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Infinite Variance Stable Gegenbaeur Arfisma Models

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Abstract. This paper develops the theory of the Gegenbauer AutoRegressive Fractionally Integrated Seasonal Moving Average (*GARFISMA*) process with α-stable innovations. We establish its conditions for causality and invertibility. This is a finite parameter process which exhibits high variability, long memory, cyclical, and seasonality in financial, hydrological data studies, and more. We perform some simulations to illustrate the behavior of our process.

Key words: Alpha stable distribution; long memory; seasonal process; Gegenbauer polynomials; cyclic time series.
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Résumé. (Abstract in French) Cet article développe la théorie du processus Gegenbauer AutoRegressive Fractionally Integrated Seasonal Moving Average (*GARFISMA*) avec des innovations à distribution α-stable. Nous établissons ses conditions de causalité et d'inversibilité. Il s'agit d'un processus de paramètre fini qui présente une grande variabilité, une mémoire longue, un caractère cyclique et saisonnier dans les études de données financières, hydrologiques, etc. Nous effectuons quelques simulations pour illustrer le comportement de notre processus.

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1. Introduction

The family of α-stable distributions is ubiquitous in statistical: such distributions appear as the limit of normalized sums of independent and identically distributed random variables. Their probability densities exist and are continuous, but they are not known in closed form, except for Gaussian distributions, Cauchy distributions, Lévy distribution. Non-Gaussian stable distributions are a model of choice for real world phenomena exhibiting jumps. Indeed, for $0 \leq \alpha < 2$, their density exhibit "heavy tails", resulting in a power law decay of the probability of extreme events. They have been used extensively in recent years for modeling in domains such that finance. Arthur and David (1996), Arthur and David (2004) and David (2005) argue that the extreme volatility in Hollywood movie revenues can be modeled with α-stable distributions. In Biomedicine, Heuvel et al. (2015) and Heuvel et al. (2018) use α-stable distributions to model proton beams in cancer treatment. In physics, fluctuation flux for plasma in a controlled fusion experiment are modeled by a α-stable distributions Yanushkevichiene and Saenko (2017). Futhermore Bollmann et al. (2017) use α-stable distributions to model network traffic and more. A stable distribution is characterized by four parameters. We write

$$X \sim S_{\alpha}(\gamma, \beta, \mu)$$

to indicate that X has a α-stable distribution with the stability index $\alpha \in]0;2]$, scale parameter $\gamma \geq 0$, skewness $\beta \in [-1;1]$ and location parameter $\mu \in \mathbb{R}$. There are several parametrizations of stable distributions, each of which having advantages and drawbacks. The following one, with characteristic function Φ, is probably the most popular:

$$\Phi(t) = \begin{cases} \exp\left\{-\gamma^{\alpha}|t|^{\alpha}\left(1 - i\beta\left(Sign(t)\right)tan\frac{\pi\alpha}{2}\right) + i\mu t\right\}, & if\, \alpha \neq 1 \\ \exp\left\{-\gamma|t|\left(1 - i\frac{2}{\pi}\beta\left(Sign(t)\right)ln|t|\right) + i\mu t\right\}, & if\, \alpha = 1, \end{cases} \tag{1}$$

where

$$Sign(t) = \begin{cases} 1 \; if\, t > 0, \\ 0 \; if\, t = 0, \\ -1 \; if\, t < 0. \end{cases} \tag{2}$$

We specify that i is complex number such that $i^2 = -1$. The Gaussian case corresponds to $\alpha = 2$ and if $0 \leq \alpha < 2$, the distributions have a heavier tail than Gaussian and do not necessarily have the first and/or second moments. These distributions are symmetric around zero when $\beta = 0$ and $\mu = 0$. Starting from an empirical data-based approach from diverse fields of application such economic, finance, hydrology, telecommunication and more. We are confronted with the phenomenon of long memory. A time series with this property has a slow and hyperbolically declining autocorrelation function (*ACF*) or, equivalently an infinite spectrum at zero frequency Boutahar et al (2007).

A most popular way to analyze a long memory model is to use AutoRegressive Fractionally Integrated Moving Average (*ARFIMA*) processes introduced by Granger et al. (1980) and Hosking et al. (1981). However, the presence of seasonal and cyclic behaviour cannot be caught by the classical *ARFIMA* process. Thus, the methodology for modelling processes with long memory behaviour has been extended to long memory time series with seasonal components. Recent contributions related to the seasonal *ARFIMA* model (hereafter denoted by *ARFISMA* model) are Porter-Hudak (1990) or Reisen et al. (2006). But, these models are very limited, insofar as they consider that the seasonal frequencies are fixed and known. Therefore, recent years have witnessed the publication of several papers dealing with long memory processes able to take into account a possible harmonic component in the data. Gray et al. (1989) proposed a new class of long memory processes, the so-called Gegenbauer AutoRegressive Moving Average (*GARMA*) models, which generalizes the class of Seasonal *ARFIMA* models, insofar as the spectral density of *GARMA* processes is not necessarily unbounded at the origin, like *ARFIMA* models, but anywhere on the interval $[0;\pi]$. Giraitis and Leipus (1995) and, later, Woodward et al. (1998) give an extension of the *GARMA* model, denoted the k-factors *GARMA* model, for which the spectral density is unbounded for a finite number of k frequencies, denoted Gegenbauer frequencies or G-frequencies, on the interval $[0;\pi]$. This k-factor extension was first suggested in the concluding works of Gray et al. (1989) and is used by Sadek and Khotanzad (2004) in a network traffic simulation. Note that Hassler and Uwe (1994) and Marius and Ooms (1995) proposed two different seasonal long memory models, which are in fact special cases of the k-factor *GARMA* model, insofar as the G-frequencies are the seasonal frequencies, which are known. They assume in their theory that the innovations are Gaussian. However, we realize that this hypothesis is too restrictive, particularly, in some domains such as finance or telecommunication in which one must take into account a high variability of the data which is translated by infinite variance. We propose to replace the Gaussian innovations by the α-stable innovations so the process obtained and called Gegenbauer *ARFISMA*-SαS process allow the modelling of long memory data exhibiting seasonal period, cyclical fluctuation and high variability components. The remainder of this paper is structured as follows. Section 2, presented the class of Gegenbauer *ARFISMA*-SαS model. In Section 3, we give conditions for causality of the Gegenbauer *ARFISMA*-SαS process. Section 4 is dedicated to invertibility of our model. The outline of the simulation study and the results are given in Section 5.

2. Gegenbauer *ARFISMA*-SαS process

Hosking et al. (1981), Porter-Hudak (1990) and Ray and Bonnie (1993) among others, proposed to use the fractional seasonal difference operator, $(1-B^s)^d$, where d is the fractionally differenced component and lies inside the interval $]-0.5;0.5[$. The parameter s is the seasonal period that is the number of observations per period ($s=1$ for annual data, $s=2$ for half-yearly data, $s=4$ for quarterly data, $s=12$ for monthly data, $s=52$ for weekly data) and B denotes the back-shift operator such that $B^sX_t = X_{t-s}$. In what follows, s is assumed to be even, which is a restriction only of the presentation. The fractional seasonal difference operator may be expanded as a binomial series, the same is true for the inverse filter:

$$(1-B^s)^d = \sum_{k=0}^{\infty} b_k B^{ks}; \tag{3}$$

$$(1-B^s)^{-d} = \sum_{k=0}^{\infty} u_k B^{ks}. \tag{4}$$

The asymptotic behaviour of the coefficients is given by

$$b_k = \frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(-d)} \sim \frac{k^{-d-1}}{\Gamma(-d)}; \quad u_k \sim \frac{k^{-d-1}}{\Gamma(d)}, k \to \infty. \tag{5}$$

Let δ_k be the unit roots in $1-\delta^s=0$, for all $\delta \in \mathbb{C}^\star$, that is,

$$\delta_k = e^{i\frac{2\pi k}{s}}, \quad k = 0, 1, \ldots, s-1. \tag{6}$$

By using the results in Reisen et al. (2006) and Hassler and Uwe (1994)

$$
\begin{aligned}
(1-B^s)^d &= (1-B)^d (1+B)^d \prod_{k=1}^{\left[\frac{s}{2}\right]-1} (1-\delta_k B)^d (1-\delta_{-k} B)^d \\
&= (1-B)^d (1+B)^d \prod_{k=1}^{\left[\frac{s}{2}\right]-1} \left\{1 - 2\cos\left(\frac{2\pi k}{s}\right) B + B^2\right\}^d, \qquad (7)
\end{aligned}
$$

where d denotes long memory parameter and $2\pi k/s$, $k = 1, \ldots, [s/2]$ are seasonal frequencies.

The operator (7) is called rigid filter because the contribution of seasonal oscillations and of the long-run behaviour to the variance are governed by one common long memory parameter d. Several economic time series can be adequately described by this filter, see for example Caporale et al. (2006) for an application on US money stock. In order to allow for different long memory parameters across different seasonal frequencies, Chan and Wei (1988), Hassler and Uwe (1994), Marius and Ooms (1995), Ferrara and Guégan (2000), Arteche et al. (2000) proposed the Seasonal and Cyclical Long Memory (*SCLM*) operator given by:

$$
\begin{aligned}
\Delta^{\tau(s)}(B) &= (1-B)^{d_0} (1+B)^{d_{\left[\frac{s}{2}\right]}} \prod_{k=1}^{\left[\frac{s}{2}\right]-1} \left\{\left(1-e^{i\lambda_{k,j}}B\right)\left(1-e^{-i\lambda_{k,j}}B\right)\right\}^{d_k} \\
&= (1-B)^{d_0} (1+B)^{\left[\frac{s}{2}\right]} \prod_{k=1}^{\left[\frac{s}{2}\right]-1} \left(1 - 2\nu_{k,j}B + B^2\right)^{d_k}, \quad with\ \tau(s) = \left(d_0, \ldots, d_{\left[\frac{s}{2}\right]}\right). \qquad (8)
\end{aligned}
$$

The long memory parameter

$$(d_k)_{k\in\left\{0,\ldots,\left[\frac{s}{2}\right]\right\}}$$

are in $]-0,5;0,5[$. The frequencies

$$\lambda_{k,j} = \cos^{-1}(\nu_{k,j})$$

are such that

$$\lambda_{k,j} = \frac{2\pi k}{s} + \frac{2\pi j}{n},\ for\ all\ \ (k,j) \in \left\{0, \ldots, \left[\frac{s}{2}\right]\right\} \times \{0, \ldots, m\},$$

where the bandwidth m is an integer between 1 and $n/2$, and in pratrice, is less than n, and for asymptomatic theory, it satisfies at least

$$\frac{1}{m} + \frac{m}{n} \to 0, \text{ as } n \to +\infty. \tag{9}$$

Condition (9) implies that the seasonal frequencies

$$\left(\frac{2\pi k}{s}\right)_{k\in\left\{0,\ldots,\left[\frac{s}{2}\right]\right\}}$$

belong to the set of harmonic frequencies $2\pi j/n$ where $j \in \{0, \ldots, m\}$. We are especially interested in the harmonic frequencies around the seasonal frequencies, for $j \in \{0, \ldots, m\}$,

$$\lambda_{0,j} = \frac{2\pi j}{n}, \quad \lambda_{1,j} = \frac{2\pi}{s} + \frac{2\pi j}{n} \quad \ldots, \quad \lambda_{\left[\frac{s}{2}\right],j} = \pi - \frac{2\pi j}{n}, \quad \lim_{n\to\infty} \lambda_{k,j} = \frac{2\pi k}{s}.$$

The Seasonal and Cyclical Long Memory (*SCLM*) filter is defined by convolution. Any of the factors in (8) may be expanded as a binomial series given by Anděl et al. (1986):

$$\left(1 - e^{i\lambda_{k,j}} B\right)^d = \sum_{l=0}^{\infty} b_l e^{i\lambda_{k,j} l} B^l \tag{10}$$

where b_l is just the coefficient in (3). This expansion is real-valued for $\lambda_{k,j} = 0, \pi$. If $0 < \lambda_{k,j} < \pi$, the real-valued expansion is given by Katayama and Naoya (2000):

$$\Delta^{-\tau(s)}(B) = \prod_{i=0}^{\left[\frac{s}{2}\right]} \left(1 - 2\nu_{i,j} B + B^2\right)^{-\omega_i} = \sum_{k=0}^{+\infty} \pi_k(\omega_i, \lambda) B^k. \tag{11}$$

The long memory parameters

$$(\omega_i)_{i\in\left\{0,\ldots,\left[\frac{s}{2}\right]\right\}}$$

are such that

$$\omega_0 = \frac{d_0}{2}, \quad \omega_{\left[\frac{s}{2}\right]} = \frac{d_{\left[\frac{s}{2}\right]}}{2}, \quad \omega_i = d_{i\in\left\{1,2,\ldots,\left[\frac{s}{2}\right]-1\right\}}.$$

For now on, we set

$$\omega := \omega_{i\in\left\{0,\ldots,\left[\frac{s}{2}\right]\right\}}$$

and real coefficients

$$\left(\pi_k\left(\omega,\lambda\right)\right)_{k\in\mathbb{N}}$$

in (11) are such that:

$$\begin{cases}\pi_0\left(\omega,\lambda\right)=1,\\ \pi_k\left(\omega,\lambda\right)=\dfrac{2}{k}\sum_{l=0}^{k-1}\sum_{i=1}^{\left[\frac{s}{2}\right]}\omega_i cos\left[(k-l)\,\nu_{i,j}\right]\pi_l\left(\omega,\lambda\right),\ \text{for all } k\geq 1.\end{cases} \tag{12}$$

Definition 1. *Let $(\epsilon_t)_{t\in\mathbb{Z}}$ be sequence of independent and identically distributed i.i.d symmetric $\alpha-$stable (SαS) random variables. $(X_t)_{t\in\mathbb{Z}}$ is said to be the Gegenbauer ARFISMA-SαS process, if it is the unique solution to the following equation:*

$$\Phi\left(B\right)\Phi_s\left(B^s\right)\Delta^{\tau(s)}\left(B\right)X_t=\Theta\left(B\right)\Theta_s\left(B^s\right)\epsilon_t, \tag{13}$$

where B is the backward operator and s is the seasonal parameter. The Seasonal-Cyclical Long Memory (SCLM) operator $\Delta^{\tau(s)}\left(B\right)$ is given in (8). $\Phi\left(.\right)$ and $\Theta\left(.\right)$ are the well known non-seasonal p-order autoregressive and q-order moving average polynomials with real coefficients defined by:

$$\begin{cases}\Phi\left(B\right)=1-\sum_{j=1}^{p}\phi_j B^j,\\ \Theta\left(B\right)=1+\sum_{j=1}^{q}\theta_j B^j.\end{cases} \tag{14}$$

The polynomials $\Phi_s\left(.\right)$ and $\Theta_s\left(.\right)$ are respectively the seasonal P-order autoregressive polynomial and the seasonal Q-order moving average polynomial which are defined by:

$$\begin{cases}\Phi_s\left(B^s\right)=1-\sum_{j=1}^{P}\Phi_{js}B^{js},\\ \Theta_s\left(B^s\right)=1+\sum_{j=1}^{Q}\Theta_{js}B^{js}.\end{cases} \tag{15}$$

Remark 1. The Gegenbauer *ARFISMA*-SαS model enables the modeling of many features of financial market returns. Since it is a direct generalization of the *ARFISMA*-SαS model of Ndongo et al. (2016), it contains several extensions:

- ▷ The linear processes with infinite variance studied by Fama and Eugene (1965) , Stuck and Kleiner (1974) or Brockwell et al. (2006) when $P=Q=0$ and $d_k=0$ for all $k\in\{0,1,2,\ldots,[s/2]\}$.
- ▷ The *ARMA*-SαS, if $P=Q=0$, $d_k=0$, $\forall k\in\{0,\ldots,[s/2]\}$, Mikosch et al. (1995).
- ▷ The Fractional *ARIMA*-SαS, if $P=Q=0$, $d_0\in\mathbb{R}$, $d_k=0$, for all $k\in\{1,2,\ldots,[s/2]\}$, introduced by Kokoszka et al. (1995).

3. Causality of the Gegenbauer *ARFISMA*-SαS process

Following equation (11), the Gegenbauer *ARFISMA*-SαS process defined in (13) can be rewritten as

$$\Phi(B)\Phi_s(B^s)X_t = \Theta(B)\Theta_s(B^s)\prod_{i=0}^{\left[\frac{s}{2}\right]}\left(1-2\nu_{i,j}B+B^2\right)^{-\omega_i}\epsilon_t, \tag{16}$$

$$= \Theta(B)\Theta_s(B^s)\sum_{k=0}^{+\infty}\pi_k(\omega,\nu)B^k\epsilon_t, \tag{17}$$

$$= \Theta(B)\Theta_s(B^s)\sum_{k=0}^{+\infty}\pi_k(\omega,\lambda)\epsilon_{t-k}. \tag{18}$$

Let the polynomials $\Phi(.)$ and $\Phi_s(.)$ be such that all its roots lie outside the unit circle. To show that Gegenbauer *ARFISMA-S*αS time series in (11) has a unique causal moving average solution, we need to define cyclical seasonal coefficients $(c_k)_{k\in\mathbb{N}}$ by the following equation:

$$\frac{\Theta(B)\Theta_s(B^s)}{\Phi(B)\Phi_s(B^s)}\Delta^{-\tau(s)}(B) = \frac{\Theta(B)\Theta_s(B^s)}{\Phi(B)\Phi_s(B^s)}\times\sum_{k=0}^{+\infty}\pi_k(\omega,\nu)B^k = \sum_{k=0}^{+\infty}c_kB^k. \tag{19}$$

$$\Phi_s(B^s)\Phi(B)\sum_{k=0}^{+\infty}c_kB^k = \Theta_s(B^s)\Theta(B)\sum_{k=0}^{+\infty}\pi_k(\omega,\nu)B^k. \tag{20}$$

We rate

$$\mathcal{A}(B) = \Phi_s(B^s)\Phi(B)\sum_{k=0}^{+\infty}c_kB^k, \tag{21}$$

$$\mathcal{B}(B) = \Theta_s(B^s)\Theta(B)\sum_{k=0}^{+\infty}\pi_k(\omega,\nu)B^k. \tag{22}$$

Lemma 1. *Let* $(c_k)_{k\in\mathbb{N}}$ *be a sequence of real numbers. If* $(c_k)_{k\in\mathbb{N}}$ *are cyclical seasonal coefficients of* s *period, then for all* $z\in\mathbb{C}$, *with* $|z|<1$

$$\sum_{k=0}^{+\infty}c_kz^k = \varrho_s\sum_{j=0}^{s-1}c_jz^j, \tag{23}$$

where $\varrho_s\in\mathbb{C}-\{0\}$ *and* $|\varrho_s|<1$.

*Proof (**Proof of Lemma** 1).* For all $z \in \mathbb{C}$ such that $|z| < 1$,

$$\sum_{k=0}^{+\infty} c_k z^k = \sum_{j=0}^{s-1} c_j z^j + c_0 \sum_{k=1}^{+\infty} z^{ks} + c_1 z \sum_{k=1}^{\infty} z^{ks} + c_2 z^2 \sum_{k=1}^{+\infty} z^{ks} + \ldots + c_{s-1} z^{s-1} \sum_{k=1}^{+\infty} z^{ks} \tag{24}$$

$$= \sum_{j=0}^{s-1} c_j z^j + \sum_{k=1}^{+\infty} z^{ks} \left(c_0 + c_1 z + c_2 z^2 + \ldots + c_{s-1} z^{s-1}\right) \tag{25}$$

$$= \varrho_s \sum_{j=0}^{s-1} c_j z^j \text{ and } \varrho_s = \left(1 + \sum_{k=1}^{+\infty} z^{ks}\right). \tag{26}$$

Since, the geometric series $1 + z^s + z^{2s} + z^{3s} + z^{4s} + \ldots$ converges if and only if $|z| < 1$, then

$$\varrho_s = \sum_{k=0}^{+\infty} z^{ks} = \frac{1}{1 - z^s}, \ |z| < 1. \tag{27}$$

This completes the proof of Lemma 1.

According equation (20) expanding and by using polynomial identification, we obtain with Lemma 1 an explicit formula for the coefficients $(c_k)_{k \in \mathbb{N}}$.

▷ For $k \in \{0, \ldots, s-1\} \cup \{(Q+1)s, \ldots\}$,

$$c_k - \sum_{i=1}^{min(k,p)} \phi_i c_{k-i} = \pi_k(\omega, \nu) + \sum_{i=1}^{min(k,q)} \theta_i \pi_{k-i}(\omega, \nu). \tag{28}$$

▷ For $k \in \{s, \ldots, (P+1)s - 1\}$,

$$c_k - \sum_{i=1}^{min(k,p)} \phi_i c_{k-i} + \tilde{\gamma}_k - \tilde{\alpha}_k = \pi_k(\omega, \nu) + \sum_{i=1}^{min(k,q)} \theta_i \pi_{k-i}(\omega, \nu) + \tilde{\zeta}_k + \tilde{\upsilon}_k. \tag{29}$$

▷ For $k \in \{(P+1)s, \ldots, (Q+1)s - 1\}$,

$$c_k - \sum_{i=1}^{min(k,p)} \phi_i c_{k-i} = \pi_k(\omega, \nu) + \sum_{i=1}^{min(k,q)} \theta_i \pi_{k-i}(\omega, \nu) + \tilde{\zeta}_k + \tilde{\upsilon}_k, \tag{30}$$

and

(a) for $1 \leq i \leq P$,

$$\begin{cases} \tilde{\alpha}_{is+j} = \sum_{k=1}^{i} \Phi_{ks} c_{(i-k)s+j}, \\ \tilde{\gamma}_{is+j} = \Phi_{is} \sum_{k=1}^{j} \phi_k c_{j-k} + \left(\sum_{l=1}^{i-1} \Phi_{ls} \right) \sum_{k=1}^{p} \phi_k c_{(i-l)s+j-k}, \end{cases} \tag{31}$$

(b) for $1 \leq i \leq Q$,

$$\begin{cases} \tilde{\zeta}_{is+j} = \sum_{k=1}^{i} \Theta_{ks} \pi_{(i-k)s+j}(\omega, \nu), \\ \tilde{v}_{is+j} = \Theta_{is} \sum_{k=1}^{j} \theta_k \pi_{j-k}(\omega, \nu) + \sum_{l=1}^{i-1} \Theta_{ls} \sum_{k=1}^{q} \theta_k \pi_{(i-l)s+j-k}(\omega, \nu). \end{cases} \tag{32}$$

Let consider the following assumptions:

(A_1) : For all $(k, j) \in \{0, 1, \ldots, [s/2]\} \times \{1, \ldots, m\}$, where m is finite and $1 < \alpha \leq 2$,

$$\omega_k < \begin{cases} 1 - \frac{1}{\alpha}, \text{ if } |\nu_{k,j}| \neq 1 \\ \frac{1}{2}\left(1 - \frac{1}{\alpha}\right), \text{ if } |\nu_{k,j}| = 1. \end{cases} \tag{33}$$

(A_2) : For all $(k, j) \in \{0, 1, \ldots, [s/2]\} \times \{1, \ldots, m\}$, where m is an integer and $1 < \alpha \leq 2$,

$$\omega_k > \begin{cases} -1 + \frac{1}{\alpha}, \text{ if } |\nu_{k,j}| \neq 1 \\ -\frac{1}{2}\left(1 - \frac{1}{\alpha}\right), \text{ if } |\nu_{k,j}| = 1. \end{cases} \tag{34}$$

(A_3) : The seasonal polynomials and non-seasonal autoregressive in (14) and (15) of the Gegenbauer *ARFISMA*-SαS process in (13) have no roots in the unit disk.

(A_4) : The seasonal polynomials and non-seasonal moving average in (14) and (15) of the Gegenbauer *ARFISMA*-SαS model in (13) have no roots in the unit disk.

(A_5) : The sequence of real numbers $\{c_k\}_{k \in \mathbb{N}}$ is such that $\sum_{k=0}^{+\infty} |c_k|^{\delta} < \infty$, where $\delta \in]0; \alpha[\cap [0; 1]$.

Remark 2. The assumptions (A_1) and (A_2) ensure convergence of the power series in (11), see Katayama and Naoya (2000). We use assumptions (A_3) and (A_4) to guarantee causality and inversibility of our process defined in (13). The assumption (A_5) allows to establish the absolute convergence of the series $\sum_{k=0}^{+\infty} c_k \epsilon_{t-k}$.

Theorem 1. *Under the assumptions (A_1),(A_3) and (A_5), the process defined by (13) has a unique causal moving average representation given by:*

$$X_t = \sum_{k=0}^{+\infty} c_k \epsilon_{t-k}, \tag{35}$$

where coefficients $(c_k)_{k \in \mathbb{N}}$ are defined by (28), (29), (30), (31) and (32).

*Proof (**Proof of Theorem** 1).* According to assumptions (A_1), (A_3) and (A_5), the power series

$$\sum_{k=0}^{+\infty} c_k \epsilon_{t-k}$$

and

$$\sum_{k=0}^{+\infty} \pi_k(\omega, \nu) \epsilon_{t-k}$$

almost surely converge. Assume that the set

$$\Lambda_0 = \left\{ \lambda : \text{for each } k, \sum_{k=0}^{+\infty} c_k \epsilon_{t-k}(\lambda) \text{ and } \sum_{k=0}^{+\infty} \pi_k(\omega, \nu) \epsilon_{t-k}(\lambda) \text{ converge} \right\} \tag{36}$$

has probability one. Fix $\lambda_0 \in \Lambda_0$ and denote, for brevity, $\epsilon_k = \epsilon_k(\lambda_0)$,

$$X_t := \sum_{k=0}^{+\infty} c_k \epsilon_{t-k}(\lambda_0)$$

and

$$Y_t := \sum_{k=0}^{+\infty} \pi_k(\omega, \nu) \epsilon_{t-k}(\lambda_0).$$

We must show that:

$$\Phi_s(B^s) \Phi(B) X_t = \Theta_s(B^s) \Theta(B) Y_t. \tag{37}$$

Using the polynomial $\mathcal{A}(.)$ in (16) and without loss of generality, $P \leq Q$, then

$$\Phi_s(B^s) \Phi(B) X_t = \left(1 - \sum_{i=1}^{P} \Phi_i B^{is}\right) \left(1 - \sum_{j=1}^{p} \phi_j B^j\right) \sum_{k=0}^{+\infty} c_k \epsilon_{t-k}, \tag{38}$$

$$= \sum_{k=0}^{+\infty} c_k \epsilon_{t-k} - \left(\sum_{j=1}^{p} \phi_j B^j\right) \left(\sum_{k=0}^{+\infty} c_k \epsilon_{t-k}\right) - \left(\sum_{i=1}^{P} \Phi_i B^{is}\right) \left(\sum_{k=0}^{+\infty} c_k \epsilon_{t-k}\right)$$
$$+ \left(\sum_{i=1}^{P} \Phi_i B^{is}\right) \left(\sum_{j=1}^{p} \phi_j B^j\right) \left(\sum_{k=0}^{+\infty} c_k \epsilon_{t-k}\right). \tag{39}$$

By lemma 1, we have,

$$\Phi_s(B^s)\Phi(B)X_t = \sum_{k=0}^{s-1}\left(c_k - \sum_{j=1}^{min(k;p)} \phi_k c_{k-j}\right)\epsilon_{t-k} + \varrho_s \sum_{i=1}^{P}\sum_{k=0}^{s-1}(\tilde{\gamma}_{is+k} - \tilde{\alpha}_{is+k})\epsilon_{t-is-k}$$
$$+ \sum_{i=1}^{+\infty}\sum_{k=0}^{s-1}\left(c_{is+k} - \sum_{j=1}^{min(is+k;p)} \phi_j c_{is+k-j}\right)\epsilon_{t-is-k}, \quad (40)$$
$$= \sum_{k=0}^{s-1}\left(c_k - \sum_{j=1}^{min(k;p)} \phi_j c_{k-j}\right)\epsilon_{t-k}$$
$$+ \varrho_s \sum_{i=1}^{P}\sum_{k=0}^{s-1}\left(c_{is+k} - \sum_{j=1}^{min(is+k,p)} \phi_j c_{is+k-j} + \tilde{\gamma}_{is+k} - \tilde{\alpha}_{is+k}\right)\epsilon_{t-is-k}$$
$$+ \sum_{i=P+1}^{+\infty}\sum_{k=0}^{s-1}\left(c_{is+k} - \sum_{j=1}^{min(is+k;p)} \phi_j c_{is+k-j}\right)\epsilon_{t-is-k}, \quad (41)$$

where $\tilde{\alpha}_{is+k}$ and $\tilde{\gamma}_{is+k}$ are defined by (31). Using (28), (29), (30) and by identification to analogous development of the polynomial $\mathcal{B}(.)$ given by (22), we obtain:

$$\Phi_s(B^s)\Phi(B)X_t = \sum_{k=0}^{s-1}\left(\pi_k(\omega,\nu) + \sum_{j=1}^{min(k,q)} \theta_j \pi_{k-j}(\omega,\nu)\right)\epsilon_{t-k}$$
$$+ \varrho_s \sum_{\lambda=1}^{Q}\sum_{k=0}^{s-1}\left(\tilde{\zeta}_{is+k} + \tilde{\upsilon}_{is+k}\right)\epsilon_{t-is-k}$$
$$+ \sum_{i=1}^{+\infty}\sum_{k=0}^{s-1}\left(\pi_{is+k}(\omega,\nu) + \sum_{j=1}^{min(is+k,q)} \theta_j \pi_{is+k-j}(\omega,\nu)\right)\epsilon_{t-is-k}, \quad (42)$$

where $\tilde{\zeta}_{is+k}$ and $\tilde{\upsilon}_{is+k}$ are given by (32). So

$$\Phi_s(B^s)\Phi(B)X_t = \sum_{k=0}^{+\infty}\pi_k(\omega,\nu)\epsilon_{t-k}$$
$$+ \left(\sum_{i=1}^{Q}\Theta_{is}B^{is}\right)\left(\sum_{k=0}^{+\infty}\pi_k(\omega,\nu)\epsilon_{t-k}\right)$$
$$+ \left(\sum_{j=1}^{q}\theta_j B^j\right)\left(\sum_{k=0}^{+\infty}\pi_k(\omega,\nu)\epsilon_{t-k}\right)$$

$$+\left(\sum_{i=1}^{Q}\Theta_{is}B^{is}\right)\left(\sum_{j=1}^{q}\theta_{j}B^{j}\right)\left(\sum_{k=0}^{+\infty}\pi_{k}\left(\omega,\nu\right)\epsilon_{t-k}\right), \tag{43}$$

$$=\Theta_{s}\left(B^{s}\right)\Theta\left(B\right)Y_{t}. \tag{44}$$

This concludes the proof.

4. Invertibility of the Gegenbauer *ARFISMA* symmetric stable process

The proof of invertibility theorem of the process given by equation (13), requires the development of sequence of real coefficients $(\tilde{c}_k)_{k\geq 0}$. Thus, under the assumption (A_4), we get:

$$\frac{\Phi\left(B\right)\Phi_{s}\left(B^{s}\right)}{\Theta\left(B\right)\Theta_{s}\left(B^{s}\right)}\prod_{k=0}^{\left[\frac{s}{2}\right]}\left(1-2\nu_{k,j}B+B^{2}\right)^{\omega_{k}}=\sum_{k=0}^{+\infty}\tilde{c}_{k}B^{k}, \tag{45}$$

and

▷ for $k\in\{0,\ldots,s-1\}\cup\{(Q+1)\,s,\ldots\}$

$$\tilde{c}_{k}+\sum_{i=1}^{min(k,q)}\theta_{i}\tilde{c}_{k-i}=\varphi_{k}\left(\omega,\nu\right)-\sum_{i=1}^{min(k,p)}\phi_{i}\varphi_{k-i}\left(\omega,\nu\right), \tag{46}$$

▷ for $k\in\{s,\ldots,(P+1)\,s-1\}$

$$\tilde{c}_{k}+\sum_{i=1}^{min(k,q)}\theta_{i}\tilde{c}_{k-i}+\tilde{\vartheta}_{k}+\tilde{\xi}_{k}=\varphi_{k}\left(\omega,\nu\right)-\sum_{i=1}^{min(k,p)}\phi_{i}\varphi_{k-i}\left(\omega,\nu\right)+\tilde{\delta}_{k}-\tilde{\tau}_{k}, \tag{47}$$

▷ for $k\in\{(P+1)\,s,\ldots,(Q+1)s-1\}$

$$\tilde{c}_{k}+\sum_{i=1}^{min(k,q)}\theta_{i}\tilde{c}_{k-i}+\tilde{\vartheta}_{k}+\tilde{\xi}_{k}=\varphi_{k}\left(\omega,\nu\right)-\sum_{i=1}^{min(k,p)}\phi_{i}\varphi_{k-i}\left(\omega,\nu\right). \tag{48}$$

The coefficients $(\varphi_k(\omega,\nu))_{k\geq 0}$ are such that $\varphi_k(\omega,\nu)=\pi_k(-\omega,\nu)$. The quantities $\tilde{\vartheta}_k$, $\tilde{\xi}_k$, $\tilde{\delta}_k$, $\tilde{\tau}_k$ are correspondingly defined by:

(a) for $1\leq i\leq P$,

$$\begin{cases}\tilde{\tau}_{is+j}=\sum_{k=1}^{i}\Phi_{ks}\varphi_{(i-k)s+j}\left(\omega,\nu\right),\\ \tilde{\delta}_{is+j}=\Phi_{is}\sum_{k=1}^{j}\phi_{k}\varphi_{j-k}\left(\omega,\nu\right)+\sum_{l=1}^{i-1}\Phi_{ls}\sum_{k=1}^{p}\phi_{k}\varphi_{(i-l)s+j-k}\left(\omega,\nu\right),\end{cases} \tag{49}$$

(b) for $1 \leq i \leq Q$,

$$\begin{cases} \tilde{\xi}_{is+j} = \sum_{k=1}^{i} \Theta_{ks} \tilde{c}_{(i-k)s+j}, \\ \tilde{\vartheta}_{is+j} = \Theta_{is} \sum_{k=1}^{j} \theta_k \tilde{c}_{j-k} + \sum_{l=1}^{i-1} \Theta_{ls} \sum_{k=1}^{q} \theta_k \tilde{c}_{(i-l)s+j-k}. \end{cases} \tag{50}$$

Theorem 2. *Under assumptions (A_2), (A_4) and (A_5), the process defined by (11) has a unique autoregressive representation given by:*

$$\epsilon_t = \sum_{k=0}^{+\infty} \tilde{c}_k X_{t-k}, \tag{51}$$

where the coefficients $(\tilde{c}_k)_{k\in\mathbb{N}}$ are defined by (46) , (47), (48), (49) and (50).

*Proof (**Proof of Theorem** .2).* The random variables $(\epsilon_t)_{t\in\mathbb{Z}}$ which are independent and identically distributed follow the SαS law. So the process $(X_t)_{t\in\mathbb{Z}}$ also follows SαS distribution. Then

$$\mathbb{E}\left[\sum_{k=0}^{+\infty} |\tilde{c}_k X_{t-k}|\right] = \sum_{k=0}^{+\infty} |\tilde{c}_k| \, \mathbb{E}\left[|X_{t-k}|\right], \tag{52}$$

$$= \sum_{k=0}^{+\infty} |\tilde{c}_k| \, \mathbb{E}\left[|X_1|\right]. \tag{53}$$

Under the assumption (A_4) ,

$$\mathbb{E}\left[|X_1|\right] < \infty, \quad \mathbb{E}\left[\sum_{k=0}^{+\infty} |\tilde{c}_k X_{t-k}|\right] < \infty$$

and we obtain

$$\sum_{k=0}^{+\infty} |\tilde{c}_k| < \infty.$$

Let

$$\begin{cases} \kappa(B) = C(B) = \dfrac{\Theta(B)\,\Theta_s(B^s)}{\Phi(B)\,\Phi_s(B^s)} \prod_{i=0}^{\left[\frac{s}{2}\right]} \left(1 - 2\nu_{i,j}B + B^2\right)^{-\omega_i}, \\ \eta(B) = C^{-1}(B) = \dfrac{\Phi(B)\,\Phi_s(B^s)}{\Theta(B)\,\Theta_s(B^s)} \prod_{i=0}^{\left[\frac{s}{2}\right]} \left(1 - 2\nu_{i,j}B + B^2\right)^{\omega_i}. \end{cases} \tag{54}$$

According to Theorem 2.3 in Kokoszka et al. (1995), there is

$$A(B)\epsilon_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-k},$$

where

$$a_j = \sum_{k=0}^{j} c_k \tilde{c}_{j-k}$$

for all $j \in \mathbb{N}$, such that

$$\kappa(B)[\eta(B)\epsilon_t] = A(B)\epsilon_t = \eta(B)[\kappa(B)\epsilon_t], \text{ almost surely.}$$

Since

$$A(B)\epsilon_t = C(B)C^{-1}(B)\epsilon_t = \epsilon_t$$

and

$$\begin{aligned} \kappa(B)[\eta(B)\epsilon_t] &= C(B)\left[C^{-1}(B)\epsilon_t\right], \\ &= C(B)X_t, \qquad (55) \\ &= \sum_{k=0}^{+\infty} \tilde{c}_k X_{t-k}, \text{ almost surely,} \qquad (56) \end{aligned}$$

then

$$\epsilon_t = \sum_{k=0}^{+\infty} \tilde{c}_k X_{t-k}, \text{ almost surely.}$$

5. Simulation study

In this section, we perform some simulations to illustrate the behaviour of the process. We first describe a way to generate Gegenbauer *ARFISMA*-SαS process data and then examine features such as long memory, seasonality, cyclical and high variability in the simulated data. Because there is no known technique for generating an exact Gegenbauer *ARFISMA* in the stable case, we will approximate the infinite moving average (35) as follows, where M is finite:

$$X_t := X_M(t) \approx \sum_{k=0}^{M} c_k \epsilon_{t-k}, \quad t = 0, 1, 2, \ldots \qquad (57)$$

where $(\epsilon_k)_{k\in\mathbb{N}}$ ∼SαS and $(c_k)_{k\in\mathbb{N}}$ are given by (28), (29), (30), (31) and (32).

We simulate a Gegenbauer *ARFISMA*-SαS process,

$$X_t = (1-B)^{d_0}(1+B)^{d_{\frac{s}{2}}} \prod_{k=1}^{(\frac{s}{2})-1} \left\{\left(1-e^{i\lambda_{k,j}}B\right)\left(1-e^{-i\lambda_{k,j}}B\right)\right\}^{d_k} \epsilon_t. \tag{58}$$

The frequencies

$$\lambda_{k,j} = \cos^{-1}(\nu_{k,j}),$$

such that

$$\lambda_{k,j} = \frac{2\pi k}{s} + \frac{2\pi j}{n}, \; for \; all \; (k,j) \in \left\{0,\dots,\frac{s}{2}\right\} \times \{0,\dots,m\},$$

where m is finite. The values of the long memory parameters and frequencies are chosen according assumption (A_1) in Theorem 1. Thus

$$\omega_k < 1 - \frac{1}{\alpha} \; if \; |\nu_{k,j}| \neq 1$$

and

$$\omega_k < \frac{1}{2}\left(1-\frac{1}{\alpha}\right) \; if \; |\nu_{k,j}| = 1.$$

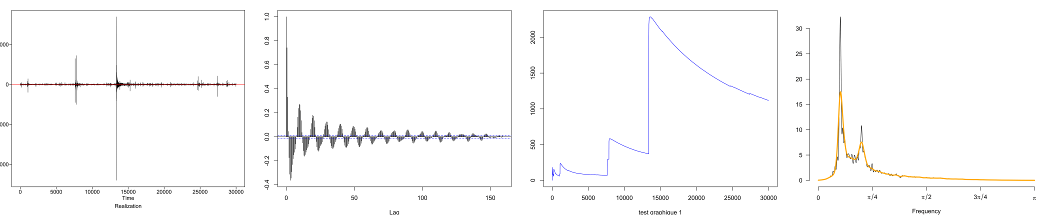
We consider sample sizes $n = 30000$, the stability index $\alpha = 1.6$, and seasonal periods $s = 2$, $s = 4$ and $s = 6$. In short, all the parameters chosen for our process are recorded in the table below.

Seasonal parameter s	$(\nu_{k,j})_j$	$(\omega_k)_{k\in\{0,1,\ldots,[\frac{s}{2}]\}}$
$s=2$	$\begin{cases}\nu_{0,j}=0.81\\ \nu_{1,j}=0.95\end{cases}$	$\begin{cases}\omega_0=0.19\\ \omega_1=0.15\end{cases}$
$s=4$	$\begin{cases}\nu_{0,j}=0.80\\ \nu_{1,j}=0.95\\ \nu_{2,j}=0.60\end{cases}$	$\begin{cases}\omega_0=0.31\\ \omega_1=0.33\\ \omega_2=0.32\end{cases}$
$s=6$	$\begin{cases}\omega_{0,j}=0.80\\ \omega_{1,j}=0.95\\ \omega_{2,j}=0.60\\ \omega_{3,j}=0.50\end{cases}$	$\begin{cases}\omega_0=0.30\\ \omega_1=0.33\\ \omega_2=0.32\\ \omega_3=0.31\end{cases}$

Table 1: Table of Parameters

The time series plot showing evidence of heavy tails (great variability) are shown in graphs $1(a)$; $2(a)$ and $3(a)$. We observe a great variability in the variance of the process. When heavy tails are present as noted above, the variances are infinite and the following heavy tailed modification of the sample autocorrelation function is more appropriate:

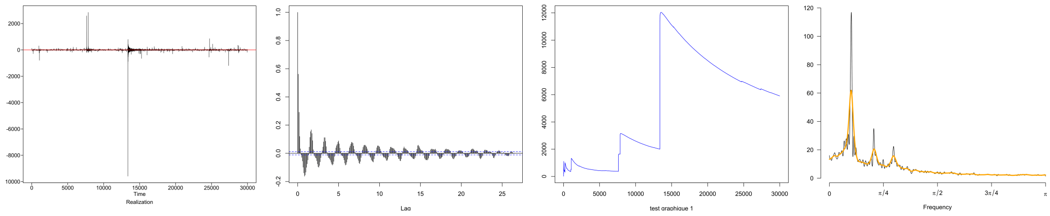
$$\hat{\rho}\left(h\right)=\frac{\sum_{i=1}^{N-|h|}X_iX_{i+h}}{\sum_{i=1}^{n}X_i^2}. \tag{59}$$



(a) (b) (c) (d)

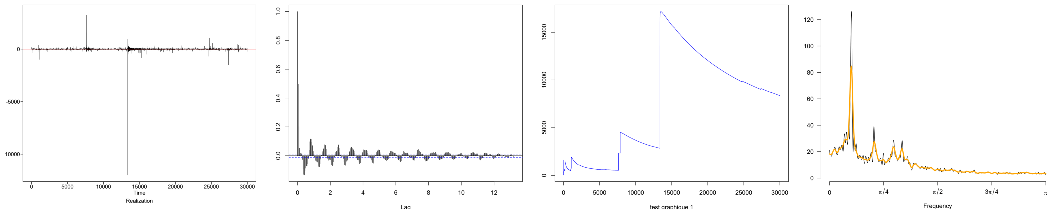
Fig. 1: (a): Trajectory, (b): Autocorrelation function, (c): Empirical variance test and (d): Normalized periodogram for $s=2$ and $\alpha=1.6$.

Looking figures $1(b)$; $2(b)$ and $3(b)$, we see that in a finite sample, the speed of convergence of $\hat{\rho}\left(h\right)$ towards 0 is very slow as $h\to\infty$. To confirm the high variability, we have performed the test for infinite variance, by the empirical variance test. One of the oldest tests for determining whether data has infinite variance is the trick



(a) (b) (c) (d)

Fig. 2: (a): Trajectory, (b): Autocorrelation function, (c): Empirical variance test and (d): Normalized periodogram for $s = 4$ and $\alpha = 1.6$.



(a) (b) (c) (d)

Fig. 3: (a): Trajectory, (b): Autocorrelation function, (c): Empirical variance test and (d): Normalized periodogram for $s = 6$ and $\alpha = 1.6$.

of plotting the sample variance S_n^2 based on the first n observations, as a function of n. If the data comes from process with finite variance, S_n^2 should converge to finite value. Otherwise, it should diverge as n grows and the graph typically shows large jumps. We notice lack of convergence of the empirical variance plot for the data in $1(c)$, $2(c)$ and $3(c)$ and the steep rise of the variance of the delays may be considered as tending towards infinity. After observing the periodogram plot in $1(d)$; $2(d)$ and $3(d)$, we note that each seasonal peak rises at its own seasonal frequency. Furthermore, the number of significant peaks emerging from the periodogram plot indicates the type of seasonality with which we are confronted.

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