Afrika Statistika

Afrika Statistika
Vol. 16 (2), 2021, pages 2749 - 2766.
DOI: http://dx.doi.org/10.16929/as/2021.2749.182

ISSN 2316-090X

Parametric Estimation of Long Memory Multivariate Gaussian random fields

Aubin Yao N'dri [1,2,*], **Amadou Kamagaté** [1,2] **and Ouagnina Hili** [2]

[1] Environmental Sciences and Technologies Laboratory, Jean Lorougnon Guédé University, Daloa, Côte d'Ivoire
[2] UMRI Mathematics and New information Technologies, Félix Houphouët-Boigny National Polytechnic Institute, Yamoussoukro, Côte d'Ivoire

Received on June 11, 2021; Accepted on August 22, 2021.

Abstract. The aim of this paper is to make a theoretically study of the Minimum Hellinger Distance Estimator of multivariate, Gaussian stationary long memory random fields. The variables are observed on a finite set of points in space. Under certain assumptions, we establish the almost sure convergence and the asymptotic distribution of this estimator.

Résumé. Le but de ce papier est d'étudier de façon théorique l'estimateur du minimum de distance de Hellinger des champs aléatoires multivariés, gaussiens stationnaires à longue mémoire. Les variables sont observées sur un ensemble fini de points de l'espace. Nous établissons sous certaines conditions, la convergence presque sûre et la distribution asymptotique de cet estimateur.

Key words: asymptotic properties; multivariate long memory; Hellinger distance estimation; random field.
AMS 2010 **Mathematics Subject Classification Objects :** $60G10$, $60G15$, $60G60$, $62F12$.

---

*Corresponding author : Aubin Yao N'dri (aubin_ndri@yahoo.com)
Amadou Kamagaté : a.kamagate@yahoo.fr
Ouagnina Hili : o_hili@yahoo.fr.

The authors.
Aubin Yao N'dri, Ph.D., is Assistant Professor of Mathematics and Statistics at Jean Lorougnon Guédé University, Daloa, Côte d'Ivoire.
Amadou Kamagaté, Ph.D., is Assistant Professor of Mathematics and Statistics at Jean Lorougnon Guédé University, Daloa, Côte d'Ivoire.
Ouagnina Hili, Ph.D., is full professor of Mathematics and Statistics at Félix Houphouët-Boigny National Polytechnic Institute, Yamoussoukro, Côte d'Ivoire.

1. Introduction

The purpose of this paper is within the framework of the estimation of the parameters of multivariate gaussian, stationary and long memory random fields using a criterion based on the minimum Hellinger distance. This estimate uses nonparametric kernel estimate of the process density. We establish the almost sure convergence and asymptotic distribution of the obtained parametric estimator. An asymptotic study of a kernel density estimator which plays an essential role in this framework of estimation is also studied. This paper enriches the literature on process statistics and contributes significantly to parametric estimation. Indeed, it extends N'dri and al. (2019) to the study of multivariate random fields. In addition, the existing work on multivariate long memory random fields are of two types. On the one hand, we find work on the modeling of long memory dependent multivariate random fields estimation procedures. Among these studies, we have the paper of Alomari and al. (2017) entitled "Ibragimov minimum contrast estimators based on tapered data of Gaussian stationary random fields". On the other hand, theoretical research on the basic tools of asymptotic statistics such as partial sums has been done by Major (2019).

Long memory phenomena are well known in various areas of applications, including notably Econometrics, Finance, and Network traffic modeling. Traditionally, a stationary stochastic process with finite second moment is considered to have long range dependence (or long memory) if, either the covariance function has the power law decay, or the spectral density has a singularity at the origin. The two approaches are referred to in the literature as the time domain approach and the frequency domain approach, respectively.

This paper uses the time domain approach to study the asymptotic properties of the Minimum Hellinger Distance Estimator (*MHDE*). The probability density function of the random field denoted by $f(x, \theta_0)$, depends on a parameter $\theta_0 \in \Theta$ a compact subset of $\mathbb{R}^q$ and $x \in \mathbb{R}^d$. We construct an estimator $\hat{\theta}_n$ of the true parameter θ_0. The values of this estimator are in the parameter space Θ and minimize the Hellinger distance between $f(., \theta_0)$ and f_n, the kernel density estimator. This study therefore falls within the field of spatial statistics. The purpose of spatial statistics is to study phenomena (temperatures, study of a population, study of cities, ...) on a spatial set $S \subset \mathbb{R}^d,\ d \geq 2$. There is a space dependency of this data. We will therefore set X, a random field (a family of random variables) on S with

$X = X_s, s \in S$ composed of variables indexed by S. We then consider the spatial data as realizations of random fields. These processes can be modeling tools in image study, Geostatistics, particle Physics or even spatial Econometrics. This paper deals with multivariate long memory gaussian spatial data. It should be noted that the study of the multivariate spatial models are vastly underdeveloped compared to multivariate gaussian time series models. Gaussian random fields with long-range dependence are known to have applications in Medical image processing Biermé and al. (2008), Major (1981) and Hydrology Benson and al. (2006), Meerschaert and al. (2013a), Physics, Engineering, Biology, Economics and Finance. For instance, multivariate Gaussian random field allows the modeling of weather variables residuals (excluding precipitation).

The rest of this paper proceeds as follows. In section 2, we define the basic tools of our study. Section 3 presents the assumptions and states our main results (Theorem 1 and Theorem 2). An important technical tool in this paper is the convergence of the kernel density estimator to the true density $f(x, \theta_0)$ in the Hellinger topology. We prove this result in Lemma 1. Then, applying this asymptotic property, we show consistency property for the Minimum Hellinger Distance Estimator(*MHDE*). The asymptotic distribution of this estimator is studied in Theorem 2. Throughout this paper, $\longrightarrow^P$ denote convergence in probability. For every site $\mathbf{i}$, $\mathbf{j} \in I_{\mathbf{n}} \subset \mathbb{N}^\nu$, $\|\mathbf{i} - \mathbf{j}\|$ is a distance between sites $\mathbf{i}$ and $\mathbf{j}$. The integer τ is the Hermite rank of the family defined in subsection 2.3.

2. Definitions and Notations

In this section, we give the mathematical definitions and the notations used for the following. The random field $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^\nu}$ is defined on some probability space $(\Omega, \mathcal{F}, P)$. Let $X_{\mathbf{i}} = (X_{\mathbf{i}}^{(1)}, X_{\mathbf{i}}^{(2)}, \ldots, X_{\mathbf{i}}^{(d)})$ and

$$\gamma(\mathbf{i}, \mathbf{j}) = cov(X_{\mathbf{i}}, X_{\mathbf{j}}) = \left\{\gamma_{(p,q)}(\mathbf{i}, \mathbf{j})\right\}_{1 \le p,q \le d}$$

be the covariance matrix function, where the functions

$$\gamma_{(p,q)}(\mathbf{i}, \mathbf{j}) = cov(X_{\mathbf{i}}^{(p)}, X_{\mathbf{j}}^{(q)}) \ for \ 1 \le p, q \le d.$$

2.1. *Multivariate Gaussian Random Field*

Gaussian random fields play an important role for several reasons. First, the specification of their finite-dimensional distributions is simple. Then, they are reasonable models for modeling many natural phenomena. Finally, estimation and inference are simple, and the model is specified by expectations and covariances. The form of the multivariate density function is

$$f(x_1, x_2, \ldots, x_d) = (2\pi)^{-d/2} \mid R \mid^{-1/2} exp\left\{-\frac{1}{2}(x - m)' R^{-1}(x - m)\right\}$$

where R is a $d \times d$ variance-covariance matrix of $X_{\mathbf{i}} = (X_{\mathbf{i}}^{(1)}, X_{\mathbf{i}}^{(2)}, \ldots, X_{\mathbf{i}}^{(d)})$, $\mid R \mid$ the determinant of R, $x' = (x_1, x_2, \ldots, x_d)$ is a row vector and $m' = (m_1, m_2, \ldots, m_d)$ the mean vector. Without loss of generality and for theoretical considerations, we assume that $\mathbb{E}(X_{\mathbf{i}}^{(p)}) = 0,\ \mathbb{E}(X_{\mathbf{i}}^{(p)})^2 = 1$ for all $1 \leq p \leq d,\ \mathbf{i} \in \mathbb{Z}^\nu$. On the over hand, we consider a site $\mathbf{i}$ with positive components.

Definition 1. A multivariate random field is second-order stationary (or just stationary) if the mean vector m is constant and the covariance function $\gamma_{(p,q)}(\mathbf{i}, \mathbf{j}) = cov(X_{\mathbf{i}}^{(p)}, X_{\mathbf{j}}^{(q)})$ is a function of the difference $\mathbf{h} = \mathbf{j} - \mathbf{i}$ only. Namely, $\gamma_{(p,q)}(\mathbf{i}, \mathbf{i+h}) = \gamma_{(p,q)}(\mathbf{h})$ for $1 \leq p, q \leq d$.

Definition 2. The multivariate random field $X_{\mathbf{i}} = (X_{\mathbf{i}}^{(1)}, X_{\mathbf{i}}^{(2)}, \ldots, X_{\mathbf{i}}^{(d)})$, is long range dependent if there exist a slowly varying function at infinity $\mathcal{L}$ such that the covariance function $\gamma_{(p,q)}(\mathbf{j})$ satisfies the relation

$$\gamma_{(p,q)}(\mathbf{j}) := \|\mathbf{j}\|^{-\alpha} b_{p,q}\left(\frac{\mathbf{j}}{\|\mathbf{j}\|}\right)\mathcal{L}(\|\mathbf{j}\|) \tag{1}$$

for all $1 \leq p,\ q \leq d$, where the parameter $0 < \alpha < \nu$. The function $\mathcal{L}$ is such that

$$\lim_{s \to +\infty} \frac{\mathcal{L}(st)}{\mathcal{L}(s)} = 1,\ t > 0$$

and $b_{p,q}(.)$ a real valued continuous function on the unit sphere $S^{\nu-1} = \{x \in \mathbb{R}^\nu,\ \|x\| = 1\}$.

Example 1. As in the Theorem 3A of Major (1981) and Ho and Sun (1990) there are measures $G_{p,q}$, for $1 \leq p, q \leq d$ such that

$$\gamma_{p,q}(\mathbf{j}) = \int_{[-\pi,\pi]^\nu} \exp(i\mathbf{j}x) dG_{p,q}(x).$$

We get a correlation function satisfying (1) with the help of a matrix valued spectral measure whose coordinates $G_{p,q}$ have a density function of the form

$$g_{p,q}(x) = \|x\|^{\alpha-\nu} b_{p,q}\left(\frac{x}{\|x\|}\right) h(\|x\|)$$

with respect to the Lebesgue measure on the cube, $x \in [-\pi, \pi]^\nu$ and $h(x)$ a non-negative smooth and even function on the torus $[-\pi, \pi]^\nu$ which does not disappear at the origin and tends to zero at infinity sufficiently fast.

2.2. *Kernel density estimator*

Denote by I_n a rectangular region defined by

$$I_n = \{\mathbf{i} = (i_1, ..., i_\nu) \in \mathbb{N}^{*\nu}, 1 \leq i_l \leq n \text{ for } l = 1, \dots, \nu\}$$

and $n^\nu = Card(I_n)$.

Assume that we observe $X_{\mathbf{i}}$ on I_n. The kernel density estimator f_n of $f(x, \theta_0)$ is defined by

$$f_n(x) = \frac{1}{n^\nu b_n^d} \sum_{\mathbf{i} \in I_n} K\left(\frac{x - X_{\mathbf{i}}}{b_n}\right) = \frac{1}{n^\nu b_n^d} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \dots \sum_{i_\nu=1}^{n} K\left(\frac{x - X_{i_1,\dots,i_\nu}}{b_n}\right), \, x \in \mathbb{R}^d$$

where K is a kernel function and $(b_n)_n$ is a sequence of bandwidths tending to zero as n tends to infinity.

The standard multivariate normal kernel for $x \in \mathbb{R}^d$ is

$$K(x) = (2\pi)^{-d/2} exp\left\{-\frac{1}{2}x'x\right\}.$$

An easy way to construct a multivariate kernel from an univariate kernel is to construct a product kernel. Let $x = (x_1, \dots, x_d)$ and K_u be an univariate kernel then

$$K(x) = \prod_{i=1}^{d} K_u(x_i).$$

An important feature is to scale kernel functions by a parameter matrix $\mathbf{H} = \{h_{ij}\}_{i,j=1,\dots,d}$ with

$$K_{\mathbf{H}}(x) = |\mathbf{H}|^{-1} K(\mathbf{H}^{-1}x).$$

2.3. *Hermite Polynomial Expansion for Multivariate Gaussian Function*

Let us consider the following function,

$$G_n(x, X_{\mathbf{i}}) = K\left(\frac{x - X_{\mathbf{i}}}{b_n}\right) - \mathbb{E}\left(K\left(\frac{x - X_{\mathbf{i}}}{b_n}\right)\right).$$

We have $\mathbb{E}(G_n(x, X_{\mathbf{i}})) = 0$ and

$$\begin{aligned}\mathbb{E}(G_n(x,X_{\mathbf{i}}))^2 &= \mathbb{E}\left(K\left(\frac{x-X_{\mathbf{i}}}{b_n}\right)-\mathbb{E}\left(K\left(\frac{x-X_{\mathbf{i}}}{b_n}\right)\right)\right)^2 \le E\left(K\left(\frac{x-X_{\mathbf{i}}}{b_n}\right)\right)^2\\ &\le \int_{\mathbb{R}^d} K^2\left(\frac{x-u}{b_n}\right)f_\theta(u)du \le \sup_{u\in\mathbb{R}^d} f_\theta(u)b_n^d\int_{\mathbb{R}^d}K^2(z)\,dz < +\infty.\end{aligned}$$

Moreover, the function $G_n : \mathbb{R}^d \to \mathbb{R}$ is a measurable function. Therefore, that function can be expand in

$$L^2\left(\mathbb{R}^d, \frac{1}{(\sqrt{2\pi})^d}\exp\left(-\frac{1}{2}(z_1^2+\ldots+z_d^2)\right)\right) \equiv$$

$$\left\{G : \frac{1}{(\sqrt{2\pi})^d}\int_{\mathbb{R}^d} G^2(z)\exp\left(-\frac{1}{2}(z_1^2+\ldots+z_d^2)\right)\,dz < +\infty\right\} \text{ space,}$$

in following series

$$G_n(x,X_{\mathbf{i}}) = \sum_{\ell_1,\ldots,\ell_d=0}^{+\infty}\frac{c_{\ell_1,\ldots,\ell_d}(x)}{\prod_{j=1}^d \ell_j!}\prod_{j=1}^d H_{\ell_j}(X_{\mathbf{i}}^{(j)})$$

where

$$c_{\ell_1,\ldots,\ell_d}(x) = \int_{\mathbb{R}^d} H_\ell(z)G_n(x,z)\phi(z)dz \text{ with } \phi(z) = \frac{1}{(\sqrt{2\pi})^d}\exp\left(-\frac{1}{2}(z_1^2+\ldots+z_d^2)\right),$$

$$H_\ell(x_1,\ldots,x_d) = \frac{(-1)^{|\ell|}}{\sqrt{\ell!}}\exp\left(\frac{x_1^2+\ldots+x_d^2}{2}\right)\frac{\partial^{|\ell|}}{\partial x_1^{\ell_1}\ldots\partial x_d^{\ell_d}}\exp\left(-\frac{x_1^2+\ldots+x_d^2}{2}\right),$$

the $\mathbb{R}^d$-valued Hermite polynomial, $\ell = (\ell_1,\ldots,\ell_d)$, $\ell! = \prod_{j=1}^d \ell_j$ and $|\ell| = \ell_1+\ldots+\ell_d$.

We say that G has Hermite rank τ if the Hermite coefficient $c_{\ell_1,\ldots,\ell_d}(x) = 0$ for $\ell_1+\ldots+\ell_d < \tau$ and $c_{\ell_1,\ldots,\ell_d}(x) \neq 0$ for some $\ell_1+\ldots+\ell_d = \tau$.

2.4. ***Multiple Wiener-Itô Integral***

Given a vector of stationary gaussian random field $X_{\mathbf{i}} = (X_{\mathbf{i}}^{(1)},\ldots,X_{\mathbf{i}}^{(d)})$, $\mathbf{i}\in\mathbb{Z}^\nu$, with expectation zero and covariance function $\gamma_{p,q}(.)$ that satisfies relation (1) with some parameter $0<\alpha<\nu$. Let us consider its matrix valued spectral measure $(G_{p,q})$, $1\le p,q\le d$ concentrated on the cube $[-\pi,\pi]^\nu$ and define

$$G_{p,q}^{(n)}(A) = \frac{n^\alpha}{\mathcal{L}(n)}\left(\frac{A}{n}\right),\ A\in\mathcal{B}^\nu,\ n=1,2,\ldots,\ \ 1\le p,q\le d,$$

concentrated on $[-n\pi, n\pi]^\nu$ for all $n = 1, 2, \ldots$ where $\mathcal{B}^\nu$ denotes the $\sigma-$algebra of the Borel measurable sets on $\mathbb{R}^\nu$.

Then, there exist a spectral measure $\left(G_{p,q}^{(0)}\right)$, $1 \leq p, q \leq d$ of generalized stationary random fields on $\mathbb{R}^\nu$ such that $G_{p,q}^{(n)}$ tend vaguely to $G_{p,q}^{(0)}$ on $\mathbb{R}^\nu$.

For details, see Proposition 1 of Dobrushin and Major (1979).

Let $\rho(x, \theta_0)$ defined in Theorem 2, $Z_{G^{(0)}} = \left(Z_{G^{(0)},1}, \ldots, Z_{G^{(0)},d}\right)$ be a vector valued random spectral measure which corresponds to the matrix valued spectral measure

$$\left(G_{p,q}^{(0)}\right), 1 \leq \ p, q \leq d$$

and define indices

$$j(s|\ell_1, \ldots, \ell_d),\ 1 \leq s \leq \tau,\ as\ j(s|\ell_1, \ldots, \ell_d) = r$$

if

$$\sum_{u=1}^{s-1} \ell_u < r \leq \sum_{u=1}^{s} \ell_u,\ 1 \leq s \leq \tau.$$

Then, the sum of multiple Wiener-Itô integrals defined by

$$\sum_{\substack{\ell_1, \ldots, \ell_d = 0 \\ \ell_1 + \ldots + \ell_d = \tau}}^{+\infty} c_{\ell_1, \ldots, \ell_d} \int_{[-\pi,\pi]^\tau} \prod_{\ell=1}^{\nu} \frac{\exp\left(i\left(x_1^{(\ell)} + \ldots + x_\tau^{(\ell)}\right)\right) - 1}{i\left(x_1^{(\ell)} + \ldots + x_\tau^{(\ell)}\right)} Z_{G^{(0)},j(1|\ell_1,\ldots,\ell_d)}(dx_1) \ldots$$
$$Z_{G^{(0)},j(\tau|\ell_1,,\ldots,\ell_d)}(dx_\tau) \int_{\mathbb{R}^d} \frac{\rho(x, \theta_0)}{2 f^{1/2}(x, \theta_0)}\, dx$$

exists.

3. Statement of Assumptions and Results

From the Kolmogorov Existence theorem on arbitrary products spaces, we suppose that there exist a random field $(X'_{\mathbf{i}})_{\mathbf{i} \in I_n}$ which is an independent copy of the random field $(X_{\mathbf{i}})_{\mathbf{i} \in I_n}$ on the same probability space $(\Omega, \mathcal{F}, P)$. Thereafter, we define the set $B'(n, d, \varepsilon)$ which uses the sequence $(X'_{\mathbf{i}})_{\mathbf{i} \in I_n}$. Thus, the set $B'(n, d, \varepsilon)$ constructed is an independent version of

$$B(n, d, \varepsilon) = \left\{ x = (x_1, x_2, \ldots, x_d), \mid \sum_{i \in I_n} G(x, X_{\mathbf{i}}) \mid \geq n^\nu b_n^d \varepsilon \right\}.$$

3.1. Assumptions

Assumptions A

(A1) Suppose that $0 < \tau\alpha < \nu$, $n^{\nu} b_n^d \to +\infty$ and

$$\lim_{n\to+\infty} n^{c\tau\alpha} b_n^2 = 0 \text{ for a constant } 0 < c < \frac{1}{4}.$$

(A2) Suppose that

$$\int_{B(n,d,\varepsilon)} \prod_{j=1}^{d} \phi(x_j)dx_j = \int_{B'(n,d,\varepsilon)} \prod_{j=1}^{d} \phi(x_j)dx_j.$$

(A3) The kernel K is bounded with compact support, such that

$$\int_{\mathbb{R}^d} u_i K(u)du = 0 \text{ and } 0 \leq \int_{\mathbb{R}^d} u_i u_j K(u)du < +\infty \text{ for } 1 \leq i,\, j \leq d.$$

Assumptions B

(B1) For each $\theta \in \Theta$, the function

$$x \mapsto f(x,\theta)$$

is twice continuously differentiable.

(B2) For each $x \in \mathbb{R}^d$, the function

$$\theta \mapsto f(x,\theta)$$

is continuous.

(B3) For each $x \in \mathbb{R}^d$, the function

$$\theta \mapsto \frac{\partial}{\partial\theta_j} f^{1/2}(x,\theta),$$

for $1 \leq j \leq q$, is continuous and for every j, the function

$$\theta \mapsto \frac{\partial}{\partial\theta_j} f^{1/2}(x,\theta)$$

is in $L^2(\mathbb{R}^q)$.

(B4) For each $x \in \mathbb{R}^d$, the function

$$\theta \mapsto \frac{\partial^2}{\partial\theta_j\partial\theta_k} f^{1/2}(x,\theta),\ 1 \le j,k \le q$$

is continuous and for every $j,\ k$, the function

$$\theta \mapsto \frac{\partial^2}{\partial\theta_j\partial\theta_k} f^{1/2}(x,\theta)$$

is in $L^2(\mathbb{R}^q)$.

(B5) For $\theta,\ \theta' \in \Theta,\ \theta \neq \theta'$ implies that

$$\{x \in \mathbb{R}^d\ f(x,\theta) \neq f(x,\theta')\}$$

is a set of positive Lebesgue measure.

Remark 3.1
(1) Assumption (A1) is satisfies when we chose $b_n = n^{\frac{\tau\alpha}{d}(2c-1)}$ when $0 < c < 1/4$ and $d \in \{2,\ 3,\ 4\}$.

(2) Assumption (B5) is the identifiability assumption on the parametrization.

3.2. ***Minimum Hellinger Distance Estimator (MHDE)***

In this subsection, we will briefly discuss minimum Hellinger distance estimation. Let $f(x)$ and $g(x)$ be any two densities ; the Hellinger distance between $f(x)$ and $g(x)$ is defined as the L_2-norm of the difference between square root of density functions, i.e.

$$HD^2(f,g) = \int_{\mathbb{R}^d} \left[(f(x))^{1/2} - (g(x))^{1/2}\right]^2 dx.$$

Let $X_{\mathbf{1}},\ X_{\mathbf{2}},\ \cdots,\ X_{\mathbf{n}}$ be long range dependent random field with density belonging to a specified parametric family $\{f(.,\theta) : \theta \in \Theta\}$. To motivate the *MHDE*, replace f by $f(.,\theta)$ and g by $f_{\mathbf{n}}$, a nonparametric estimator of the density. Therefore, the Hellinger distance in our question becomes the distance between the true density $f(.,\theta_0)$ and the non-parametric density estimator of the $X_{\mathbf{i}}$'s, which can be expressed as follows :

$$HD^2(f(.,\theta), f_{\mathbf{n}}) = \int_{\mathbb{R}^d} \left[(f(x,\theta))^{1/2} - (f_{\mathbf{n}}(x))^{1/2}\right]^2 dx\,. \quad (2)$$

The minimum Hellinger distance estimator of θ is defined to be the value $\widehat{\theta_n}$ (in the parameter space Θ), if it exists, that minimizes (2).

Beran (1977) showed that the *MHDE* is more robust than maximum likelihood estimator when data contaminations are present. Furthermore *MHDE* is known to be asymptotically efficient under a specified parametric family of densities and is minimax robust in a small Hellinger metric neighborhood of the given family (see Beran (1977), Hili (1995)).

Let $\mathcal{G}$ denote the class of densities metrized by the L_1 distance. We define the Minimum Hellinger Distance Functional (*MHDF*) to be the functional $T : \mathcal{G} \to \Theta$ such that

$$T(g) = arg \min_{\theta \in \Theta} HD^2(f(.,\theta), g).$$

Applying Assumption (B2) and (B5), Beran (1977) has shown the existence of *MHDE* for Θ compact and discussed the extension of the result for noncompact Θ. Moreover $T(f(.,\theta))$ may have multiple values, so we shall assume that it stands for any one of those values.

*3.3. **Main Results***

In this subsection, we study in Theorem 1, the efficiency property of the *MHDE*. For the proof of this Theorem, we use Lemma 1 below and the continuity of the functional T (see Beran (1977)). The study of asymptotic distribution property in Theorem 2 is very important. Indeed, this property is useful in the selection criteria of estimators. Knowing the asymptotic distribution of the estimator can solve the problem of estimation by confidence interval and hypothesis testing.

Lemma 1. *If the Assumptions (A1)-(A3), (B1), (B2) and (B5) are satisfied, then f_n almost surely converges to $f(.,\theta_0)$ in the Hellinger topology.*

*Proof (**Proof of Lemma** 1).* We have $f_n(x) - f(x,\theta_0) = f_n(x) - \mathbb{E}f_n(x) + (\mathbb{E}f_n(x) - f(x,\theta_0))$. Now, we show that

$$f_n(x) - \mathbb{E}f_n(x) = \frac{1}{n^\nu b_n^d} \sum_{\mathbf{i} \in I_n} \left(K\left(\frac{x - X_{\mathbf{i}}}{b_n}\right) - \mathbb{E}K\left(\frac{x - X_{\mathbf{i}}}{b_n}\right)\right) \to 0 \text{ a.s.}, \ n \to +\infty.$$

We have for $\varepsilon \geq 0$,

$$\mathbb{P}\{|f_n(x) - \mathbb{E}f_n(x)| \geq \varepsilon\} = \mathbb{P}\left\{\left|\frac{1}{n^\nu b_n^d} \sum_{\mathbf{i} \in I_n} \left(K\left(\frac{x - X_{\mathbf{i}}}{b_n}\right) - \mathbb{E}K\left(\frac{x - X_{\mathbf{i}}}{b_n}\right)\right)\right| \geq \varepsilon\right\}.$$

Thus, for $\varepsilon \geq 0$,

$$\mathbb{P}\{|f_n(x) - \mathbb{E}f_n(x)| \geq \varepsilon\} = \int_{B(n,d,\varepsilon)} f(x_1, x_2, \ldots, x_d)dx_1dx_2 \ldots dx_d.$$

For large n, $\gamma(n) = \sup_{\|\mathbf{j}\| \geq n} \left(\gamma_{(p,q)}(\mathbf{j})\right) \to 0$.
This enables us to use Lemma 3.3 of Taqqu (1977) to express the density f as a uniformly convergent series over $\mathbb{R}^d$. So for $n \geq n_0$,

$$f(x_1, x_2, \ldots, x_d) = \sum_{q=0}^{\infty} \sum_{\substack{k_1+\ldots+k_d=2q \\ 0<k_1,\ldots,k_d<q}} \left\{ E \prod_{j=1}^{d} H_{k_j}(X_i^{(j)}) \right\} \prod_{j=1}^{d} \frac{H_{k_j}(x_j)}{k_j!} \phi(x_j)$$

where

$$H_k(x) = (-1)^k \exp\left(\frac{x^2}{2}\right) \frac{d^k}{dx^k} \exp\left(-\frac{x^2}{2}\right)$$

denotes the $k-th$ Hermite polynomial of

$$K\left(\frac{x-X}{b_n}\right) - \mathbb{E}K\left(\frac{x-X}{b_n}\right)$$

in $L^2(\phi)$ and

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

By using Holder's inequality and the fact that

$$\int H_k^2(x)\phi(x)dx = k!,$$

we get

$$\int_{B(n,d,\varepsilon)} f(x_1, x_2, \ldots, x_d)dx_1dx_2 \ldots dx_d \leq \left\{ \sum_{q=0}^{\infty} \sum_{\substack{k_1+\ldots+k_d=2q \\ 0<k_1,\ldots,k_d<q}} \left| \mathbb{E} \prod_{j=1}^{d} \frac{H_{k_j}(X_i^{(j)})}{\sqrt{k_j!}} \right| \right\} \times \left\{ \int_{B(n,b,\varepsilon)} \prod_{j=1}^{d} \phi(x_j)dx_j \right\}^{1/2}.$$

By applying Lemma 3.1 of Taqqu (1977)(cf. the proof of Lemma 3.1 of the paper), we obtain

$$\sum_{q=0}^{\infty} \sum_{\substack{k_1+\ldots+k_d=2q \\ 0<k_1,\ldots,k_d<q}} \left| \mathbb{E} \prod_{j=1}^{d} \frac{H_{k_j}(X_i^{(j)})}{\sqrt{k_j!}} \right| \leq \sum_{q=0}^{\infty} \sum_{\substack{k_1+\ldots+k_d=2q \\ 0<k_1,\ldots,k_d<q}} \gamma(n)^{(k_1+\ldots+k_d)/2} \prod_{j=1}^{d} (d-1)^{k_j/2}$$
$$\leq \left(\sum_{k=0}^{\infty} ((d-1)\gamma(n))^{k/2} \right)^d = \exp(-d\ln(1-U(n))), \quad (3)$$

where $U(n) = ((d-1)\gamma(n))^{1/2}$.

By using Bernstein-type inequality in Dedecker (2001)(See Corollary 3) and Assumption $(A2)$, we obtain for some $C(d) = d\exp\left(\frac{1}{e}\right) > 0$

$$\left\{ \int_{B(n,d,\varepsilon)} \prod_{j=1}^{d} \phi(x_j)dx_j \right\}^{1/2} = \left\{ \int_{B'(n,d,\varepsilon)} \prod_{j=1}^{d} \phi(x_j)dx_j \right\}^{1/2}$$
$$\leq C(d)\exp\left(-\frac{n^{2\nu} b_n^{2d} \varepsilon^2}{4 b_n^d n^{\nu}}\right) = C(d)\exp\left(-\frac{n^{\nu} b_n^d \varepsilon^2}{4}\right). \quad (4)$$

By combining (3) and (4), we get for $\varepsilon > 0$,

$$\mathbb{P}(|f_n(x) - \mathbb{E}f_n(x)| \geq \varepsilon) \leq C(d)\exp\left(-\frac{n^{\nu} b_n^d \varepsilon^2}{4} - d\ln(1-U(n))\right)$$
$$= C(d)\exp\left(-n^{\nu} b_n^d \left(\frac{\varepsilon^2}{4} + \vartheta(n)\right)\right)$$

where

$$\vartheta(n) = \frac{d\ln(1-U(n))}{n^{\nu} b_n^d}.$$

A one order expansion of $\ln(1-U(n))$ in the neighborhood of zero gives
$\ln(1-U(n)) \sim -U(n) = -((d-1)\gamma(n))^{1/2}$. So,

$$\lim_{n\to+\infty} \vartheta(n) = \lim_{n\to+\infty} \frac{d\ln(1-U(n))}{n^{\nu} b_n^d} = 0.$$

Hence there is n_1 such that, for all $n \geq n_1$

$$\mathbb{P}(|f_n(x) - \mathbb{E}f_n(x)| \geq \varepsilon) \leq C(d)\exp\left(-\left(n^{\nu} b_n^d \left(\frac{\varepsilon^2}{4}\right)\right)\right).$$

Then

$$\sum_{n\geq 0}\mathbb{P}\left(|f_n(x)-\mathbb{E}f_n(x)|>\varepsilon\right)\leq \sum_{n\geq 0}\exp\left(-\left(n^{\nu}b_n^d\left(\frac{\varepsilon^2}{4}\right)\right)\right)<\infty.$$

By using Borel-Cantelli lemma, we get $f_n(x)-\mathbb{E}(f_n(x))\to\ 0$ a.s.
On the other hand, $\mathbb{E}f_n\left(x\right)-f\left(x,\theta_0\right)=\int_{\mathbb{R}^d}K\left(u\right)\left(f\left(x-b_nu,\theta_0\right)-f\left(x,\theta_0\right)\right)du$ and for each $x\in\mathbb{R}^d$, $K\left(u\right)|f\left(x-b_nu,\theta_0\right)-f\left(x,\theta_0\right)|\leq cK\left(u\right)$.
By the continuity of the density and by the dominated convergence theorem, we conclude that $\mathbb{E}f_n\left(x\right)-f\left(x,\theta_0\right)\to 0$ as $n\to+\infty$ for each $x\in\mathbb{R}^d$.
Then, for all $x\in\mathbb{R}^d$, $f_n(x)$ almost surely (a.s.) converges to $f(x,\theta_0)$. Thus,

$$\mathbb{P}\left(\lim_{n\to+\infty}f_n^{1/2}(x)=f^{1/2}(x,\theta_0),\ \forall x\in\mathbb{R}^d\right)=1.$$

Furthermore, since $\int_{\mathbb{R}^d}f_n(x)dx=\int_{\mathbb{R}^d}f(x,\theta_0)dx=1$, hence

$$\lim_{n\to+\infty}\left(\int_{\mathbb{R}^d}|f_n^{1/2}(x)-f^{1/2}(x,\theta_0)|^2dx\right)^{1/2}=0\ \ a.s.$$

Therefore $f_n\to f$ a.s. when $n\to+\infty$ in the Hellinger topology.

Theorem 1 (Almost sure convergence). *Let Assumptions (A1)-(A3), (B1), (B2) and (B5) be fulfilled. If θ_0 is in the interior of Θ, then $\hat{\theta}_n$ almost surely converges to θ_0 when $n\to+\infty$.*

*Proof (**Proof of Theorem** 1).* From Lemma 1 and from the continuity of the functional T (see Theorem 1 in Beran (1977)), we deduce that $\hat{\theta}_n=T(f_{\mathbf{n}})\to T(f(.,\theta))=\theta_0$ a.s. as $n\to+\infty$.

This completes the proof.

For the following theorem, we adopt the following notations, $S(.,\theta)=f^{1/2}(.,\theta)$, $\dot{S}(.,\theta)=\left(\frac{\partial}{\partial\theta_1}f^{1/2}(.,\theta),\ldots,\frac{\partial}{\partial\theta_q}f^{1/2}(.,\theta)\right)^T$, where $\dot{S}(.,\theta)^T$ is the transpose of $\dot{S}(.,\theta)$ and

$$\rho(x,\theta)=\left[\int_{\mathbb{R}^d}\dot{S}(x,\theta)\dot{S}(x,\theta)^Tdx\right]^{-1}\dot{S}(x,\theta).$$

Theorem 2 (Asymptotic distribution). *Let Assumptions (A1)-(A3) and (B1)-(B5) be fulfilled. If θ_0 lies in the interior of Θ and if $\int_{\mathbb{R}^d}\dot{S}(x,\theta_0)\dot{S}(x,\theta_0)^Tdx$ is a non singular $(q\times q)$-matrix, then $\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}(\hat{\theta}_n-\theta_0)$ converges in distribution to*

$$\sum_{\substack{l_1,\ldots,l_d=0\\ l_1+\ldots+l_d=\tau}}^{+\infty} c_{l_1,\ldots,l_d} \int_{[-\pi,\pi]^\tau} \prod_{l=1}^{\nu} \frac{\exp\left(i\left(x_1^{(l)}+\ldots+x_\tau^{(l)}\right)\right)-1}{i\left(x_1^{(l)}+\ldots+x_\tau^{(l)}\right)} Z_{G^{(0)},j(1|l_1,\ldots,l_d)}(dx_1)\ldots$$

$$Z_{G^{(0)},j(\tau|l_1,,\ldots,l_d)}(dx_\tau) \int_{\mathbb{R}^d} \frac{\rho(x,\theta_0)}{2f^{1/2}(x,\theta_0)}\, dx.$$

*Proof (**Proof of Theorem** 2).* From Theorem 2 in Beran (1977), we deduce that

$$\begin{aligned}\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}(\hat\theta_n-\theta_0) &= \frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}\int_{\mathbb{R}^d}\rho(x,\theta_0)(f_n^{1/2}(x)-f^{1/2}(x,\theta_0))\,dx + \\ & V_n\int_{\mathbb{R}^d}\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}\dot S(x,\theta_0)(f_n^{1/2}(x)-f^{1/2}(x,\theta_0))\,dx,\end{aligned}$$

where V_n is a $(q\times q)$-matrix whose components tends to zero in probability as $n\to+\infty$.

For $b\geq 0,\ a>0$ we have the algebraic identity

$$b^{1/2}-a^{1/2}=(b-a)/(2a^{1/2})-(b-a)^2/\left[2a^{1/2}(\ b^{1/2}+a^{1/2})^2\right].$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}^d}\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}\rho(x,\theta_0)(f_n^{1/2}(x)-f^{1/2}(x,\theta_0))\,dx\\ &\qquad = \int_{\mathbb{R}^d}\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}\frac{\rho(x,\theta_0)}{2f^{1/2}(x,\theta_0)}(f_n(x)-f(x,\theta_0))\,dx + \mathcal{R}_n(\theta_0) \qquad (5)\end{aligned}$$

where

$$\begin{aligned}|\ \mathcal{R}_n(\theta_0)\ | &\leq \int_{\mathbb{R}^d}\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}\left|\frac{\rho(x,\theta_0)}{f^{3/2}(x,\theta_0)}\right|(f_n(x)-f(x,\theta_0))^2\,dx\\ &\leq C_1\left\{\int_{\mathbb{R}^d}\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}|\rho(x,\theta_0)|\,(f_n(x)-\mathbb{E}f_n(x))^2\,dx\right\}\\ &\quad + C_1\left\{\int_{\mathbb{R}^d}\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}|\rho(x,\theta_0)|\,(\mathbb{E}f_n(x)-f(x,\theta_0))^2\,dx\right\}\end{aligned}$$

and C_1 a positive constant. We have,

$$\sum_{n\geq 0} n^{c\alpha\tau}\mathbb{P}\left(|f_n(x)-\mathbb{E}f_n(x)|^2>\varepsilon\right) \leq \sum_{n\geq 0} n^{c\alpha\tau}\exp\left(-\left(n^\nu b_n^d\left(\frac{\varepsilon^2}{4}\right)\right)\right)<\infty. \qquad (6)$$

Therefore by the dominated convergence Theorem, we get

$$\int_{\mathbb{R}^d} \frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)} |\rho(x,\theta_0)| (f_n(x) - \mathbb{E}f_n(x))^2 dx \to 0 \ \ a.s., \ \ n \to +\infty.$$

By using Assumptions (A3) and (B1), Taylor's formula in several variables gives for $x \in U_x \subset \mathbb{R}^d$ a neighborhood of x

$$\begin{aligned} b_n^{-2}(\mathbb{E}f_n(x) - f(x,\theta_0)) &= b_n^{-2} \int_{\mathbb{R}^d} K(u)[f(x-ub_n) - f(x,\theta_0)]du \\ &= \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} K(u)u_iu_j \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(x - ub_nt, \theta_0)(1-t)dtdu. \end{aligned}$$

Assumption (B1) gives,

$$\lim_{n\to+\infty} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} K(u)u_iu_j \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(x - ub_nt, \theta_0)(1-t)dtdu$$

$$= \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(x,\theta_0) \int_{\mathbb{R}^d} K(u)u_iu_jdu = \gamma(x,\theta_0),$$

and

$$\lim_{n\to+\infty} n^{c\tau\alpha}(\mathbb{E}f_n(x) - f(x,\theta_0))^2 = \lim_{n\to+\infty} n^{c\tau\alpha} b_n^4 \gamma^2(x,\theta_0) = 0. \tag{7}$$

Consequently, by (7) and by the dominated convergence theorem,

$$\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)} \int_{\mathbb{R}^d} \left| \frac{\rho(x,\theta_0)}{f^{3/2}(x,\theta_0)} \right| (\mathbb{E}f_n(x) - f(x,\theta_0))^2 dx \to 0 \ \ \text{as} \ \ n \to +\infty.$$

So,

$$| \mathcal{R}_n(\theta_0) | \to 0 \text{ a.s when } n \to +\infty.$$

Now, we are studying the first term on the right hand side of relation (5).
By using Assumptions (A1), (A3) and (B1), we show that

$$n^{c\tau\alpha} \int_{\mathbb{R}^d} |\rho(x,\theta_0)| (\mathbb{E}f_n(x) - f(x,\theta_0))dx \to 0 \ \ \text{as} \ \ n \to +\infty.$$

On the other hand, we have

$$
\begin{aligned}
G_n(x, X_{\mathbf{i}}) &= \sum_{\ell_1,\ldots,\ell_d=0}^{+\infty} \frac{c_{\ell_1,\ldots,\ell_d}(x)}{\prod_{j=1}^{d} \ell_j!} \prod_{j=1}^{d} H_{\ell_j}(X_{\mathbf{i}}^{(j)}) \\
&= \sum_{\substack{\ell_1,\ldots,\ell_d=0 \\ \ell_1+\ldots+\ell_d=\tau}}^{+\infty} \frac{c_{\ell_1,\ldots,\ell_d}(x)}{\prod_{j=1}^{d} \ell_j!} \prod_{j=1}^{d} H_{\ell_j}(X_{\mathbf{i}}^{(j)}) + \sum_{\substack{\ell_1,\ldots,\ell_d=0 \\ \ell_1+\ldots+\ell_d\geq\tau+1}}^{+\infty} \frac{c_{\ell_1,\ldots,\ell_d}(x)}{\prod_{j=1}^{d} \ell_j!} \prod_{j=1}^{d} H_{\ell_j}(X_{\mathbf{i}}^{(j)}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}(f_n(x) - \mathbb{E}f_n(x)) &= \frac{n^{c\tau\alpha}}{n^{\nu} b_n^{d/2} \mathcal{L}^{\tau/2}(n)} \sum_{\mathbf{i}\in I_n} \frac{1}{b_n^{d/2}} \left(K\left(\frac{x - X_{\mathbf{i}}}{b_n}\right) - \mathbb{E}K\left(\frac{x - X_{\mathbf{i}}}{b_n}\right)\right) \\
&= \frac{\mathcal{L}^{-\tau/2}(n)}{n^{\nu-\tau\alpha/2}} \sum_{\mathbf{i}\in I_n} \sum_{\substack{\ell_1,\ldots,\ell_d=0 \\ \ell_1+\ldots+\ell_d=\tau}}^{+\infty} \frac{c_{\ell_1,\ldots,\ell_d}(x)}{\prod_{j=1}^{d} \ell_j!} \prod_{j=1}^{d} H_{\ell_j}(X_{\mathbf{i}}^{(j)}) + \\
&\quad \frac{\mathcal{L}^{-\tau/2}(n)}{n^{\nu-\tau\alpha/2}} \sum_{\mathbf{i}\in I_n} \sum_{\substack{\ell_1,\ldots,\ell_d=0 \\ \ell_1+\ldots+\ell_d\geq\tau+1}}^{+\infty} \frac{c_{\ell_1,\ldots,\ell_d}(x)}{\prod_{j=1}^{d} \ell_j!} \prod_{j=1}^{d} H_{\ell_j}(X_{\mathbf{i}}^{(j)}).
\end{aligned}
$$

By using the boundedness property of the kernel, Theorem 1.2A, Theorem 1.2B in Major (2019) and Lemma 8 in Arcones (2000), we deduce that

$$
\frac{\mathcal{L}^{-\tau/2}(n)}{n^{\nu-\tau\alpha/2}} \sum_{\mathbf{i}\in I_n} \sum_{\substack{\ell_1,\ldots,\ell_d=0 \\ \ell_1+\ldots+\ell_d\geq\tau+1}}^{+\infty} \frac{c_{\ell_1,\ldots,\ell_d}(x)}{\prod_{j=1}^{d} \ell_j!} \prod_{j=1}^{d} H_{\ell_j}(X_{\mathbf{i}}^{(j)}) \to^{P} 0
$$

when $n \to +\infty$.

So

$$
\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}(\hat{\theta}_n - \theta_0)
$$

and

$$
\frac{\mathcal{L}^{-\tau/2}(n)}{n^{\nu-\tau\alpha/2}} \sum_{\mathbf{i}\in I_n} \sum_{\substack{\ell_1,\ldots,\ell_d=0 \\ \ell_1+\ldots+\ell_d=\tau}}^{+\infty} \frac{c_{\ell_1,\ldots,\ell_d}}{\prod_{j=1}^{d} \ell_j!} \prod_{j=1}^{d} H_{\ell_j}(X_{\mathbf{i}}^{(j)}) \int_{\mathbb{R}^d} \frac{\rho(x,\theta_0)}{2f^{1/2}(x,\theta_0)}\,dx
$$

where

$$
c_{\ell_1,\ldots,\ell_d} = \max_{x\in supp(K)} c_{\ell_1,\ldots,\ell_d}(x)
$$

have the same distribution.

By using Theorem 1.3 in Major (2019) and Theorem 6 in Arcones (1994), we conclude that

$$\frac{n^{c\tau\alpha}}{\mathcal{L}^{\tau/2}(n)}(\hat{\theta}_n - \theta_0)$$

converge in distribution to

$$\sum_{\substack{\ell_1,\ldots,\ell_d=0 \\ \ell_1+\ldots+\ell_d=\tau}}^{+\infty} c_{\ell_1,\ldots,\ell_d} \int_{[-\pi,\pi]^\tau} \prod_{\ell=1}^{\nu} \frac{\exp\left(i\left(x_1^{(\ell)}+\ldots+x_\tau^{(\ell)}\right)\right)-1}{i\left(x_1^{(\ell)}+\ldots+x_\tau^{(\ell)}\right)} Z_{G^{(0)},j(1|\ell_1,\ldots,\ell_d)}(dx_1)\ldots$$
$$Z_{G^{(0)},j(\tau|\ell_1,,\ldots,\ell_d)}(dx_\tau)\int_{\mathbb{R}^d}\frac{\rho(x,\theta_0)}{2f^{1/2}(x,\theta_0)}\,dx.$$

This conclude the proof.

Acknowledgements

The authors would like to thank the anonymous referees, an Associate Editor and the Editor for their constructive comments that improved the quality of this paper.

References

Alomari, H. M., Frías, M. P., Leonenko, N. N., Ruiz-Medina, M, D, Sakhno, L. and Torres, A. (2017). Asymptotic properties of parameter estimates for random fields with tapered data. *Electron. J. Statist*, **11(2)**, 3332-3367.

Arcones, M. A. (1994). Limit theorems for nonlinear functionals of a stationary gaussian sequence of vectors. *Annals of Probability*, **22**, 2242-2274.

Arcones, M. A. (2000). Distributional limit theorem over stationary Gaussian sequence of random vectors. *Stochastic Processes and their Applications*, **88**, 135-159.

Beran, R. (1977). Minimum Hellinger distance estimates for parametric models. *Annals of Statistics*, **5**, 445-463.

Benson, D. A., Meerschaert, M. M., Baeumer, B. and Scheffler, H. P. (2006). Aquifer operator scaling and the effect on solute mixing and dispersion. *Water Resources Research*, 42(1).

Biermé, H., Richard, F., Rachidi, M. and Benhamou, C. L. (2008). Anisotropic texture modeling and applications to medical image analysis. *ESAIM Proc., EDP Sci., Les Ulis*, **26**,100-122.

Dedecker, J. (2001). Exponential inequalities and functional central limit theorems for random fields. *ESAIM : Probability and Statistics*, **5**, 77-104.

Dobrushin, R. L. and Major, P. (1979). Non-central limit theorems for non-linear functionals of gaussian fields. *Z. Wahr. Verw. Gebiete*, **50**, 27-52.

Hili, O. (1995). On the estimation of nonlinear time series models. *Stochastics and Stochastic Reports* **52**, 207-226.

Ho, H. C and Sun, T. C., (1990). Limit distributions of non-linear vector functions of stationary Gaussian processes. *Annals of Probability*, **18**, 1159-1173.

Lavancier, F. (2006). Processus empirique de fonctionnelles de champs gaussiens à longue mémoire. *C. R. Acad. Sci. Paris, Ser. I*, **342**, 345-348.

Lopes, R. and Betrouni, N. (2009). Fractal and multifractal analysis. *Medical image analysis*, **13(4)**, 634-649.

Major, P. (1981). *Multiple Winner-Itô Integrals*, Lecture Notes in Math. 849, Springer, New York.

Major, P. (2019). Non-central limit theorem for non-linear functionals of vector valued Gaussian stationary random fields, *Submitted to Probability Surveys*.

Meerschaert, M. M., Dogan, M., Dam, R. L., Hyndman, D. W. and Benson, D. A. (2013a). Hydraulic conductivity fields : Gaussian or not? *Water resources research*, **49(8)** : 4730-4737.

N'dri, A., Hili, O. and Okou, C. (2019). Hellinger Distance Estimation of Strongly Dependent Gaussian random fields. *Afrika Statistika*, **14**, 1937-1950.

Taqqu, M. (1977). Law of the Iterated Logarithm for Sums of Non-Linear Functions of Gaussian Variables that Exhibit a Long Range Dependence. *Z. Wahr. Verw. Gebiete*, **40**, 203-238.