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The New Odd Lindley-G Power Series Class of Distributions: Theory, Properties and Applications

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Abstract. We propose a new generalized class of distributions called the odd Lindley-G Power Series (*OL-GPS*) family of distributions and a special class, namely, odd Lindley-Weibull power series (*OL-WPS*) family of distributions. We also derive the structural properties of the *OL-GPS* family of distributions including moments, order statistics, Rényi entropy, mean and median deviations, Bonferroni and Lorenz curves, and maximum likelihood estimates. Sub-models of the special cases were also obtained together with their structural properties. A simulation study to examine the consistency of the maximum likelihood estimators for each parameter is presented. Finally, real data examples are presented to illustrate the applicability and usefulness of the proposed model.

Key words: Lindley distribution; Odd Lindley-G distribution; power series distribution; Poisson distribution; geometric distribution; maximum likelihood estimation.
AMS 2010 Mathematics Subject Classification Objects : 60E05, 62E15, 62F30.

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Résumé. (Abstract in French) Nous proposons une nouvelle classe généralisée de distributions appelée la famille de distributions impaires Lindley-G Power Series (*OL-GPS*) et une classe spéciale, à savoir la famille de distributions des séries de puissance impaires Lindley-Weibull (*OL-WPS*). Nous dérivons également les propriétés structurelles de la famille de distributions *OL-GPS*, y compris les moments, les statistiques d'ordre, l'entropie Rényi, les écarts moyens et médians, les courbes de Bonferroni et de Lorenz et les estimations du maximum de vraisemblance. Des sous-modèles des cas particuliers ont également été obtenus avec leurs propriétés structurelles. Une étude de simulation visant à examiner la cohérence des estimateurs du maximum de vraisemblance pour chaque paramètre est présentée. Enfin, des exemples de données réelles sont présentés pour illustrer l'applicabilité et l'utilité du modèle proposé.

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1. Introduction

Increasing demand for extended models in areas of biology, life time analysis, reliability, insurance and economics, has motivated statisticians to work on improving classical models by adding some extra shape and scale parameters on these models. The extra parameters are good in handling skewness and kurtosis. For example, survival time data is generally highly skewed and as such, modelling it using classical distributions may result in over or under-fitting of the data. The generalized or modified distributions provides more flexibility in modelling real life data in areas such as lifetime analysis, reliability, finance and insurance. Several methods for generating new families of distributions have been studied, these include the work by Eugene et al. (2002), Cordeiro et al. (2011), Alexander et al. (2012), Zografos and Balakrishnan (2009), Ristić and Balakrisihnan (2012), Alzaatreh et al. (2013), Alzaghal et al.(2013), Cordeiro et al. (2013), Cordeiro et al. (2014b), Bourguignon et al. (2014b), Gomes-Silva et al. (2017), and Marshall and Olkin (1997), to mention a few.

In real life applications, empirical hazard rate curves often exhibit non-monotonic shapes such as a bathtub, upside-down bathtub (uni-modal) and others. So, increased interest in generating new families of distributions that can provide more flexibility in lifetime modelling. In this paper, we compound the odd Lindley-G distribution and power series distributions to produce a new class of distributions and its sub-models. The new class of distributions is called the odd Lindley-G Power Series (*OL-GPS*) class of distributions and its sub-model odd Lindley-Weibull power series (*OL-WPS*) class of distributions.

Lindley (1958)developed the Lindley distribution by mixing the exponential and length-biased exponential distributions to illustrate the difference between fiducial and posterior distributions. Ghitany et al. (2005) investigated the properties of this distribution. Nadarajah et al. (2011) studied the mathematical and statistical properties of the generalized Lindley (GL) distribution. Sankaran (1970) studied the discrete Poisson-Lindley distribution, Ghitany et al. (2013) developed and studied the power Lindley and associated inference, Oluyede et al. (2015) developed the log-generalized Lindley-Weibull distribution with application, Gomes-Silva et al. (2017) developed the odd Lindley-G family of distributions, Oluyede and Yang (2015) presented the beta generalized Lindley distribution. Zakerzadeh and Dolati (2009) presented and studied another generalization of the Lindley distribution. Chipepa et al. (2019a) studied the beta odd Lindley-G distribution. Chipepa et al. (2019) also developed important and useful results on the Kumaraswamy odd Lindley-G distribution. These generalizations of the Lindley distribution are considered to be useful life distribution models and are suitable for modelling data with different types of hazard rate functions: increasing, decreasing, bathtub and uni-modal.

Similarly, many generalizations of the power series distributions are available in the literature, these include works by Chahkandi and Ganjali (2009), Morais and Barreto-Souza (2011), Mahmoudi and Jafari (2012), Silva et al. (2013), Bidram and Nekoukhou (2013), Silva et al. (2015), Alamatsaz and Harandi (2016), Bourguignon et al. (2014), Mahmoudi and Jafari (2017), and Cordeiro et al. (2014). For compounding continuous distributions with discrete distributions, Nadarajah et al. (2013) introduced the package Compounding in R software (R Development Core Team (2014)).

In this paper, we propose a new distribution, referred to as the odd Lindley-G power series class of distributions and its sub-model odd Lindley-Weibull power series. This paper is organized as follows: In section 2, we present the generalized class of distributions, the corresponding probability density function (*pdf*) and sub-models. Some structural properties including the hazard and reverse hazard functions, quantile function, and various sub-models, moments, moment generating function, conditional moments, mean deviations, Bonferroni and Lorenz curves, the distribution of the order statistics, Rényi entropy and estimates of model parameters are presented. In section 3, the special cases of the odd Lindley-Weibull power series distribution are presented. Monte Carlo simulation study is conducted to examine the consistency of the maximum likelihood estimators for each parameter in section 4. Applications of the proposed model to real data are given in section 5, followed by summary remarks.

2. The Model and Properties

In this section, we derive the new model, the odd Lindley-G power series (*OL-GPS*) and its statistical properties which includes expansion of the density, hazard function, quantile function, moments, mean deviations, Lorenz and Bonferroni curves, order statistics, Rényi entropy and maximum likelihood estimation of model parameters.

Gomes-Silva et al. (2017) proposed the odd Lindley-G (OL-G) distribution whose cumulative distribution function (*cdf*), probability density function (*pdf*) and survival function are given by

$$
\begin{aligned}
F_{OL-G}(x;a,\xi) &= \int_0^{\frac{G(x;\xi)}{1-G(x;\xi)}} \frac{a^2(1+t)}{1+a}\exp\{-at\}dt \\
&= 1-\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}, \qquad (1)
\end{aligned}
$$

$$
f_{OL-G}(x;a,\xi)=\frac{a^2}{(1+a)}\frac{g(x;\xi)}{[\overline{G}(x;\xi)]^3}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}, \qquad (2)
$$

and

$$
\overline{F}(x;a,\xi)=\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}, \qquad (3)
$$

respectively, for $a>0$, where $G(x;\xi)$ and $g(x;\xi)$ is the baseline *cdf* and *pdf*, respectively, and $\overline{G}(x;\xi)=1-G(x;\xi)$ is the survival function. We were motivated by the structural property of this family, which allows for the inclusion of any continuous baseline distribution in the new family and also the traceability of the statistical properties of this new family of distributions. Another important motivation for the *OL-GPS* family of distributions, particularly for use in survival and reliability studies is as follows: Suppose the failure of a device is due to the presence of an unknown number of initial defects of the same kind say N, which is identifiable only after causing failure and are repaired perfectly. Let Y_i, $i=1,...,N$, denote the time to the failure of the device due to the i^{th} defect and assume the Y_i's are independent and identically distributed (iid) OL-G random variables independent of N which is a truncated power series random variable, then the time to the first failure can be modelled by a distribution in the class of *OL-GPS* distributions. The proposed class of distributions can be used for series systems with identical components, which is often the case in many industrial applications and biological organisms.

Now, consider a sequence of N iid random variables, say Y_i, $i=1,\ldots,N$, from the OL-G distribution. Also, let N be a discrete random variable following a power series distribution assumed to be truncated at zero, whose probability mass function (pmf) is given by

$$P(N=n)=\frac{a_n\theta^n}{C(\theta)},\quad n=1,2,\ldots, \tag{4}$$

where $C(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$ is finite, $\theta>0$, and $\{a_n\}_{n\geq 1}$ a sequence of positive real numbers. The power series family of distributions includes binomial, Poisson, geometric and logarithmic distributions (see Johnson et al (1994)).

We mix the OL-G distribution and power series distribution to obtain a new class of distributions, namely, odd Lindley-G power series (*OL-GPS*) distribution. Given N, let $X_1, X_2, ..., X_N$ be identically and independently distributed (iid) random variable following OL-G distribution. Let $X=Y_{(1)}=\min(Y_1,\ldots,Y_N)$. The conditional distribution of X given $N=n$ is given by

$$\begin{aligned}G_{X|N=n}(x) &= 1-\prod_{i=1}^{n}(1-G(x))=1-S^n(x)\\ &= 1-\left[\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right]^n.\end{aligned}$$

Thus, the *cdf* of the life length of the whole system, X, say F_θ, is given by

$$F_\theta(x)=1-\frac{C(\theta S(x))}{C(\theta)}=1-\frac{C\left(\theta\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)\right)}{C(\theta)},$$

that is, the *cdf* of the *OL-GPS* distribution denoted by *OL-GPS*$(a,\theta;\xi)$ is given by the marginal distribution of $X_{(1)}$, that is,

$$\begin{aligned}F_\theta(x) &= \sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)}\left(1-\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)^n\right)\\ &= 1-\frac{C\left(\theta\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)\right)}{C(\theta)}, x>0.\end{aligned} \tag{5}$$

The *pdf* is given by

$$f_\theta(x) = \frac{\theta a^2}{(1+a)}\frac{g(x;\xi)}{\overline{G}^3(x;\xi)}\exp\left(-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)\frac{C'\left(\theta\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)\right)}{C(\theta)}. \tag{6}$$

Table 1. Special Cases of the *OL-GPS*

Distribution	a_n	$C(\theta)$	*cdf*
OL-G Poisson	$(n!)^{-1}$	$e^{\theta}-1$	$1-\frac{\exp\left(\theta\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)\right)-1}{e^{\theta}-1}$
OL-G Geometric	1	$\theta(1-\theta)^{-1}$	$1-\frac{(1-\theta)\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)}{1-\theta\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)}$
OL-G Logarithmic	n^{-1}	$-\log(1-\theta)$	$1-\frac{\log\left(1-\theta\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)\right)}{\log(1-\theta)}$
OL-G Binomial	$\binom{m}{n}$	$(1+\theta)^m-1$	$1-\frac{\left(1+\theta\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)\right)^m-1}{(1+\theta)^m-1}$

Several sub-classes of the *OL-GPS* distribution can be obtained by varying the baseline distribution function $G(x;\xi)$ and the power series distribution.

Remark: Let $C'(\theta)$ be the derivative of $C(\theta)$, that is, $C'(\theta)=\sum_{n=1}^{\infty} na_n\theta^{n-1}$, then the density of F_θ, say f_θ, is given by

$$f_\theta(x)=\frac{dF_\theta(x)}{dx}=\frac{\theta g(x)C'(\theta S(x))}{C(\theta)}.$$

The hazard and reverse hazard rate functions are given by

$$h_\theta(x)=\frac{f_\theta(x)}{S_\theta(x)}=\theta g(x)\frac{C'(\theta S(x))}{C(\theta S(x))},\quad \text{and}\quad \tau_\theta(x)=\frac{f_\theta(x)}{F_\theta(x)}=\theta g(x)\frac{C'(\theta S(x))}{C(\theta)-C(\theta S(x))},$$

respectively, where $S_\theta(x)=1-F_\theta(x)$. Therefore, the hazard function of *OL-GPS* distribution is given in equation (7). That is,

$$h_\theta(x) = \frac{C'\left(\theta\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)\right)}{C\left(\theta\left(\frac{a+\overline{G}(x;\xi)}{(1+a)\overline{G}(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}\right)\right)}\frac{\theta a^2}{(1+a)}\frac{g(x;\xi)}{\overline{G}^3(x;\xi)}\exp\left\{-a\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right\}. \qquad (7)$$

Table 1 shows some special cases of the *OL-GPS* distribution.

2.1. Quantile Function

The quantile function of the *OL-GPS* distribution is obtained by inverting $F_\theta(x) = u$, $0 \le u \le 1$. This is equivalent to solving the equation

$$-\ln C(\theta S(x)) + \ln C(\theta) + \ln(1-u) = 0, \tag{8}$$

which can be expressed as

$$G(x) + \frac{\overline{G}(x)}{a} \ln\left[\frac{(1+a)\overline{G}(x)C^{-1}((1-u)C(\theta))}{\theta(a+\overline{G}(x))}\right] = 0. \tag{9}$$

where C^{-1} is the inverse function. The solution of the non-linear equation (8) gives the quantiles of the *OL-GPS* class of distributions.

2.2. Expansion of Density

In this sub-section, we present the series expansion of the *OL-GPS* distribution. The *OL-GPS* class of distributions can be expresses as an infinite linear combination of exponentiated-G (*Exp-G*) distribution as follows:

$$f_\theta(x) = \sum_{p,q=0}^{\infty} v_{p,q}\, g_{p+q+1}(x;\xi), \tag{10}$$

where $g_{p+q+1}(x;\xi) = (p+q+1)g(x;\xi)[G(x;\xi)]^{p+q}$ is the *Exp-G* density function with power parameter $(p+q+1)$ and

$$v_{p,q} = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{a_n\theta^n}{C(\theta)} \frac{(-1)^q n^{q+1}}{p!q!} \binom{n-1}{k} \frac{a^{1+q+n-k}}{(1+a)^n} \frac{\Gamma(p+2+n+q-k)}{\Gamma(2+n+q-k)} \times \frac{1}{p+q+1}, \tag{11}$$

see appendix for details. Thus, the statistical properties of the *OL-GPS* family of distributions can be obtained directly from those of the *Exp-G* class of distributions.

2.3. Moments

We assume that $W_{p+q+1} \sim$ *Exp-G*$(p+q+1)$ and let $X \sim$ *OL-GPS*$(a, \theta; \xi)$, then the s^{th} moment can be obtained from equation (10) as follows:

$$E(X^s) = \sum_{p,q=0}^{\infty} v_{p,q} E(W_{p+q+1}^s), \tag{12}$$

where $E(W_{p+q+1}^s)$ denotes the s^{th} moment of W_{p+q+1} which follows an *Exp-G* distribution with parameter $(p+q+1)$ and $v_{p,q}$ is as defined in equation (11). Furthermore, the incomplete moments can be obtained as follows

$$I_X(t) = \int_0^t x^s f_\theta(x; a, \theta, \xi) dx = \sum_{p,q=0}^{\infty} v_{p,q} I_{p+q+1}(t),$$

where $I_{p+q+1}(t) = \int_0^t x^s g_{p+q+1}(x; a, \theta, \xi) dx$. The moment generating function (mgf) of X is given by

$$M_X(t) = \sum_{p,q=0}^{\infty} v_{p,q} E(e^{tW_{p+q+1}}),$$

where $E(e^{tW_{p+q+1}})$ is the mgf of the *Exp-G* distribution and $v_{p,q}$ is as defined in equation (11). Furthermore, we can obtain the characteristic function and is given by $\phi(t) = E(e^{itX})$, where $i = \sqrt{-1}$, that is

$$\phi(t) = \sum_{p,q=0}^{\infty} v_{p,q} \phi_{p+q+1}(t),$$

where $\phi_{p+q+1}(t)$ is the characteristic function of *Exp-G* distribution and $v_{p,q}$ is as defined in equation (11).

2.4. Mean Deviation, Lorenz and Bonferroni Curves

Let $X \sim$ *OL-GPS*(a, θ, ξ), the mean deviation about the mean and about the median are defined by

$$\delta_1(x) = \int_0^\infty |x - \mu| \, f_{_\theta}(x; a, \theta, \xi) dx \quad \text{and} \quad \delta_2(x) = \int_0^\infty |x - M| \, f_{_\theta}(x; a, \theta, \xi) dx,$$

respectively, where $\mu = E(X)$ and $M = Median(X)$. The deviations can also be expressed as

$$\delta_1(x) = 2\mu F_{_\theta}(\mu) - 2\int_0^\mu x f_{_\theta}(x; a, \theta, \xi) dx = 2\mu F_{_\theta}(\mu) - 2\sum_{p,q=0}^{\infty} v_{p,q} I_{p+q+1}(t), \tag{13}$$

and

$$\delta_2(x) = \mu - 2\int_0^M x f_\theta(x; a, \theta, \xi)dx = \mu - \sum_{p,q=0}^{\infty} v_{p,q} Ip + q + 1(t). \quad (14)$$

Bonferroni and Lorenz curves are given by

$$B(m) = \frac{1}{m\mu}\int_0^t \sum_{p,q=0}^{\infty} x v_{p,q} g_{p+q+1}(x;\xi)dx = \frac{1}{m\mu}\sum_{p,q=0}^{\infty} v_{p,q} I_{p+q+1}(t), \quad (15)$$

and

$$L(m) = \frac{1}{\mu}\int_0^t \sum_{p,q=0}^{\infty} x v_{p,q} g_{p+q+1}(x;\xi)dx = \frac{1}{\mu}\sum_{p,q=0}^{\infty} v_{p,q} I_{p+q+1}(t), \quad (16)$$

where $I_{p+q+1}(t) = \int_0^t x g_{p+q+1}(x;\xi)dx$, is the first incomplete moment of the *Exp-G* distribution and $v_{p,q}$ is as given in equation (11).

2.5. Order Statistics and Rényi Entropy

In this section, we present the distribution of the order statistic and Rényi entropy for the *OL-GPS* class of distributions.

2.5.1. Order Statistics

Order statistics play an important role in probability and statistics. The *pdf* of the i^{th} order statistic from the *OL-GPS pdf* $f_\theta(x)$ is given by

$$g_{i:n}(x) = \sum_{p,q,h,w,z=0}^{\infty} v^*_{p,q,h,w,z} g_{p+q+h+z+1}(x;\xi), \quad (17)$$

where $g_{p+q+h+z+1}(x;\xi) = (p+q+h+z+1)g(x;\xi)G^{p+q+h+z}(x;\xi)$ is the *Exp-G* distribution with power parameter $(p+q+h+z+1)$ and

$$v^*_{p,q,h,w,z} = \frac{n!}{(i-1)!(n-i)!}\sum_{s=0}^{j+i-1}\sum_{j=0}^{n-i}\sum_{k=0}^{w}\frac{(-1)^{j+s+h}\theta^w w^h a^{h+w-k}}{h!z!C^s(\theta)(1+a)^w} \times \binom{n-i}{j}\binom{j+i-1}{s}\binom{w}{k}\frac{\Gamma(z+h+w-k)}{\Gamma(h+w-k)}\frac{p+q+1}{p+q+h+z+1}d_{w,s}, \quad (18)$$

see appendix for details. It follows that the i^{th} order statistic of the *OL-GPS* series can be expressed as an infinite linear combination of *Exp-G* densities.

2.5.2. Rényi Entropy

Entropy is a measure of variation of uncertainty for a random variable X with *pdf* $f(x)$. There are two popular measures of entropy, namely Shannon entropy and Rényi entropy. Shannon entropy is due to Shannon, C.E. (1951) and Rényi entropy is due to Rényi (1960). Rényi entropy is defined by

$$I_R(\nu) = (1-\nu)^{-1} \log \left[\int_0^\infty f_\theta^\nu(x)dx \right], v \neq 1, v > 0, \tag{19}$$

and Shannon entropy is given by $E\{-\log[f(X)]\}$. Shannon entropy is a special case of Rényi entropy, we therefore, derive expressions for Rényi entropy for the *OL-GPS* distribution.

Therefore, Rényi entropy of the *OL-GPS* class of distributions can be expressed as

$$I_R(\nu) = (1-\nu)^{-1} \log \left[\sum_{p,s,z=0}^{\infty} w^*_{p,s,z} e^{(1-\nu)I_{REG}} \right], \tag{20}$$

where

$$w^*_{p,s,z} = \sum_{k=0}^{s} \frac{(-1)^z \theta^{\nu+s} a^{2\nu+z+s-k} (\nu+s)^z}{C^\nu(\theta)(1+a)^{\nu+s} z! p!} \frac{\Gamma(p+3\nu+s+z-k)}{\Gamma(3\nu+s+z-k)} \binom{s}{k} d_{s,v} \left(\frac{\nu}{p+z} + 1 \right)^\nu,$$

and $I_{REG} = \int_0^\infty \left[\left(\frac{p+z}{\nu}+1\right) g(x;\xi)[G(x;\xi)]^{\frac{p+z}{\nu}} \right]^\nu dx$ is Rényi entropy of *Exp-G* distribution with parameter $\left(\frac{p+z}{\nu}+1\right)$. Details of the derivation are given in the appendix section. It follows that Rényi entropy of *OL-GPS* class of distributions can be derived directly from Rényi entropy of *Exp-G* distribution.

2.6. Maximum Likelihood Estimation

Let $X_i \sim$ *OL-GPS*$(a, \theta; \xi)$ and $\Delta = (a, \theta; \xi)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned} \ell &= \ell(\Delta) = 2n \ln a + n \ln \theta - n \ln(1+a) + \sum_{i=1}^{n} g(x_i;\xi) - 3 \sum_{i=0}^{n} \overline{G}(x_i;\xi) \\ &- a \sum_{i=0}^{n} \frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)} + \sum_{i=1}^{n} \ln \left(C' \left(\frac{a + \overline{G}(x_i;\xi)}{(1+a)\overline{G}(x_i;\xi)} e^{-a \frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}} \right) \right) - n \ln C(\theta). \end{aligned} \tag{21}$$

Elements of the score vector $U = (\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \xi_k})$ are given in the appendix section.

The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ are obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \xi})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $\mathbf{I}(\mathbf{\Delta}) = [\mathbf{I}_{\theta_i,\theta_j}]_{(2+q)\times(2+q)} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i,j = 1,2,...,2+q$ can be numerically obtained by using MATLAB or NLMIXED in SAS or R software. The total Fisher information matrix $n\mathbf{I}(\mathbf{\Delta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Delta}}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j}\bigg|_{\mathbf{\Delta}=\hat{\mathbf{\Delta}}} \right]_{(2+q)\times(2+q)}, \quad i,j = 1,2,...,2+q, \tag{22}$$

where q is the number of components in the vector of parameters ξ. Note that for a given set of observations, the matrix given in equation (22) is obtained after the convergence of the Newton-Raphson procedure via NLMIXED in SAS or R software.

3. Some Special Cases

In this section, we look at some special cases of the *OL-GPS* distribution. These special cases are the odd Lindley-Weibull Poisson (OL-WP) and odd Lindley-Weibull geometric distributions. We derive the statistical properties for the special cases, which include quantile and hazard rate functions, and moments.

3.1. The Odd Lindley-Weibull Power Series Distribution, Sub Models and Some Properties

We further derive the odd Lindley-Weibull power series (*OL-WPS*) distribution, its sub models and statistical properties. We applied the *OL-W* geometric (*OL-WG*) distribution to two real data sets, to illustrate the flexibility of the new class of distributions. The *cdf* and *pdf* of the *OL-WPS* distribution are given by

$$F_{OL-WPS}(x) = 1 - \frac{C\left(\theta\left(\frac{a+e^{-(x/c)^k}}{(1+a)e^{-(x/c)^k}}e^{-a\frac{1-e^{-(x/c)^k}}{e^{-(x/c)^k}}}\right)\right)}{C(\theta)} \tag{23}$$

and

$$f_{OL-WPS}(x) = \frac{\theta a^2 (k/c)(x/c)^{k-1} e^{-(x/c)^k}}{(1+a)e^{-3(x/c)^k}} e^{-a\frac{(1-e^{-(x/c)^k})}{e^{-(x/c)^k}}} \frac{C'\left(\theta\left(\frac{a+e^{-(x/c)^k}}{(1+a)e^{-(x/c)^k}}e^{-a\frac{1-e^{-(x/c)^k}}{e^{-(x/c)^k}}}\right)\right)}{C(\theta)},$$

respectively, for $\theta, a, c, k > 0$.

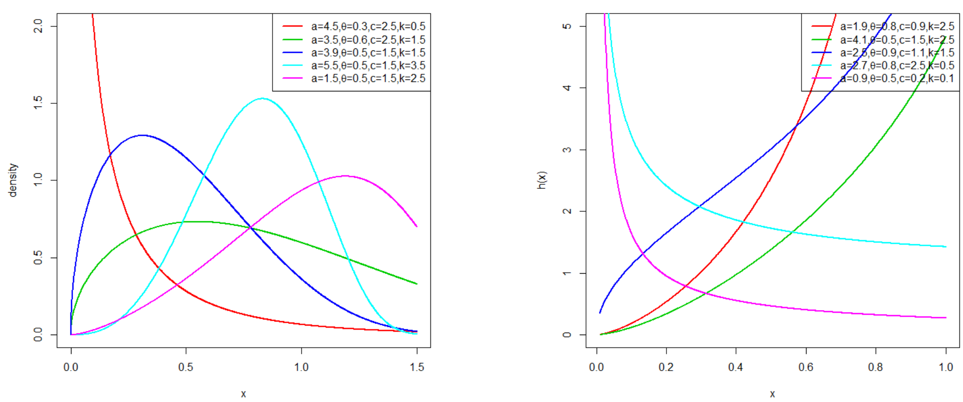

Fig. 1. *pdf* and *hrf* plots for *OL-WP* distribution

3.1.1. Sub Models of *OL-WPS* Distribution

In this section, we consider two sub-models of the *OL-WPS* distribution, namely, OL-W Poisson (OL-WP) and OL-W geometric (*OL-WG*) distributions.

- ***OL-WP* Distribution**

  We present the *OL-WP* distribution and its statistical properties. The *cdf* and *pdf* of the *OL-WP* distribution are given by

$$F_{OL-WP}(x) = 1 - \frac{\exp\left(\theta\left(\frac{a+e^{-(x/c)^k}}{(1+a)e^{-(x/c)^k}} e^{-a\frac{1-e^{-(x/c)^k}}{e^{-(x/c)^k}}}\right)\right) - 1}{(\exp(\theta) - 1)} \tag{24}$$

  and

$$f_{OL-WP}(x) \tag{25}$$
$$= \frac{\theta a^2 (k/c)(x/c)^{k-1} e^{-(x/c)^k}}{(1+a)e^{-3(x/c)^k}} e^{-a\left(\frac{1-e^{-(x/c)^k}}{e^{-(x/c)^k}}\right)} \frac{\exp\left(\theta\left(\frac{a+e^{-(x/c)^k}}{(1+a)e^{-(x/c)^k}} e^{-a\frac{1-e^{-(x/c)^k}}{e^{-(x/c)^k}}}\right)\right)}{(\exp(\theta) - 1)},$$

  respectively, for $\theta, a, c, k > 0$.

Figure 1 shows the plots of *pdf*'s and *hrf*s of *OL-WP* distribution for selected parameter values. The *pdf* can take various shapes including uni-modal, left and right skewed. Graphs of the hazard function exhibit increasing and decreasing shapes for selected values of the parameters.

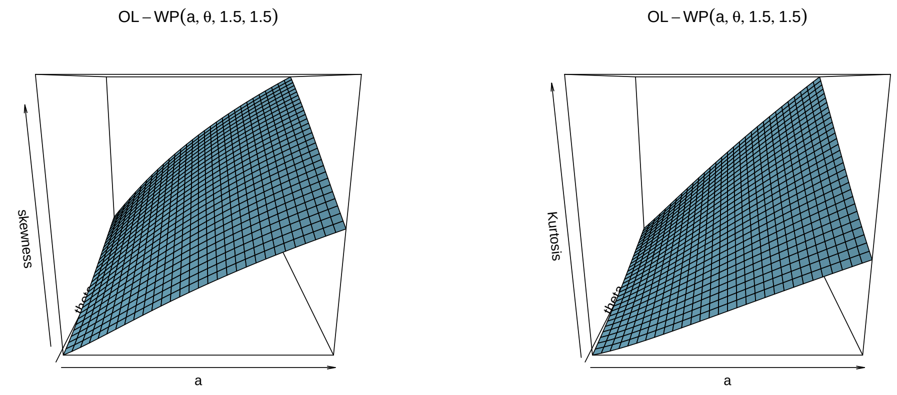

Fig. 2. Plots of skewness and kurtosis for *OL-WP* distribution

- *Quantile Function*

  We obtain the quantile function of the *OL-WP* distribution by solving the non linear equation (26)

$$(e^{-(x/c)^k})\ln\left[\frac{(1+a)e^{-(x/c)^k}\ln[(1-u)(e^{\theta}-1)+1]}{\theta(a+e^{-(x/c)^k})}\right]+a(1-e^{-(x/c)^k})=0. \tag{26}$$

- *Hazard Function*

  The hazard function of the *OL-WP* distribution is given by

$$h_{OL-WP}(x)=\frac{f_{OL-WP}(x)}{1-F_{OL-WP}(x)}, \tag{27}$$

  where $F_{OL-WP}(x)$ and $f_{OL-WP(x)}$ are given in equations (24) and (25) respectively.

- *Moments*

  Let $X\sim OL-WP(a,\theta,c,k)$, the s^{th} moment can be obtained from equation (12) as

$$E(X^s)=\sum_{p,q=0}^{\infty}v_{p,q}E(W_{p+q+1}^s),$$

  where $E(W_{p+q+1}^s)$ denotes the s^{th} moment of W_{p+q+1} which follows an Exp-W distribution with parameter $(p+q+1)$ and $v_{p,q}$ is as defined in equation (11).

  We present in Figures 2 and 3, 3D plots of skewness and kurtosis for the *OL-WP* distribution.

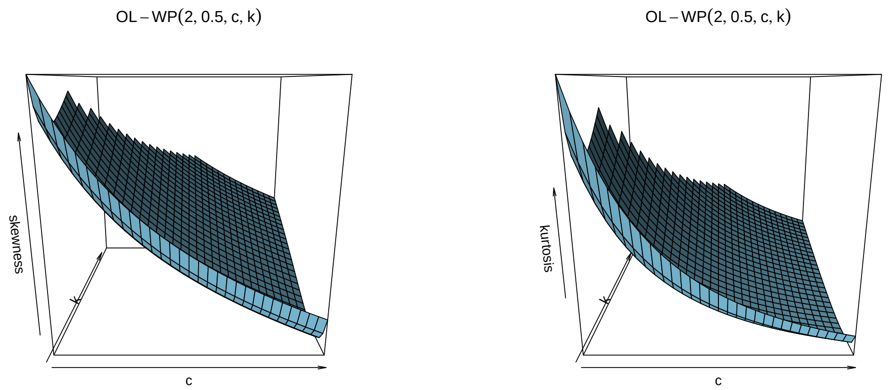

Fig. 3. Plots of skewness and kurtosis for *OL-WP* distribution

- When we fix the parameters c and k, skewness and kurtosis of *OL-WP* distribution increase as the parameters a and θ increase.
- When we fix the parameters a and θ, skewness and kurtosis of *OL-WP* distribution decrease as the parameters c and k increase.

- ***OL-WG*** **Distribution**

The *cdf* and *pdf* of *OL-WG* distribution are given by

$$F_{OL-WG}(x) = 1 - \frac{(1-\theta)\left(\frac{a+e^{-(x/c)^k}}{(1+a)e^{-(x/c)^k}} e^{-a\frac{1-e^{-(x/c)^k}}{e^{-(x/c)^k}}}\right)}{1-\theta\left(\frac{a+e^{-(x/c)^k}}{(1+a)e^{-(x/c)^k}} e^{-a\frac{1-e^{-(x/c)^k}}{e^{-(x/c)^k}}}\right)} \tag{28}$$

and

$$f_{OL-WG}(x) = \frac{\theta a^2 (k/c)(x/c)^{k-1} e^{-(x/c)^k}}{(1+a)e^{-3(x/c)^k}} e^{-a\frac{(1-e^{-(x/c)^k})}{e^{-(x/c)^k}}} \frac{(1-\theta)}{\theta\left(1-\theta\left(\frac{a+e^{-(x/c)^k}}{(1+a)e^{-(x/c)^k}} e^{-a\frac{1-e^{-(x/c)^k}}{e^{-(x/c)^k}}}\right)\right)}, \tag{29}$$

respectively, for $a, c, k > 0$, $0 < \theta < 1$.

Plots of the density and hazard function for selected parameter values are given in Figure 4. The density and hazard functions reveal different shapes for various values of the parameters, as shown in these plots. The graphs of the hazard function exhibit increasing, decreasing and upside down bathtub followed by bathtub

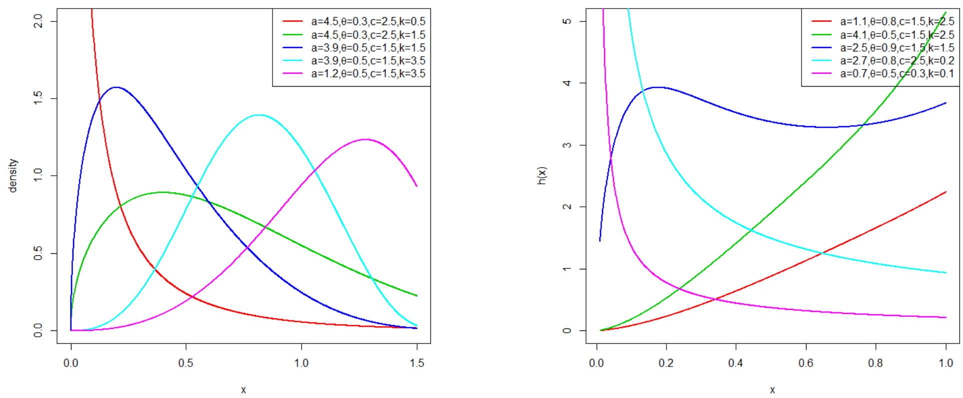

Fig. 4. *pdf* and *hrf* plots for *OL-WG* distribution

shapes for selected values of the parameters. This flexibility makes the *OL-WG* hazard function useful and suitable for non-monotonic empirical hazard behaviours which are likely to be encountered in real life problems.

– *Quantile Function*

Furthermore, we can obtain the quantile function of the *OL-WG* distribution by solving the non linear equation (30)

$$(e^{-(x/c)^k})\ln\left[\frac{(1-u)(1+a)e^{-(x/\lambda)^k}}{(1-\theta)(a+e^{-(x/c)^k})}\left(1-\theta\left(\frac{(a+e^{-(x/c)^k})}{(1+a)e^{-(x/c)^k}}e^{-a\frac{1-e^{-(x/c)^k}}{e^{-(x/c)^k}}}\right)\right)\right]$$
$$+\ a(1-e^{-(x/c)^k})=0. \qquad (30)$$

– *Hazard Function*

The hazard function of the *OL-WG* distribution is given by

$$h_{OL-WG}(x)=\frac{f_{OL-WG}(x)}{1-F_{OL-WG}(x)}, \qquad (31)$$

where $F_{OL-WG}(x)$ and $f_{OL-WG(x)}$ are given in equations (28) and (29) respectively.

– *Moments*

Let $X\sim$ *OL-WG*(a,θ,c,k), the s^{th} moment can be obtained from equation (12) as

$$E(X^s)=\sum_{p,q=0}^{\infty}v_{p,q}E(W^s_{p+q+1}),$$

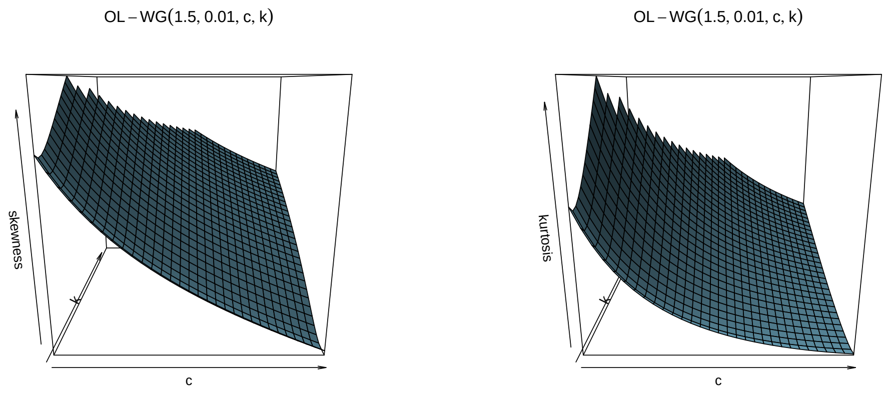

Fig. 5. Plots of skewness and kurtosis for *OL-WG* distribution

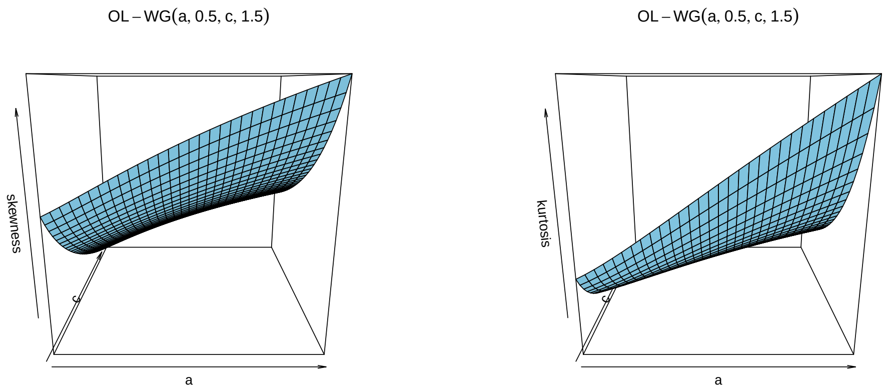

Fig. 6. Plots of skewness and kurtosis for *OL-WG* distribution

3D plots of skewness and kurtosis for the *OL-WG* distribution are shown in Figures 5 and 6

- When we fix the parameters a and θ, skewness and kurtosis of *OL-WG* distribution decrease as the parameters c and k increase.
- When we fix the parameters θ and k, skewness and kurtosis of *OL-WG* distribution increase as the parameters a and c increase.

4. Simulation Study

In this section, we examine the performance of the *OL-WG* distribution by conducting various simulations for different sizes (n = 25, 50, 100, 200, 400, 800, 1000) via the mle package in R software. We simulate 1000 samples for the true parameter values given in the Table 3. Table 3 lists the mean MLEs of the four model parameters along with the respective root mean squared errors (RMSEs). The bias and RMSE are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

Table 2. Monte Carlo Simulation Results for *OL-WP* Distribution: Mean, RMSE and Average Bias

		a=1.5,c=2.5,k=0.5,θ=0.5			a=1.5,c=1.5,k=0.5,θ=0.5		
Parameter	n	Mean	RMSE	Average Bias	Mean	RMSE	Average Bias
	25	7.783579	10.272393	6.283579	8.274155	14.341830	6.774155
	50	4.736499	6.508272	3.236499	4.547003	6.770187	3.047003
	100	3.288397	4.721441	1.788397	3.037911	4.490376	1.537911
a	200	2.739259	3.133344	1.239259	2.248669	2.325205	0.748669
	400	2.289573	2.207100	0.789573	2.116530	1.728799	0.616530
	800	2.095694	1.876525	0.595694	1.869231	1.232147	0.369231
	1000	1.668904	1.088960	0.168904	1.732841	1.165697	0.232841
	25	11.691982	14.156839	9.191982	8.892843	12.928910	7.392843
	50	9.065003	11.413652	6.565003	6.206994	9.501229	4.706994
	100	6.333609	7.894797	3.833609	4.056575	6.329552	2.556575
c	200	4.502284	5.486437	2.002284	2.419534	3.874855	0.919534
	400	3.769070	3.710749	1.269070	1.821268	1.652147	0.321268
	800	3.061289	2.444649	0.561289	1.546996	0.834321	0.046996
	1000	2.913359	1.547843	0.413359	1.502427	0.631997	0.002427
	25	0.605430	0.465537	0.105430	0.688197	0.662477	0.188197
	50	0.659221	0.490990	0.159221	0.685997	0.652259	0.185997
	100	0.603962	0.433812	0.103962	0.612533	0.473966	0.112533
k	200	0.516532	0.350328	0.016532	0.521943	0.418722	0.021943
	400	0.531757	0.278765	0.031757	0.510751	0.362302	0.010751
	800	0.497284	0.232856	-0.002716	0.503048	0.317897	0.003048
	1000	0.530897	0.193017	0.030897	0.520226	0.296641	0.020226
	25	0.772885	1.219599	0.272885	0.780942	1.124962	0.280942
	50	0.685033	1.033124	0.185033	0.748589	1.006121	0.248589
	100	0.758690	1.042655	0.258690	0.723410	0.852227	0.223410
θ	200	0.817472	0.928553	0.317472	0.796958	0.793460	0.296958
	400	0.643159	0.626722	0.143158	0.722928	0.644062	0.222928
	800	0.651751	0.553663	0.151751	0.662675	0.468781	0.162675
	1000	0.543941	0.365118	0.043941	0.618455	0.388563	0.118455

From the results, we can verify that as the sample size n increase, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs and bias decrease for all the parameter values.

5. Applications

To illustrate the flexibility of the *OL-WPS* family of distributions, we present applications of the special case; *OL-WG* distribution to two real data sets. Goodness-of-fit of the model was assessed by the use of the following goodness-of-fit statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC),

Table 3. Monte Carlo Simulation Results for *OL-WP* Distribution: Mean, RMSE and Average Bias

		a=1.5,c=2.5,k=2.5,θ=0.5			a=0.5,c=2.5,k=1.5,θ=0.5		
Parameter	n	Mean	RMSE	Average Bias	Mean	RMSE	Average Bias
	25	3.978856	4.669299	2.478856	7.159171	10.081066	6.659171
	50	3.676939	4.691505	2.176939	5.332959	7.622215	4.832959
	100	2.910398	3.267302	1.410398	3.713096	5.657598	3.213096
a	200	2.621227	2.557964	1.121227	2.665682	3.770198	2.165682
	400	2.295367	1.905574	0.795367	2.059650	2.752570	1.559650
	800	2.060386	1.537827	0.560386	1.607547	1.969328	1.107547
	1000	2.001673	1.643630	0.501673	1.400141	1.748855	0.900141
	25	5.512436	6.403858	3.012436	15.902946	23.424029	13.402945
	50	5.019606	5.455636	2.519606	11.939488	18.355883	9.439488
	100	4.090070	4.104111	1.590070	7.886983	12.953901	5.386983
c	200	3.382790	3.066108	0.882790	5.339687	9.013946	2.839687
	400	2.947418	2.029009	0.447418	3.472341	4.405391	0.972341
	800	2.664078	1.393817	0.164078	2.505293	1.897862	0.005293
	1000	2.591227	1.138175	0.091227	2.476166	1.389003	-0.023834
	25	205.608429	1064.102069	203.108429	173.008961	1205.290199	171.508961
	50	104.986878	853.111401	102.486878	15.407953	67.076549	13.907953
	100	36.571459	122.453736	34.071459	5.546790	19.909882	4.046790
k	200	18.874447	50.690475	16.374447	3.325044	12.696226	1.825044
	400	8.779868	20.859722	6.279868	2.023790	2.421740	0.523790
	800	5.300147	9.128474	2.800147	1.728432	0.906381	0.228432
	1000	4.634789	8.256746	2.134789	1.639579	0.804412	0.139579
	25	1.059129	1.197303	0.559129	1.758597	3.134165	1.258597
	50	0.888354	0.996717	0.388354	1.419508	2.416036	0.919508
	100	0.795832	0.846371	0.295832	1.215797	1.871092	0.715797
θ	200	0.783856	0.738833	0.283856	1.153559	1.564572	0.653559
	400	0.682706	0.563026	0.182706	0.962849	1.170288	0.462849
	800	0.633923	0.430374	0.133923	0.863184	0.855389	0.363184
	1000	0.604903	0.371908	0.104903	0.759425	0.692873	0.259425

Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*), Andersen-Darling (A^*), Kolmogorov-Simirnov (K-S) and sum of squares (SS) from the probability plots.

We used the subroutine *NLMIXED* in SAS as well as the function *nlm* in R to compute the maximum likelihood estimates (MLEs) of the *OL-WG* model parameters. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), and Consistent Akaike Information Criterion ($AICC = AIC + 2\frac{p(p+1)}{n-p-1}$), where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Tables 4 and 5.

We used the likelihood ratio (*LR*) test to compare the fit of the *OL-WG* distribution with it's sub-models for a given data set. For example, to test $\alpha = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{a}, \hat{\theta}, \hat{c}, \hat{k})) - \ln(L(\tilde{a}, \tilde{\theta}, \tilde{c}, 1))]$, where $\hat{a}$, $\hat{\theta}$, $\hat{c}$, and $\hat{k}$ are the unrestricted estimates, and $\tilde{a}$, $\tilde{\theta}$, and $\tilde{c}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 1 degrees of freedom.

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan (1985) are also presented in Tables 4 and 5. These

Table 4. Parameter estimates and goodness of fit statistics for various models fitted for active repair times data set

	Estimates				Statistics							
Model	a	θ	c	k	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	$P-value$
OL-WG	1.2740	0.9893	29.9610	1.6291	183.1	191.1	192.2	197.9	0.0817	0.5713	0.1114	0.7032
	(3.0729)	(0.02934)	(25.8805)	(0.2392)								
OL-WG$(a,\theta,c,1)$	8.3314	0.6343	59.0934	1	190.1	196.1	196.7	201.1	0.1116	0.8417	0.1525	0.3102
	(20.9255)	(0.3800)	(117.45)	-								
OL-WG$(a,\theta,1,k)$	0.02654	0.9933	1	0.4645	191.9	197.9	198.6	203.0	0.1561	1.0647	0.1413	0.4016
	(0.02953)	(0.01424)	-	(0.03832)								
OL-WG$(1,\theta,c,1)$	1	0.9113	15.9131	1	253.5	257.5	257.8	260.9	0.1101	0.8469	0.1559	0.2856
	-	(0.03554)	(4.0165)	-								
	a	b	λ	α								
BGL	0.6328	0.1750	1.6205	2.3077	191.0	199.0	200.1	205.7	0.1406	1.0202	0.1985	0.0856
	(0.6203)	(0.03378)	(0.04227)	(2.9060)								
	a	b	λ	θ								
BOL-U	1.1045	1.0599	6.7740×10^{5}	2.7006×10^{6}	190.9	198.9	200.1	205.7	0.1490	1.0715	0.1577	0.2728
	(0.2302)	(0.2191)	(3.2121×10^{-7})	(8.0572×10^{-8})								
	α	β	λ	γ								
EWP	0.7404	2.3877	5.9609×10^{-8}	0.6644	186.0	194.0	195.1	200.8	0.1020	0.7413	0.1298	0.5102
	(0.3743)	(1.2723)	(0.0304)	(0.1558)								

statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^*, A^*, KS^* and SS the better the fit.

The *OL-WG* distribution was compared to other four parameter non nested models, namely; beta generalized Lindley (BGL) (see Oluyede and Yang (2015)), beta odd Lindley-uniform (BOL-U) by Chipepa et al. (2019a) and eponentiated Weibull-Poisson (EWP) (see Mahmoudi and Sepahdar (2013)) distributions.

5.1. Active Repair Times Data

The first data set are data on active repair times (hours) for an airborne communication transceiver (see Chhikara and Folks (1977)). The data are as follows: 0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.
The variance-covariance matrix is given by

$$\begin{bmatrix} 9.4428 & -0.07894 & 58.7471 & 0.03104 \\ -0.07894 & 0.000861 & -0.2799 & 0.001991 \\ 58.7471 & -0.2799 & 669.80 & 1.1802 \\ 0.03104 & 0.001991 & 1.1802 & 0.05722 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$a \in [-4.9366, 7.4846]$, $\theta \in [0.9300, 1.0486]$, $c \in [-22.3454, 82.2673]$ and $k \in [1.1456, 2.1125]$.

The *LR* test statistic results for the *OL-WG* model for active repair times data are as follows; H_0: *OL-WG* against H_a: *OL-WG*$(a,\theta,c,1)$ are 7.0 (p-value = 0.00815), H_0: *OL-WG* against H_a:*OL-WG*$(a,\theta,1,k)$ 8.8 (p-value = 0.00301) and H_0: *OL-WG* against H_a: *OL-WG*$(1,\theta,c,1)$ 70.4 (p-value < 0.00001). We can conclude that there *OL-WG* distribution is also better than the non-nested BGL, BOL-U and EWP distributions on active repair times data set, based on all the statistics presented in Table 4.

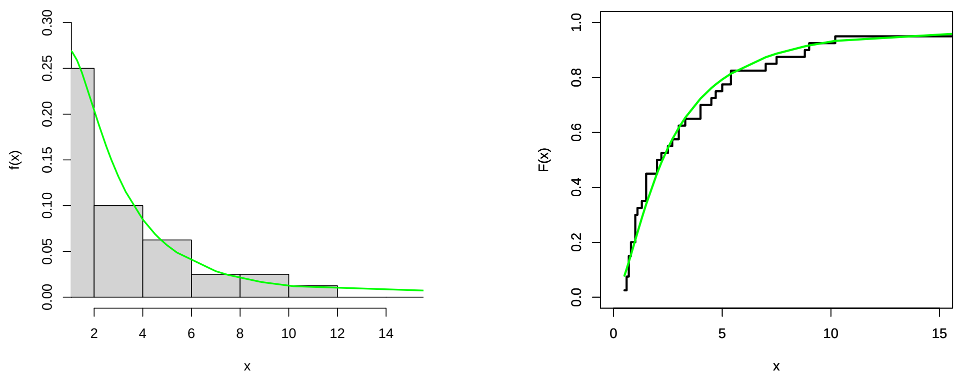

Fig. 7. Fitted *pdf* and *cdf* for repair times data

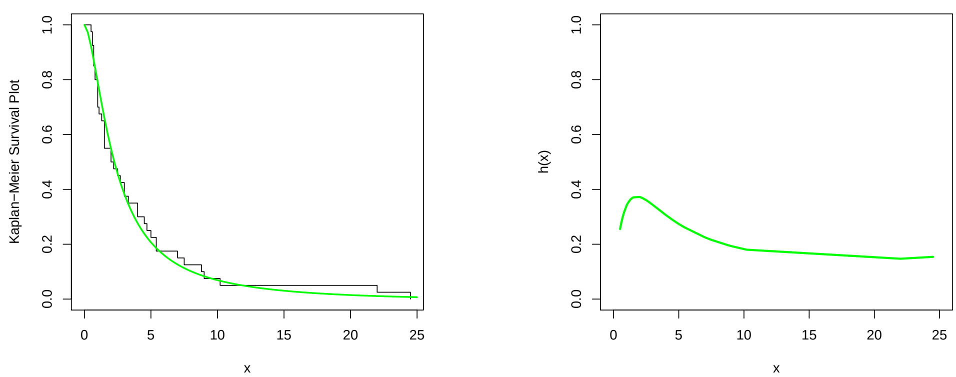

Fig. 8. *KM* and *hrf* plots for repair times data

Figures 7, 8 and 9 show the fitted *pdf* and *cdf*, Kaplan-Meier (*KM*), hazard rate function (*hrf*),and probability and *TTT* scaled plots, with the distribution of the data given in black and our distribution in green. We conclude that our model fit the data set well and applies to non-monotonic hazard rate.

5.2. Run Off Data

The second data set is by Chhikara and Folks (1977). The data represents run off amounts at Jug Bridge, Maryland. The data are shown below: 0.17, 0.19, 0.23, 0.33, 0.39, 0.39, 0.4, 0.45, 0.52, 0.56, 0.59, 0.64, 0.66, 0.7, 0.76, 0.77, 0.78, 0.95, 0.97, 1.02, 1.12, 1.24, 1.59, 1.74, 2.92.

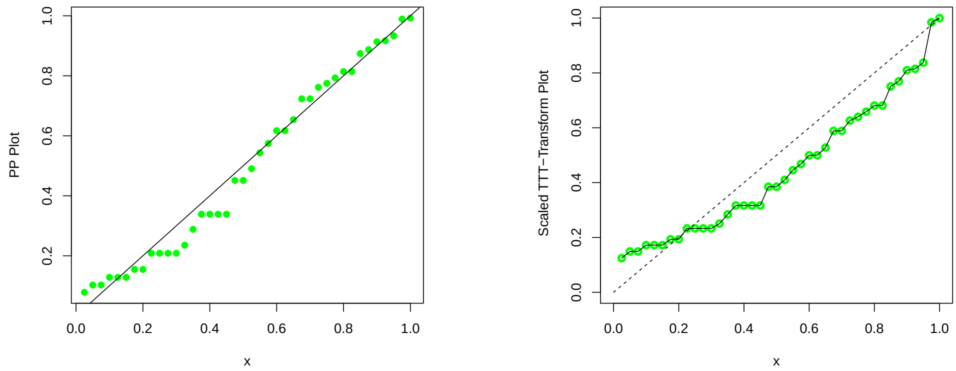

Fig. 9. Probability and *TTT* scaled plots for repair times data

Table 5. Parameter estimates and goodness of fit statistics for various models fitted for run off data

	Estimates				Statistics							
Model	a	θ	c	k	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	$P-value$
OL-WG	0.1810	0.9996	3.3008	2.7095	29.4	37.4	39.4	42.30	0.0147	0.1207	0.0939	0.9800
	(0.2963)	(0.0008)	(2.2377)	(0.5217)								
OL-WG$(a,\theta,c,1)$	0.0272	0.9972	0.6734	1	33.6	39.6	40.7	43.2	0.0706	0.4825	0.1475	0.6485
	(0.0382)	(0.0072)	(0.1537)	-								
OL-WG$(a,\theta,1,k)$	0.04337	0.9974	1	1.2568	33.7	39.7	40.8	43.4	0.0717	0.4912	0.1506	0.6225
	(0.0575)	(0.0067)	-	(0.1821)								
OL-WG$(1,\theta,c,1)$	1	0.2819	1.1559	1	46.7	50.7	51.3	53.2	0.4280	2.5497	0.2491	0.0899
	-	(0.2841)	(0.1269)	-								
	a	b	λ	α								
BGL	1.5175	0.3077	6.56522	4.0216	29.6	37.6	39.6	42.5	0.0219	0.1650	0.1151	0.8949
	(13.0748)	(0.4060)	(7.1424)	(37.9026)								
	a	b	λ	θ								
BOL-U	2.7069	10.2290	1.6768 $\times 10^5$	5.7638 $\times 10^5$	30.8	38.8	40.8	43.7	0.0296	0.2216	0.1082	0.9317
	(0.7246)	(2.9318)	(1.9065 $\times 10^{-4}$)	(5.5463 $\times 10^{-5}$)								
	α	β	λ	γ								
EWP	1.7129	1.7275	4.9547 $\times 10^{-8}$	1.1954	30.6	38.6	40.6	43.5	0.0350	0.2597	0.1131	0.9062
	(0.7232)	(1.1335)	(0.0366)	(0.3782)								

The variance-covariance matrix is given by

$$\begin{bmatrix} 0.08779 & -0.00016 & 0.4535 & 0.07189 \\ -0.00016 & 7.159\times 10^{-7} & 0.000075 & 5.312\times 10^{-6} \\ 0.4535 & 0.000075 & 5.0074 & 0.3737 \\ 0.07189 & 5.312\times 10^{-6} & 0.3737 & 0.2722 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $a \in [-0.4293, 0.7912]$, $\theta \in [0.9978, 1.0013]$, $c \in [-1.3079, 7.9095]$ and $k \in [1.6350, 3.7839]$. The *LR* test statistic results for the *OL-WG* model for run off data are as follows; H_0: *OL-WG* against H_a: *OL-WG*$(a,\theta,c,1)$ are 4.2 (p-value = 0.04042), H_0: *OL-WG* against H_a:*OL-WG*$(a,\theta,1,k)$ 4.3 (p-value = 0.03811), and H_0: *OL-WG* against H_a: *OL-WG*$(1,\theta,c,1)$ 17.3 (p-value = 0.00018). We can conclude that there are significant differences between *OL-WG* distribution and its nested models for run off data.

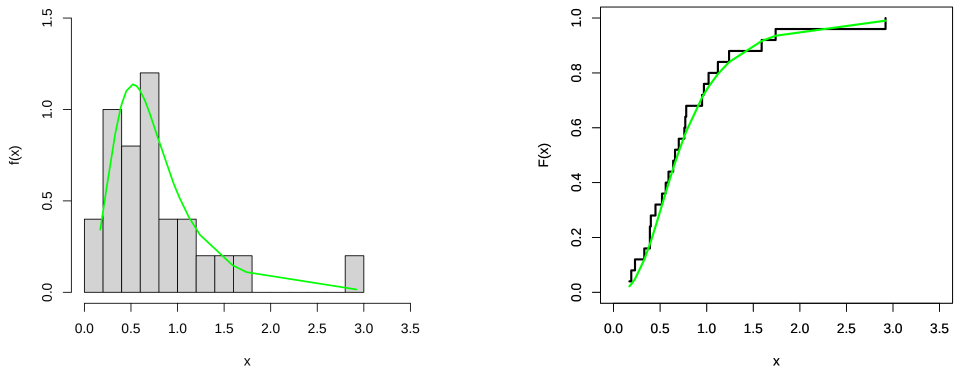

Fig. 10. Fitted *pdf* and *cdf* for run off data

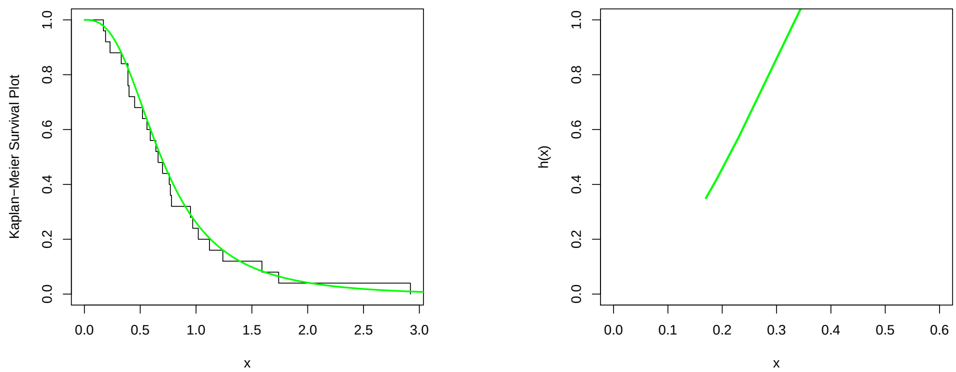

Fig. 11. *KM* and *hrf* plots for run off data

The *OL-WG* distribution is also better than the non-nested BGL, BOL-U and EWP distributions on run off data set, based on all the statistics presented in Table 5.

Figures 10, 11 and 12 show the fitted *pdf* and *cdf*, Kaplan-Meier (*KM*), hazard rate function (*hrf*),and probability and *TTT* scaled plots, with the distribution of the data given in black and our distribution in blue. We conclude that our model fit the data set well and applies to monotonic hazard rate as well.

6. Conclusions

We presented a new class of distributions called the *OL-GPS* distribution and a special case, *OL-WPS* distributions that are suitable for applications in various areas including reliability, survival analysis, income distribution, actuarial sciences just

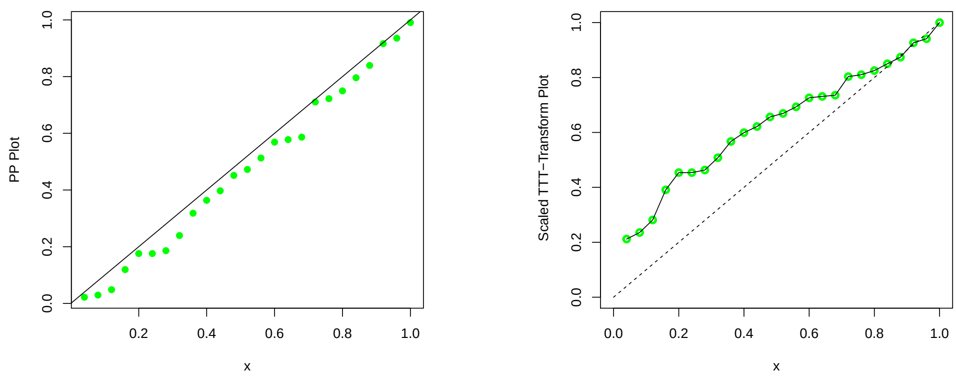

Fig. 12. Probability and *TTT* scaled plots for run off data

to mention a few areas. The proposed distribution and some of its structural properties including hazard and reverse hazard functions, quantile function, moments, conditional moments, mean deviations, Bonferroni and Lorenz curves, Rényi entropy, distribution of order statistics, maximum likelihood estimates, asymptotic confidence intervals are presented. We applied the odd Lindley Weibull Geometric to two data sets, one with non-monotonic hazard rate and the other with monotonic hazard rate in order to illustrate the applicability and usefulness of the proposed class of distributions. The *OL-WG* distribution outperformed the selected models BGL, BOL-U and EWP distributions and its nested models.

Appendix

The following *url* contain the derivations as indicated in the main text: https://drive.google.com/file/d/1b2q8_LS2Mxj0vf4dUoJOXW4d1yaKQjJW/view?usp=sharing

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References

Alamatsaz, M. H. and Shams Harandi, S. (2016). *Generalized Linear Failure Rate Power Series Distribution*. Communications in Statistics - Theory and Methods, 45(8), 2204-2227.

Alexander, C., Cordeiro, G.M., Ortega, E.M.M. and Sarabia, J.M. (2012). Generalized Beta-Generated Distributions. *Computational Statistics and Data Analysis*, 56, 1880-1897.

Alzaatreh, A., Lee, C. and Famoye, F. (2013). A New Method for Generating Families of Continuous Distributions. *Metron*, 71, 63-79.

Alzaghal A., Felix F. and Lee C. (2013). Exponentiated T-X Family of Distributions with Some Applications. *International Journal of Statistics and Probability*, 2, 31-49.

Bourguignon, M., Silva, R. B., and Cordeiro, G. M. (2014). A New Class of Fatigue Life Distributions. *Journal of Statistical Computation and Simulation*, 84(12), 2619-2635.

Bourguignon, M., Silva, R. B. and Cordeiro, G. M. (2014). The Weibull-G Family of Probability Distributions. *Journal of Data Science*, 12, 53-68.

Bidram, H. and Nekoukhou, V. (2013). Double Bounded Kumaraswamy-Power Series Class of Distributions. *Statistics and Operations Research Transactions*, 37(2), 211-230.

Chen, G. and Balakrishnan, N. (1985). A General Purpose Approximate Goodness-of-fit Test. *Journal of Quality Technology*, 27(2), 154-161.

Chhikara, R. S. and Folks, J. L. (1977). The Inverse Gaussian Distribution as a Lifetime Model. *Technometrics*, 19, 461-468.

Chipepa, F., Oluyede, B.O. and Makubate, B. (2019). A New Generalized Family of Odd Lindley-G Distributions with Application. *International Journal of Statistics and Probability*, 8(6). doi:10.5539/ijsp.v8n6p1

Chipepa, F., Oluyede, B.O., Makubate, B. and Fagbamigbe, A.F. (2019). The Beta Odd Lindley-G Family of Distributions with Applications. *Journal of Probability and Statistical Science*, 17(1), 51-84.

Cordeiro, G. M. and Silva, R. B. (2014). The Complementary Extended Weibull Power Series Class of Distributions. *Ciễncia e Natura*, 36(3).

Cordeiro, G. M., Gomes, A., da Silva. C., and Ortega, M. (2013). The Beta Exponentiated Weibull Distribution. *Journal of Statistical Computation and Simulations*, 38(1), 114-138.

Cordeiro, G.M., Ortega, E.M.M., Popović, B.V. and Pescim, R.R. (2014). The Lomax Generator of Distributions: Properties, Minification Process and Regression Model. *Applied Mathematics and Computation*, 247, 465-486.

Cordeiro, G.M. and de Castro, M. (2011). A New Family of Generalized Distributions. *Journal of Statistical Computation and Simulation*, 81:7, 883-898.

Chahkandi, M. and Ganjali, M. (2009). On Some Lifetime Distributions with Decreasing Failure Rate. *Computational Statistics and Data Analysis*, 53(12), 4433-4440.

Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal Distribution and its Applications. *Communications in Statistics-Theory and Methods*, 31, 497-512.

Ghitany, M. E, AL-Hussaini, E. K., and AL-Jarallah. (2005). Marshall-Olkin Extended Weibull Distribution and its Application to Censored Data. *Journal of Applied Statistics*, 32(10), 1025-1034.

Ghitany, M.E., Al-Mutairi, D.K., Balakrishnan, N., Al-Enezi, L.J. (2013). Power Lindley Distribution and Associated Inference. *Computational Statistics and Data Anallysis*, 64, 20-33.

Gomes-Silva, F., Percontini, A., de Brito, E. Ramos, M.W., Venáncio, R. and Cordeiro, G. M. (2017). The Odd Lindley-G Family of Distributions. *Austrian Journal of Statistics*, 46, 65-87.

Gradshetyn, I. S., and Ryzhik, I. M. (2000). Tables of Integrals, Series and Products, sixth edition, Academic Press, San Diego.

Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). Continuous Distributions, Volume 1, John Wiley & Sons, New York, NY.

Lindley, D.V. (1958). Fiducial Distributions and Bayes theorem. *Journal of Royal Statistical Society. Series B (Methodological)*. 20(1), 102-107.

Morais, A. L. and Barreto-Souza, W. (2011). A Compound Class of Weibull and Power Series Distributions. *Computational Statistics and Data Analysis*, 55(3), 1410-1425.

Mahmoudi, E. and Sepahdar, A. (2013). Exponentiated Weibull?Poisson distribution: Model, properties and applications. *Mathematics and Computers in Simulation*, 92, 76-97.

Mahmoudi, E. and Jafari, A. A. (2012). Generalized Exponential-Power Series Distributions. *Computational Statistics and Data Analysis*, 56(12), 4047-4066.

Mahmoudi, E and Jafari, A. A. (2017). The Compound Class of Linear Failure Rate-Power Series Distributions: Model, Properties, and Applications. *Communications in Statistics - Simulation and Computation*, 46(2), 1414-1440.

Marshall, A.W. and Olkin I. (1997). A New Method for Adding a Parameter to a Family of Distributions with Applications to the Exponential and Weibull Families. *Biometrika*, 84, 641-652.

Nadarajah, S., Bakouch, H. S., Tahmasbi, R. (2011). A Generalized Lindley Distribution, *Sankhya B*, 73, 331-359.

Nadarajah, S., Popović, B. V., and Ristić, M. M. (2013). Compounding: An R Package for Computing Continuous Distributions obtained by Compounding a Continuous and a Discrete Distribution. *Computational Statistics*, 28(3), 977-992.

Oluyede, B. O., and Yang, T. (2015). A New Class of Generalized Lindley Distributions with Applications. *Journal of Statistical Computation and Simulation*, 10, 2072-2100, DOI: 10.1080/00949655.2014.917308

Oluyede B.O. Mutiso F., Huang S. (2015). The Log Generalized Lindley-Weibull Distribution with Application. *Journal of Data Science*, 13, 281-310.

R Development Core Team (2014). *A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria.

Silva, R. B., Bourguignon, M., Dias, C. R. B. and Cordeiro, G. M. (2013). The Compound Class of Extended Weibull Power Series Distributions. *Computational Statistics and Data Analysis*, 58, 352-367.

Silva, R. B. and Cordeiro, G. M. (2015). The Burr XII Power Series Distributions: A New Compounding Family. *Brazilian Journal of Probability and Statistics*, 29(3), 565-589.

Rényi, A. (1960). On Measures of Entropy and Information, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, 547 - 561.

Ristić, M.M. and Balakrisihnan, N. (2012). The Gamma Exponentiated Exponential Distribution. *Journal of Computation and Simulation*, 82, 1191-1206.

Sankaran M. (1970). The Discrete Poisson-Lindley Distribution. *Biometrics*, 26, 145-149.

Shannon, C.E. (1951). Predection and Entropy of Printed English. *The Bell System Technical Journal*, 30, 50-64.

Zakerzadeh, H., and Dolati, A. (2009) Generalized Lindley Distribution. *Journal of Mathematical Extension*, 3(2), 13-25.

Zografos, K. and Balakrishnan, N. (2009). On Families of Beta- and Generalized Gamma-Generated Distributions and Associated Inference. *Statistical Methodology*, 6, 344-362.