that

$$\sup_{t\in[0,1]} |\mathrm{B}_n(t) - \dot{D}_\theta(t)\Lambda_n| \xrightarrow{\mathbb{P}} 0.$$

Secondly, it will be proved that $\mathrm{B}_n(t)$ can be arbitrarily close to $\dot{D}_\theta(t)\Lambda_n$, as long as n is large enough. With that in mind, that is $\alpha > 0$, arbitrary. By hypothesis $\mathcal{A}_4$, the sequence (Λ_n) is tight as it converges in law to Λ. Thus, for all $\beta > 0$, there exists $M = M_\beta \in \mathbb{R}_+$ and $N_0 \in \mathbb{N}$ such that $\mathbb{P}(|\Lambda_n| > M) < \beta$ for all $n \geq N_0$. For any n, we have

$$\begin{aligned}
\left\{\sup_{t\in[0,1]} |\mathrm{B}_n(t) - \Lambda_n\dot{D}_\theta(t)| > \alpha\right\} &= \left\{\sup_{t\in[0,1]} |\mathrm{B}_n(t) - \Lambda_n\dot{D}_\theta(t)| > \alpha\right\} \cap \left\{(|\Lambda_n| \leq M) \cup (|\Lambda_n| > M)\right\} \\
&= \left\{(\sup_{t\in[0,1]} |\mathrm{B}_n(t) - \Lambda_n\dot{D}_\theta(t)| > \alpha) \cap (|\Lambda_n| \leq M)\right\} \\
&\cup \left\{(\sup_{t\in[0,1]} |\mathrm{B}_n(t) - \Lambda_n\dot{D}_\theta(t)| > \alpha) \cap (|\Lambda_n| > M)\right\}.
\end{aligned}$$

It follows that

$$\mathbb{P}\left\{\sup_{t\in[0,1]}|\mathrm{B}_n(t)-\Lambda_n\dot{D}_\theta(t)|>\alpha\right\}=\mathbb{P}\left\{(\sup_{t\in[0,1]}|\mathrm{B}_n(t)-\Lambda_n\dot{D}_\theta(t)|>\alpha)\cap(|\Lambda_n|\leq M)\right\}$$
$$+\,\mathbb{P}\left\{(\sup_{t\in[0,1]}|\mathrm{B}_n(t)-\Lambda_n\dot{D}_\theta(t)|>\alpha)\cap(|\Lambda_n|>M)\right\}.$$

Furthermore

$$\left\{(\sup_{t\in[0,1]}|\mathrm{B}_n(t)-\Lambda_n\dot{D}_\theta(t)|>\alpha)\cap(|\Lambda_n|>M)\right\}\subset(|\Lambda_n|>M),$$

we have

$$\mathbb{P}\left\{(\sup_{t\in[0,1]}|\mathrm{B}_n(t)-\Lambda_n\dot{D}_\theta(t)|>\alpha)\cap(|\Lambda_n|>M)\right\}\leq\mathbb{P}(|\Lambda_n|>M).$$

This implies that

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,1]}|\mathrm{B}_n(t)-\Lambda_n\dot{D}_\theta(t)|>\alpha\right\}\ &\leq\ \mathbb{P}\left\{(\sup_{t\in[0,1]}|\mathrm{B}_n(t)-\Lambda_n\dot{D}_\theta(t)|>\alpha)\cap(|\Lambda_n|\leq M)\right\}\\ &+\ \mathbb{P}(|\Lambda_n|>M)\\ &\leq\ \mathbb{P}\left\{(\sup_{t\in[0,1]}|\mathrm{B}_n(t)-\Lambda_n\dot{D}_\theta(t)|>\alpha)\cap(|\Lambda_n|\leq M)\right\}+\beta,\end{aligned}$$

because of

$$\mathbb{P}(|\Lambda_n|>M)\leq\beta.$$

D_θ being continuous on $[0.1]$ and derivable on (0.1), the mean-value theorem involves that for any realization (Λ_n), there exists θ_n^* between $\hat{\theta}_n$ and θ with $|\theta_n^*-\theta|\leq\Lambda_n/\sqrt{n}$ such that $D_{\hat{\theta}_n}(t)-D_\theta(t)=\dot{D}_{\theta_n^*}(t)(\hat{\theta}_n-\theta)$.

This implies that

$$\mathrm{B}_n(t)=\Lambda_n\dot{D}_{\theta_n^*}(t),$$

because of

$$\mathrm{B}_n(t) = \sqrt{n}(D_{\hat{\theta}_n}(t) - D_\theta(t))$$

and

$$\Lambda_n = \sqrt{n}(\hat{\theta}_n - \theta).$$

By replacing $\mathrm{B}_n(t)$ by $\Lambda_n \dot{D}_{\theta_n^*}(t)$, we have:

$$|\mathrm{B}_n(t) - \Lambda_n \dot{D}_\theta(t)| = |\Lambda_n \dot{D}_{\theta_n^*}(t) - \Lambda_n \dot{D}_\theta(t)| = |\Lambda_n||\dot{D}_{\theta_n^*}(t) - \dot{D}_\theta(t)|.$$

Consider $A_n = \sup_{t\in[0,1]} |\mathrm{B}_n(t) - \Lambda_n \dot{D}_\theta(t)| > \alpha$. Hence using hypothesis $\mathcal{A}_3$, we have

$$\begin{aligned}
\lim_{n\to+\infty} \mathbb{P}\left\{A_n \cap (|\Lambda_n| \le M)\right\} &= \lim_{n\to+\infty} \mathbb{P}\left\{(|\Lambda_n| \sup_{t\in[0,1]} |\dot{D}_{\theta_n^*}(t) - \dot{D}_\theta(t)| > \alpha) \cap (|\Lambda_n| \le M)\right\} \\
&\le \lim_{n\to+\infty} \mathbb{P}\left\{\sup_{|\theta^*-\theta|\le M/\sqrt{n}} \sup_{t\in[0,1]} |\dot{D}_{\theta^*}(t) - \dot{D}_\theta(t)| > \frac{\alpha}{M}\right\} \\
&= 0.
\end{aligned}$$

Consequently, limit of the process B_n is the same as that of $\Lambda_n \dot{D}_\theta$, namely $\Lambda \dot{D}_\theta$, which completes the demonstration.

3.2. Estimation by the method of moments

Since the copula admits no density, moment-based methods can provide a "fast" way to estimate θ, or at least produce an appropriate starting value for the pseudo-likelihood estimate. Of these methods, Kendall's tau inversion is the most popular. In the special case where θ, another common procedure considered by Wang and Wells (2000), consists of estimating θ by $\hat{\theta}_n = \tau_\theta^{-1}(\tau_n)$, where τ_θ is the population version of Kendall's tau of the Durante's copula defined by

$$\tau_\theta = 3 - 4\int_0^1 D_\theta(t)\, dt,$$

and τ_n is an asymptotical unbiased estimator of the population version of Kendall's tau where

$$\tau_n = \frac{4}{n(n-1)} P_n - 1, \quad P_n = \sum_{i<j} \mathbf{1}\left\{(X_i - X_j)(Y_i - Y_j) > 0\right\},$$

with P_n the number of concordant pair in a bivariate sample of size n. Two pairs are concordant if

$$(X_i - X_j)(Y_i - Y_j) > 0$$

Assume that the mapping $\theta \mapsto \tau_\theta$ has a continuous non-vanishing derivative

$$\dot{\tau}_\theta = -4 \int_0^1 \dot{D}_\theta(t)\, dt,$$

in Θ. we have,

$$\tau_n = 3 - 4 \int_0^1 D_n(t)\, dt,$$

one can remark that $\sqrt{n}(\tau_n - \tau_\theta)$ is related to Kendall's process $\delta_{n,\theta}$ through the linear functional

$$\begin{aligned}
\sqrt{n}(\tau_n - \tau_\theta) &= \sqrt{n}\left\{3 - 4\int_0^1 D_n(t)\, dt - 3 + 4\int_0^1 D_\theta(t)\, dt\right\} \\
&= \sqrt{n}\left\{-4\int_0^1 D_n(t)\, dt + 4\int_0^1 D_\theta(t)\, dt\right\} \\
&= -4\int_0^1 \sqrt{n}\{D_n(t) - D_\theta(t)\}\, dt \\
&= -4\int_0^1 \delta_{n,\theta}(t)\, dt.
\end{aligned}$$

Furthermore, a Taylor expansion of order one allows to write

$$\Lambda_n = \sqrt{n}(\hat{\theta}_n - \theta) = \frac{1}{\dot{\tau}_\theta}\sqrt{n}\{\tau_n - \tau_\theta\} + o_{\mathbb{P}}(1)$$

and under hypotheses $\mathcal{A}_1$ and $\mathcal{A}_2$,

$$\Lambda_n = \sqrt{n}(\hat{\theta}_n - \theta) = -\frac{4}{\dot{\tau}_\theta}\int_0^1 \delta_{n,\theta}(t)\, dt + o_{\mathbb{P}}(1),$$

converges in law to

$$\Lambda_\theta = -\frac{4}{\dot{\tau}_\theta}\int_0^1 \delta_\theta(t)\, dt.$$

As in Genest *and al.* (2006), the consequence of the main result is defined as follows.

Theorem 2. *If $\theta \in \Theta \subseteq \mathbb{R}$ is estimated by $\hat{\theta}_n = \tau_\theta^{-1}(\tau_n)$, then under hypotheses $\mathcal{A}_1$, $\mathcal{A}_3$ and $\mathcal{A}_4$, one has the empirical process $\Delta_n = \sqrt{n}(D_n - D_{\hat{\theta}_n})$ which weakly converge to the weak limit Δ_θ, a Gaussian centered process of the form:*

$$\Delta_\theta(t) = \delta_\theta(t) + \frac{4}{\dot{\tau}_\theta}\dot{D}_\theta(t)\int_0^1 \delta_\theta(u)\,du, \quad t \in]0,1]. \tag{6}$$

4. Some computations of Kendall function

4.1. Models of Durante's copulas

This subsection presents some examples of Durante's copulas that check the above hypotheses. However, The verification of hypotheses $\mathcal{A}_2$ and $\mathcal{A}_3$ is deferred to appendix (see 7).

4.1.1. Durante 1

It represents a member of the Fréchet family of copulas.

The generator f_θ of model is $f_\theta(t) = (1-\theta)t+\theta$, $\theta \in [0,1]$ thus giving a bivariate copula, a mixture of the independence copula $C_\Pi(u,v) = uv$ and the Fréchet-Hoeffding upper bound $C_M(u,v) = \min(u,v)$ which is,

$$C_\theta(u,v) = (1-\theta)uv + \theta\min(u,v).$$

Then, its Kendall dependence function is defined as follows:

$$D_\theta(t) = t - t\log(t) + t\log(\Gamma_\theta(t)), \quad t \in]0,1]$$

with $\Gamma_\theta(t) = 4t/\{I_\theta(t)+\theta\}^2$ and $I_\theta(t) = \{\theta^2 + 4t(1-\theta)\}^{1/2}$.

Note that Γ_θ is continuous on $[0,1]^2$ and bounded above by 1. Hence, the density associated with $D_\theta(t)$ is given by $d_\theta(t) = -\log(t) + \log(\Gamma_\theta(t)) + \theta/I_\theta(t)$. We have:

$$\begin{aligned}
d_\theta(t) \times t^{1/2}\log^{1/2+\epsilon}(1/t) &= -t^{1/2}\log^3(t) \times \log^{1/2+\epsilon-2}(1/t) \\
&+ t^{1/2}\log^2(t) \times \log(\Gamma_\theta(t)) \times \log^{1/2+\epsilon-2}(1/t) \\
&+ \theta/I_\theta(t) \times t^{1/2}\log^2(t) \times \log^{1/2+\epsilon-2}(1/t) \\
&= -t^{1/2}\log^3(t) \times \log^{1/2+\epsilon-2}(1/t) \\
&+ t^{1/4}\log^2(t) \times \log^{1/2+\epsilon-2}(1/t)\left\{t^{1/4}\log(4t) - 2t^{1/4}\log(I_\theta(t)+\theta)\right\} \\
&+ \theta/I_\theta(t) \times t^{1/2}\log^2(t) \times \log^{1/2+\epsilon-2}(1/t).
\end{aligned}$$

It follows that

$$\lim_{t\to 0}\left\{d_\theta(t)\times t^{1/2}\log^{1/2+\epsilon}(1/t)\right\}=0,$$

because of

$$\lim_{t\to 0}\left(\theta/I_\theta(t)\right)=1$$

and

$$\lim_{t\to 0}\left(t^{1/4}\log(I_\theta(t)+\theta)\right)=0.$$

Hence, the hypothesis $\mathcal{A}_1$ is verified.

4.1.2. Durante 2

It corresponds to the Cuadras-Augé copula family and its generator is $f_\theta(t)=t^{1-\theta}$ for all $\theta\in[0,1]$. Then, the copula associated is

$$C_\theta(u,v)=uv\left[\max(u,v)\right]^{-\theta}.$$

For

$$0\leq\theta\leq 1,\quad D_\theta(t)=t-2(1-\theta)(2-\theta)^{-1}t\log(t),\quad t\in]0,1],$$

the Kendall dependence function of model. So, its density associated is

$$d_\theta(t)=\theta(2-\theta)^{-1}-2(1-\theta)(2-\theta)^{-1}\log(t).$$

The latter function exists and continuous on $]0,1]\times[0,1]$. For $0<\epsilon\leq 3/2$, one has

$$\begin{aligned}\lim_{t\to 0}\left\{d_\theta(t)\times t^{1/2}\log^{1/2+\epsilon}(1/t)\right\} &= \lim_{t\to 0}\{\theta(2-\theta)^{-1}t^{1/2}\log^2(t)\times\log^{1/2+\epsilon-2}(1/t)\\ &\quad - 2(1-\theta)(2-\theta)^{-1}t^{1/2}\log^3(t)\times\log^{1/2+\epsilon-2}(1/t)\}\\ &= 0.\end{aligned}$$

Hence, the hypothesis $\mathcal{A}_1$ holds for all $\theta\in[0,1]$.

4.1.3. Durante 3

This positively ordered family is generated by the function

$$f_\theta(t) = (1+\theta)t/(\theta t+1),$$

for every $\theta \in [0, +\infty[$ and its copula has the following form:

$$C_\theta(u,v) = (1+\theta)uv/\left[\theta \max(u,v)+1\right]$$

Letting,

$$J_\theta(t) = \{(t\theta)^2 + 4t(1+\theta)\}^{1/2}$$

and

$$\Omega_\theta(t) = \frac{1}{1+\theta} + \frac{\theta^2 t + \theta J_\theta(t)}{2(1+\theta)^2},$$

the Kendall dependence function is

$$D_\theta(t) = t - t\log(t) + t\log(\Omega_\theta(t)).$$

Therefore, the density associated at Durante 3 is:

$$d_\theta(t) = -\log(t) + \log(\Omega_\theta(t)) + \frac{\theta t}{J_\theta(t)}.$$

In order to show that the hypothesis $\mathcal{A}_1$ is well verified. First we have

$$\begin{aligned} t^{1/2}\log^{1/2+\epsilon}(1/t) \times \log(\Omega_\theta)(t) &= t^{1/2}\log^{1/2+\epsilon}(1/t) \times \log\left(\frac{1}{1+\theta} + \frac{\theta^2 t + \theta J_\theta(t)}{2(1+\theta)^2}\right) \\ &= t^{1/2}\log^{1/2+\epsilon}(1/t) \times \left[\log\left(\frac{1}{\theta+1}\right) + \log\left\{1 + \frac{t\theta^2 + \theta J_\theta(t)}{2(\theta+1)}\right\}\right]. \end{aligned}$$

In the neighborhood of 0, we have:

$$\log\left\{1 + \frac{t\theta^2 + \theta J_\theta(t)}{2(\theta+1)}\right\} = \frac{t\theta^2 + \theta J_\theta(t)}{2(\theta+1)} + t\varepsilon(t),$$

with $\lim_{t\to 0}\varepsilon(t) = 0$. Since

Table 1. Generator, Kendall function, Kendall tau for some families of bivariate Durante copulas

Model	$f_\theta(t)$	$D_\theta(t)$	τ_θ	Θ
Durante 1	$(1-\theta)t+\theta$	$t - t\log(4)$ $-2t\log\left(\sqrt{\theta^2+4t(1-\theta)}+\theta\right)$	$\frac{\theta(\theta+2)}{3}$	$[0,1]$
Durante 2	$t^{1-\theta}$	$t-\frac{2(1-\theta)}{2-\theta}t\log(t)$	$\frac{\theta}{2-\theta}$	$(0,1]$
Durante 3	$\frac{(1+\theta)t}{\theta t+1}$	$t-\log(t)+$ $t\log\{\frac{1}{1+\theta}$ $+\frac{\theta^2 t+\theta\sqrt{(t\theta)^2+4t(1+\theta)}}{2(1+\theta)^2}\}$	$1-\frac{12}{\theta^3}$ $-\frac{18}{\theta^2}-\frac{4}{\theta}+$ $\frac{12(1+\theta)^2}{\theta^4}$ $\times\log(\theta^4)$	$(0,\infty)$

$$\lim_{t\to 0}\left\{t^{1/2}\log^{1/2+\epsilon}(1/t)\times\log\left(\frac{1}{\theta+1}\right)\right\}=0$$

and

$$\lim_{t\to 0}\left\{t^{1/2}\log^{1/2+\epsilon}(1/t)\times\left[\frac{t\theta^2+\theta J_\theta(t)}{2(\theta+1)}\right]\right\}=0.$$

Hence

$$\lim_{t\to 0}\left\{t^{1/2}\log^{1/2+\epsilon}(1/t)\times\log(\Omega_\theta)(t)\right\}=0.$$

Moreover, for $0<\epsilon\leq\frac{3}{2}$ and using the Hospital theorem, we have

$$\lim_{t\to 0}\left\{t^{1/2}\log^{1/2+\epsilon}(1/t)\times\frac{\theta t}{J_\theta(t)}\right\}=0.$$

Finally by similar arguments, we have

$$\lim_{t\to 0}\left\{t^{1/2}\log^{1/2+\epsilon}(1/t)\log(t)\right\}=\lim_{t\to 0}\left\{t^{1/2}\log^{3}(t)\log^{1/2+\epsilon-2}(1/t)\right\}=0.$$

It follows that

$$d_\theta(t)=o\{t^{-1/2}\log^{-1/2-\epsilon}(1/t)\},$$

for $\theta>0$ and $0<\epsilon\leq\frac{3}{2}$, which implies that the Kendall process converges asymptotically.

The characteristics of Durante's copula models are summarized in Table 1.

Table 2. Generator, Kendall function, Kendall tau for some families of bivariate Archimedean copulas

Model	$f_\theta(t)$	$D_\theta(t)$	τ_θ	Θ
Clayton	$(t^{-\theta}-1)/\theta$	$t+t(1-t^\theta)/\theta$	$\theta/(\theta+2)$	$(0,\infty)$
Gumbel-Hougaard	$(-\log(t))^\theta$	$t-(t\log(t)/\theta)$	$(\theta-1)/\theta$	$(0,\infty)$
Frank	$\log\left(\frac{1-e^{-\theta}}{1-e^{-t\theta}}\right)$	$t-\frac{1-e^{t\theta}}{\theta}\log\left(\frac{1-e^{-\theta}}{1-e^{-t\theta}}\right)$	$1-\frac{4}{\theta}+\frac{4\mathcal{D}}{\theta}$	$(0,1)$

$$\mathcal{D}=\theta^{-1}\int_0^\theta \frac{x}{e^x-1}\,dx$$

stands for the Debye function.

4.2. Bivariate Archimedean copulas

Let

$$\phi_\theta :]0,1] \to [0,+\infty[$$

a convex, continuous, strictly decreasing function with $\phi(1)=0$. Then the function

$$C:[0,1]^2 \to [0,1]$$

and defined by

$$C(u,v)=\phi_\theta^{-1}(\phi_\theta(u)+\phi_\theta(v)),$$

is a copula, called bivariate Archimedean copula where ϕ_θ^{-1} is a pseudo-inverse of ϕ_θ, the generator. As proved by Genest and Rivest (1993), the ϕ_θ generator can be inferred from the Kendall's dependence function D_θ, because

$$D_\theta(t)=t-\frac{\phi_\theta(t)}{\phi_\theta'(t)},\quad t\in]0,1].$$

In this Table 2, we present three families of bivariate Archimedean copula that satisfy hypothesis $\mathcal{A}_1$. We can find the proof in Barbe *and al.*(1996).

5. Implementation of the goodness-of-fit test

The weak convergence of the statistic Φ_n defined in (3) derives immediately from theorem 1 and also from the continuous applications theorem (see, for example, Van der Vaart and Wellner (1996), Theorem 1.3.6). Specifically, the Cramér-von Mises distance will provide a consistent goodness-of-fit test statistic for families of copulas for which the hypotheses $\mathcal{A}_1$ - $\mathcal{A}_4$ are verified.

An interesting feature of the Cramér-von Mises distance Φ_n is the fact that this fit statistic admits an explicit expression as a finite sum involving the ranks of the observations. In particular, developing the factor

$$\{D_n(t) - D_{\hat{\theta}_n}(t)\}^2$$

of the equation (3), we have

$$\Phi_n = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[1 - D_{\hat{\theta}_n}(\xi_{i,n} \vee \xi_{j,n})\right] - \sum_{i=1}^{n}\left\{1 - D^2_{\hat{\theta}_n}(\xi_{i,n})\right\} + \frac{n}{3}, \tag{7}$$

where $a \vee b = \max(a, b)$ et for more details see (7)

In general, the decision rule for the goodness-of-fit assumptions $\mathcal{H}_0$ and $\mathcal{H}_1$ would consist in rejecting $\mathcal{H}_0$ when the value of the test statistic Φ_n is greater than the order quantile $(1 - \alpha)$ of its distribution under the null hypothesis with α at the nominal level (see, Genest *and al.* (2006)). There are, however, obstacles to the use of such a decision rule dealing mainly, on the one hand, with the form of the distribution and, on the other hand, with its parameter. This test statistic is easily computed when D_θ admits an explicit representation.

However, since the limiting distribution of Φ_n under $\mathcal{H}_0$ depends on the unknown value of the parameter θ and cannot be tabulated, a parametric bootstrap procedure can be used to compute the approximate p-values.

Parametric bootstrap procedure:

The p-values for the test statistic Φ_n based on the goodness-of-fit process Δ_n, are computed generating a large number M of independent samples of size n from $C_{\hat{\theta}_n}$. Then we compute the corresponding values of the statistics $\Phi_{n,i}$, $1 \leq i \leq M$ each time. Its validity derives from the regularity conditions proven by Genest and Rémillard (2008). Therefore, this procedure is as follows:

1. Consider

$$\xi_{i,n} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{1}\left(\frac{R_{j,n}}{n} \leq x, \frac{S_{j,n}}{n} \leq y\right)$$

   with
   $R_{j,n}$ and $S_{j,n}$ respectively the ranks of X_j among $X_1, \cdots, X_n$ and Y_j among $Y_1, \cdots, Y_n$.

   compute

$$D_n(t) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(\xi_{i,n} \leq t), \quad t \in [0, 1]$$

and estimate θ using a rank-based estimator,

$$\hat{\theta}_n = \tau_\theta^{-1}(\tau_n)$$

where

$$\tau_n = \frac{4}{n(n-1)} \sum_{i<j} \mathbf{1}_{\{(X_i - X_j)(Y_i - Y_j) > 0\}} - 1.$$

2. Compute the test statistic Φ_n defined (7):

$$\Phi_n = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[1 - D_{\hat{\theta}_n}(\xi_{i,n} \vee \xi_{j,n})\right] - \sum_{i=1}^{n} \left\{1 - D^2_{\hat{\theta}_n}(\xi_{i,n})\right\} + \frac{n}{3}.$$

3. For some large integer M, repeat the following steps for every $k \in \{1, \cdots, M\}$:

   (a) generate a random sample $(X^\star_{1,k}, Y^\star_{1,k}), \cdots, (X^\star_{n,k}, Y^\star_{n,k})$ from copula $C_{\hat{\theta}_n}$ and compute their associated rank vectors $(R^\star_{1,k}, S^\star_{1,k}), \cdots, (R^\star_{n,k}, S^\star_{n,k})$.

   (b) Compute:

$$\xi^\star_{i,k} = \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{\left(\frac{R^\star_{j,k}}{n} \leq x, \frac{S^\star_{j,k}}{n} \leq y\right)}, \quad x, y \in [0,1]$$

   where $R^\star_{j,k}$ is the rank of $X^\star_{j,k}$ among the sample $X^\star_{1,k}, \cdots, X^\star_{n,k}$ and $S^\star_{j,k}$ is the rank of $Y^*_{j,k}$ among the sample $Y^*_{1,k}, \cdots, Y^\star_{n,k}$.

$$D^\star_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(\xi^\star_{i,k} \leq t)}, \quad t \in [0,1]$$

   and build $\hat{\theta}^\star_{n,k}$ de θ by the same rank-based method as in step 1.

   (c) compute:

$$\Phi^\star_{n,k} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[1 - D_{\hat{\theta}^\star_{n,k}}(\xi^\star_{i,k} \vee \xi^\star_{j,k})\right] - \sum_{i=1}^{n} \left\{1 - D^2_{\hat{\theta}^\star_{n,k}}(\xi^\star_{i,k})\right\} + \frac{n}{3}.$$

4. An approximate p-value for the test is then given by

$$\frac{1}{M} \sum_{k=1}^{M} \mathbf{1}_{(\Phi^\star_{n,k} \geq \Phi_n)}.$$

6. Simulations study

Simulation studies were performed to investigate the finite-samples properties of the proposed goodness-of-fit test for several choices of dependence structures under the null hypothesis and under the alternative hypothesis. Three families of bivariate Durante copula are used under $\mathcal{H}_0$, namely the Durante 1, Durante 2 and Durante 3 copulas. They cover all possible degrees of positive dependence, measured by the Kendall's tau. In addition to the Durante copulas, three families of bivariate Archimedean copulas are also used as an alternative, namely the families of Clayton, Frank and Gumbel-Hougaard, and finally another family of copula belonging neither to the Durante nor the Archimedean family, the families of Plackett. Three levels of dependence as measured by Kendall's tau were considered, namely $\tau \in \{0.3; 0.5; 0.7\}$ as in Mesfioui *and al.* (2009).

In each case, a sample of bivariate dimension and size $n \in \{150; 200; 250\}$ is drawn from the assumed copula under $\mathcal{H}_1$ with the corresponding dependence parameter at τ. The goodness-of-fit test statistic is then computed under the null hypothesis and the p-values are estimated.

This entire procedure is repeated 10000 times to estimate the nominal level, arbitrarily fixed at 5% throughout the study and power of the test under various alternatives. In addition, for each of these 10000 samples, the dependence parameter of the copula model considered was estimated by inverting the Kendall's tau. With the empirical version τ_n of τ_θ, this involved solving for θ in the equation

$$4\int_0^1\int_0^1 C_\theta(u,v)\; dC_\theta(u,v) - 1 = \tau_n.$$

In all considered models, the solution is unique. But for some copula models, such as Frank's, Plackett's and Durante 3, the inversion has to be performed numerically.

Table 3 report the values of the parameter θ used throughout the simulation study according to the three levels of dependence τ_θ.

Tables 4 and 5 presents the level and power of the test and each line of it indicates the percentage of rejection of $\mathcal{H}_0 : C \in C_\theta$.

Analyzing Table 4, we can say that the ability of test to keep the nominal level 5% under the null hypothesis is quite good for the low level of dependence, that is, $\tau = 0.3$. On the other hand, when this level goes from 0.5 to 0.7, the ability is not very appreciable, because the estimated power moves away from the nominal level 5%. When testing Durante 1 model, we note that the percentage of rejection of the null hypothesis is very low, even zero, especially when the data come from the Durante 2 copula. This shows a similarity in their Kendall dependence function which confirms that these copula models belong to the same family, that

Table 3. Kendall's tau according to the parameter θ of the underlying copula

Kendall tau τ_θ	0.3	0.5	0.7
Durante 1	0.38	0.58	0.80
Durante 2	0.46	0.67	0.86
Durante 3	1.16	2.89	9.64
Clayton	0.86	2.00	6.00
Gambel-Hougaard	1.43	2.00	4.00
Frank	2.91	5.74	14.14
Plackett	3.99	11.40	68.47

Table 4. Percentage of rejection of the null hypothesis *Durante 1 copula* under many copula models and three levels de dependance

Families under $\mathcal{H}_1$	τ=0.3			τ=0.5			τ=0.7		
	n=150	n=200	n=250	n=150	n=200	n=250	n=150	n=200	n=250
Durante 1	**2.07**	**5.79**	**5.36**	**0.41**	**7.54**	**34.79**	**10.94**	**58.77**	**10.95**
Durante 2	0.01	0.00	0.01	0.01	0.01	0.00	0.00	0.00	0.00
Durante 3	99.97	100.00	99.99	99.99	100.00	100.00	100.00	100.00	100.00
Clayton	70.74	76.85	95.80	100.00	100.00	100.00	100.00	100.00	100.00
Frank	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Gambel-H	99.97	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Plackett	99.98	100.00	99.98	100.00	100.00	100.00	99.99	100.00	100.00

is the Durante copula family. However, Durante 3 seem to be very different of the model under the null hypothesis with respect to their Kendall function. Given the very high proportion of rejection, this confirms the idea of Genest *and al.* (2009) that the Kendall process is generally inconsistent. In contrast, with the other alternatives, the high power of the test is noted when the sample size increases below the three levels of dependence.

In the light of these results in Table 5, one can notice that the test procedure maintains its nominal level 5%, which is very low or even zero in the different levels of dependence. However, when testing the Durante 2 copula, using data from the Durante 1 copula, the proportions rejecting the null hypothesis are mostly very low. Thus, this result highlights the similarity of these copula in their Kendall function. However, as in Table 4, we note that the behavior of Durante 3 differs from that of the copula under the null hypothesis. Under the alternatives, all the statistics are very powerful. Indeed, the rejection percentages are 100% in almost all situations.

We can conclude from the results in Table 4 and 5 that the goodness-of-fit test seems to have problem maintaining its nominal level under $\mathcal{H}_0$. Indeed, this could probably be explained by the particular form of the Durante models with its singular component. In addition, this test is powerful under a wide variety of copula alternatives and this powers increase with the sample size. A quick look at the rejection percentages compared to Mesfioui *and al.* (2009) leads to the conclusion that the test based on the Kendall process is better than the one based on the Spearman process.

Table 5. Percentage of rejection of the null hypothesis *Durante* 2 *copula* under many copula models and three levels de dependance

Families under $\mathcal{H}_1$	τ=0.3			τ=0.5			τ=0.7		
	n=150	n=200	n=250	n=150	n=200	n=250	n=150	n=200	n=250
Durante 1	3.71	6.12	13.10	28.58	45.54	2.03	9.48	13.08	32.90
Durante 2	**0.02**	**0.00**	**0.01**	**0.00**	**0.00**	**0.01**	**0.00**	**0.00**	**0.00**
Durante 3	99.99	99.95	100.00	99.98	100.00	100.00	99.99	100.00	100.00
Clayton	0.16	4.32	0.97	100.00	100.00	100.00	100.00	100.00	100.00
Frank	99.98	100.00	100.00	99.99	100.00	100.00	100.00	100.00	100.00
Gambel-H	10.00	99.98	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Plackett	100.00	100.00	100.00	100.00	100.00	99.99	10.00	100.00	100.00

7. Appendix

A1 - Determination of the expression of the statistic Φ_n

Proof. We have:

$$\Phi_n = n \int_0^1 \{D_n(t) - D_{\hat{\theta}_n}(t)\}^2 \, dD_{\hat{\theta}_n}(t),$$

where

$$D_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(\xi_{i,n} \leq t)},$$

hence

$$\Phi_n = n \int_0^1 D_n^2(t) d_{\hat{\theta}_n}(t) \, dt - 2n \int_0^1 D_n(t) D_{\hat{\theta}_n}(t) \, d_{\hat{\theta}_n}(t) \, dt + n \int_0^1 D_{\hat{\theta}_n}^2(t) \, d_{\hat{\theta}_n}(t) \, dt.$$

Let be

$$A = n \int_0^1 D_n^2(t) \, d_{\hat{\theta}_n}(t) \, dt, \quad B = n \int_0^1 D_n(t) \, D_{\hat{\theta}_n}(t) \, d_{\hat{\theta}_n}(t) \, dt, \quad C = n \int_0^1 D_{\hat{\theta}_n}^2(t) \, d_{\hat{\theta}_n}(t) \, dt.$$

Compute:

$$\begin{aligned} A &= \frac{1}{n}\int_0^1 \sum_{i=1}^n \mathbf{1}_{(\xi_{i,n}\le t)} \sum_{j=1}^n \mathbf{1}_{(\xi_{j,n}\le t)}\; d_{\hat{\theta}_n}(t)\; dt \\ &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \int_0^1 \mathbf{1}_{\{t\ge \xi_{i,n}\vee\xi_{j,n}\}}\; d_{\hat{\theta}_n}(t)\; dt \\ &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \int_{\xi_{i,n}\vee\xi_{j,n}}^1 d_{\hat{\theta}_n}(t)\; dt \\ &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \Big[1 - D_{\hat{\theta}_n}(\xi_{i,n}\vee\xi_{j,n})\Big] \end{aligned}$$

$$\begin{aligned} B &= n\int_0^1 \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{(\xi_{i,n}\le t)}\; D_{\hat{\theta}_n}(t)\; d_{\hat{\theta}_n}(t)\; dt \\ &= \sum_{i=1}^n \int_{\xi_{i,n}}^1 D_{\hat{\theta}_n}(t)\; d_{\hat{\theta}_n}(t)\; dt \\ &= \frac{1}{2}\sum_{i=1}^n \Big\{1 - D^2_{\hat{\theta}_n}(\xi_{i,n})\Big\} \end{aligned}$$

and

$$C = n\int_0^1 D^2_{\hat{\theta}_n}(t)\; d_{\hat{\theta}_n}(t)\; dt = n\left[\frac{1}{3}D^3_{\hat{\theta}_n}(t)\right]_0^1 = \frac{n}{3}.$$

The expression stated for Φ_n is finally obtained by putting the three terms together.

A2 - Proof of continuity of τ_θ

For $(\theta, t) \in [0, +\infty[\times[0, 1]$, let

$$g_\theta(t) = t f_\theta^2(t),$$

so that for any $\theta \in [0, +\infty[$,

$$\tau_\theta = 4\int_0^1 g_\theta(t)\; dt - 1$$

Let us show that τ_θ is continuous on $[0, +\infty[$.

We have for all $t \in [0;1], t \leq f_\theta(t) \leq 1$ and the function f_θ is derivable on $[0,1]$.

Let the set $\mathcal{G}$ = $\{h_\theta : \theta \longmapsto h_\theta(t)$ such that $h_\theta(t)$ is continuous on Θ, the parameter space$\}$ for all $t \in [0;1]$.

- For every $\theta \in [0;+\infty[$, the function $t \longmapsto g_\theta(t)$ being the product of two piecewise continuous functions, then it is piecewise continuous on $[0;1]$;
- we have $g_\theta \in \mathcal{G}$, then for every $t \in [0;1]$, la function $\theta \longmapsto g_\theta(t)$ is continuous on $[0;+\infty[$;
- for every $(\theta,t) \times [0;+\infty[$, we have $t^3 \leq t f_\theta^2(t) \leq t$, that is $g_\theta(t) \leq t$. Moreover, the function $t \longmapsto t$ is piecewise continuous, positive and integrable on $[0;1]$.

According to the continuity theorem for parameter integrals, the function τ_θ is continuous on $[0,+\infty[$.

Examples of Kendall's tau functions

- Durante 1: $\forall t \in [0;1], f_\theta(t) = (1-\theta)t + \theta$, for all $\theta \in [0;1]$ and this function:

$$\tau_\theta :\ \theta \longmapsto \frac{\theta(\theta+1)}{3}$$

  is continuous on $[0;1]$.

- Durante 2: $\forall t \in [0;1], f_\theta(t) = t^{1-\theta}$, for all $\theta \in]0;1]$ and we have the function:

$$\tau_\theta :\ \theta \longmapsto \frac{\theta}{\theta-2}$$

  is continuous on $]0;1]$.

- Durante 3: $\forall t \in [0,1]$, $f_\theta(t) = (1+\theta)t/(\theta t+1)$, for all $\theta \in]0,+\infty[$ and this Kendall tau

$$\tau_\theta :\ \theta \longmapsto 1 - \frac{12}{\theta^3} - \frac{18}{\theta^2} - \frac{4}{\theta} + \frac{12(1+\theta)^2}{\theta^4}\log(1+\theta)$$

  is continuous on $]0,+\infty[$.

A3 - Verification of Hypothesis $\mathcal{A}_2$ for models of Durante copulas

For the verification of hypothesis $\mathcal{A}_2$, we note that these bivariate copulas, Durante 1, Durante 2 and Durante 3 are mixtures of the independence copula $\Pi(u,v) = uv$ and of the upper frechet-Hoeffding $M(u,v) = \min(u,v)$. As noted by Barbe *and al.*(1996), this hypothesis is automatically satisfied for all copulas whose density function is continuous and strictly positive. We notice that the density function of the different copulas listed in table 1 depends on the derivative $f_\theta^{'}$ of the generator f_θ (see (5)). Since $f_\theta^{'}$ of these Durante copulas is continuous and strictly positive on $(0,1)^2$, then Hypothesis $\mathcal{A}_2$ holds true.

A4 - Verification of Hypothesis $\mathcal{A}_3$ for models of Durante copulas

For Durante 1, one has

$$\dot{D}_\theta(t) = t\frac{\dot{\Gamma}(\theta,t)}{\Gamma(\theta,t)} = -2\frac{t}{I(\theta,t)} + 4\frac{t^2}{I(\theta,t)\{I(\theta,t)+\theta\}} = -2\frac{t}{I(\theta,t)} + \frac{t\Gamma(\theta,t)\{I(\theta,t)+\theta\}}{I(\theta,t)},$$

where

$$\Gamma(\theta,t) = \frac{4t}{\{I(\theta,t)+\theta\}^2}$$

and

$$I(\theta,t) = \{\theta^2 + 4t(1-\theta)\}^{1/2}.$$

Note that for any $(\theta,t) \in [0,1]^2$, $t/I(\theta,t)$ is continuous, hence Hypothesis $\mathcal{A}_3$ is satisfied on $(0,1)$. This condition is also checked at $\theta = 1$, by observing that $\dot{D}_\theta(t) \longrightarrow -t$ as $\theta \longrightarrow 1$. For more details, (see Genest *and al.* (2006), Appendix B5).

Consider the case of Durante 2. We have, for all $\theta \in [0,1]$ and $t \in (0,1]$:

$$\dot{D}_\theta(t) = \frac{2}{(2-\theta)^2} t\log(t).$$

The function $\dot{D}_\theta$ is defined and continuous on $[0,1] \times (0,1]$, because it is a product of continuous functions. Thus the verification of the hypothesis $\mathcal{A}_3$.

For the Durante 3 copula, we have for all $\theta \in [0,+\infty)$,

$$\dot{D}_\theta(t) = \frac{2\theta t - 2(1+\theta) + (1-\theta)J_\theta(t)}{2(1+\theta)\{2(1+\theta)+\theta^2 t + \theta J_\theta(t)\}} + \frac{\theta t(\theta+2)}{2J_\theta(t)\{2(1+\theta)+\theta^2 t + \theta J_\theta(t)\}},$$

the sum of rational functions with non-zero denominators. Furthermore, for all $(\theta, t) \in [0, +\infty) \times [0, 1]$, the function

$$J_\theta(t) = \sqrt{(t\theta)^2 + 4t(1+\theta)}$$

is positive and continuous. Consequently, hypothesis $\mathcal{A}_3$ holds true on $[0, +\infty)$.
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