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The alpha power Teissier distribution and its applications

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Abstract. Statistical distribution that represents the true characteristics of real-life data is paramount to data analysis. Thus, this study introduces a tractable alpha power Teissier distribution (*APOT*). Some statistical properties of the proposed model like moments, probability generating function, moment generating function and order statistic were examined. The shape of the hazard rate and survival functions were investigated. The shapes of the hazard rate function indicated increasing, decreasing, J-shaped and bathtub shapes. A Monte Carlo simulation and a real-life data analysis were conduction on the proposed model to examine the flexibility and performance of the new model. The results of the data analysis indicated that the *APOT* model performed better when compared to some existing classical statistical distributions.

Key words: alpha power distribution; Gompertz distribution; maximum likelihood function; moments; Teissier distribution
AMS 2010 Mathematics Subject Classification Objects : 62E10; 62E15.

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Résumé. (Abstract in French) La distribution statistique qui représente les caractéristiques réelles des données de survie réelle primordiale à l'analyse des données. Ainsi, cette étude introduit une distribution traçable de Teissier de puissance alpha (*APOT* en Anglais). Certaines propriétés statistiques du modèle proposé, comme les moments, la fonction génératrice de probabilités, la fonction génératrice de moments et les statistiques d'ordre, ont été examinées. La forme du taux de risque et les fonctions de survie ont été étudiées. Les formes de la fonction de taux de risque indiquaient des formes croissantes, décroissantes, en forme de J et de baignoire. Une simulation de Monte Carlo et une analyse de données réelles ont été conduites sur le modèle proposé pour examiner la flexibilité et les performances du nouveau modèle. Les résultats de l'analyse des données ont indiqué que le modèle *APOT* est plus performant que certaines distributions statistiques classiques existantes.

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1. Introduction

It is intuitive that statistical distribution has received tremendous attentions over the years. However, the flexibility, tractability and applicability of such statistical distribution become more paramount in data analysis. Hence, more tremendous attentions are given to the distributions that represent the true characteristics of the data set. These true characteristics will enhance a reliable inference about the given data set. One of such important statistical distributions is the parsimonious Teissier distribution.

The Teissier distribution is an extension of the Gompertz distribution proposed by a French biologist in Teissier(1934). However, the Teissier distribution has a great deal of applications in hydrology, failure rate of computer codes, lifespan, economics and actuarial sciences because of its exponentially monotonic increasing failure rate. Thus, to improve the Teissier distribution flexibility and applicability, statistical researchers in distribution and probability theory have either added an additional parameter or extend its parameters. Hence, this study seeks to improve the tractability and flexibility of the Teissier model by using the alpha power characterization.

Stochastic processes vary from one unit to another as a result of non-stationarity in lifetime activities. Thus, there is need to modify the statistical distribution associated with them; so that they can reflect their true nature. Hence, several classical distributions have been modified using the alpha power characterization of Mahdavi and Kundu(2017). For examples, the alpha power Gompertz by Eghwerido *et al.*(2021), Marshall-Olkin Sujatha distribution by Agu and Eghwerido(2021b), alpha power Weibull Frechet by Burton *et al.*(2020), the alpha power inverted exponential by Unal *et al.*(2018), exponentiated Teissier distribution by Sharma *et al.*(2020), Weibull alpha power inverted exponential distribution by Eghwerido *et al.*(2020), alpha power Marshall-Olkin-G by Eghwerido *et al.*(2021b), transmuted alpha power-G by Eghwerido *et al.*(2020b), Gompertz alpha power inverted exponential by Eghwerido *et al.*(2020a), Kumaraswamy Alpha power inverted exponential distribution by Zelibe *et al.*(2019), alpha power Weibull distribution by Nassar *et al.*(2017), alpha power transformed generalized exponential distribution by Nadarajah and Okorie(2017), alpha power shifted exponential by Eghwerido *et al.*(2021d), Marshall–Olkin alpha power family of distributions by Nassar *et al.*(2017a), Topp-Leone Gompertz distribution by Nzei *et al.*(2020), Weibull Frechet distribution by Afify *et al.*(2016), Type 11 Topp-Leone Generalized Power Ishita distribution by Agu *et al.*(2020c), Inverse odd Weibull generated family of distributions by Eghwerido *et al.*(2020c), Zubair Gompertz distribution by Eghwerido *et al.*(2021a), Gompertz extended generalized exponential distribution by Eghwerido *et al.*(2020d), quasi Xgamma-Poisson distribution by Sen *et al.*(2019), Agu-E Distribution by Burton *et al.*(1986), and a two parameter exponential distribution based on progressive type II censored data by Belaghi *et al.*(2015).

Let X be a random variable. Then, the Teissier probability density function (*pdf*) is characterized by

$$f(x) = \beta\Big(e^{\beta x} - 1\Big) exp\Big(\beta x - e^{\beta x} + 1\Big)\ 1_{(x\geq 0)}, \qquad \beta\ > 0, \tag{1}$$

with a cumulative density function (*cdf*) given as

$$F(x) = \Big(1 - exp\Big(\beta x - e^{\beta x} + 1\Big)\Big)\ 1_{(x\geq 0)}, \qquad \beta\ > 0. \tag{2}$$

However, Mahdavi and Kundu(2017) defined the *cdf* of the alpha power transformation as

$$G(x) = \begin{cases} (\alpha - 1)^{-1}(\alpha^{F(x)} - 1)\ 1_{(x\geq 0)}, & \text{if } \alpha \in (\mathbb{R}^+ - 1) \\ \\ F(x), & \text{otherwise} \quad \alpha = 1. \end{cases} \tag{3}$$

The corresponding *pdf* is given as

$$g(x) = \begin{cases} \frac{f(x)\log\alpha}{(\alpha-1)}\alpha^{F(x)}\ 1_{(x\geq 0)}, & \text{if } \alpha \in (\mathbb{R}^+ - 1) \\ \\ f(x), & \text{otherwise} \quad \alpha = 1. \end{cases} \tag{4}$$

The motivation of this study is to improve the tractability and flexibility of the Teissier distribution by using the alpha power characterization with a bathtub, increasing, decreasing hazard rate functions whose characteristics combined Weibull, exponential and Gompertz distributions.

The aim of this article is to introduce a class of statistical distribution called *APOT* for improving the flexibility and goodness-of-fit of continuous real life data sets in data analysis.

2. *APOT* distribution

Let X be a random variable in the real domain. Then, the *pdf* and *cdf* of the *APOT* distribution is expressed as

$$g(x) = \begin{cases} \frac{\beta(e^{\beta x}-1)exp(\beta x - e^{\beta x}+1)\log\alpha}{(\alpha-1)}\alpha^{1-exp(\beta x-e^{\beta x}+1)} \ 1_{(x\geq 0)}, & \text{if } \alpha \in (\mathbb{R}^{+}-1) \\ \\ f(x), & \text{otherwise } \alpha = 1. \end{cases} \tag{5}$$

and

$$G(x) = \begin{cases} (\alpha-1)^{-1}(\alpha^{1-exp(\beta x-e^{\beta x}+1)}-1), & \text{if } \alpha \in (\mathbb{R}^{+}-1) \ \ 1_{(x\geq 0)} \\ \\ F(x), & \text{otherwise } \ \alpha = 1. \end{cases} \tag{6}$$

3. Mathematical characteristics and parameter estimation

3.1. The APOT quantile function

The quantile function of the *APOT* function for a uniform $u \in (0,1)$ can be obtained using the Lambert W function $W(x) = exp(W(x)) = x$. Thus, it is given as

$$x_u = \frac{1}{\beta}\log\left[-W_{-1}\left(\frac{1-(\log\alpha)^{-1}\log[u(\alpha-1)+1]}{e}\right)\right]. \tag{7}$$

3.2. APOT model reliability

The *APOT* model reliability function is expressed as

$$S(x) = \begin{cases} 1-(\alpha-1)^{-1}(\alpha^{1-exp(\beta x-e^{\beta x}+1)}-1), & \text{if } \alpha \in (\mathbb{R}^{+}-1) \\ \\ 1-F(x), & \text{otherwise } \ \alpha = 1. \end{cases} \tag{8}$$

3.3. APOT model hazard rate function

The *APOT* model hazard rate function is expressed as

$$h(x) = \begin{cases} \frac{\frac{\beta(e^{\beta x}-1)exp(\beta x - e^{\beta x}+1)\log\alpha}{(\alpha-1)}\alpha^{1-exp(\beta x - e^{\beta x}+1)}}{1-(\alpha-1)^{-1}(\alpha^{1-exp(\beta x-e^{\beta x}+1)}-1)}, & \text{if } \alpha \in (\mathbb{R}^+ - 1) \\ \frac{f(x)}{1-F(x)}, & \text{otherwise } \alpha = 1. \end{cases} \tag{9}$$

Figure 1 shows the *pdf*, *cdf* and hazard rate function of the *APOT* model. The graph of *APOT* hazard rate function is increasing, J-shape and bathtub. Also, the shapes of the *APOT pdf* is decreasing and increasing.

3.4. Order statistics and Entropies

Let $X_1, X_2, \ldots, X_n$ be APOT random sample of size n and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ the order statistics of the processes. Then, the *pdf* of the k^{th} order statistic $X_{(k)}$, say $g_k(x)$ is defined as

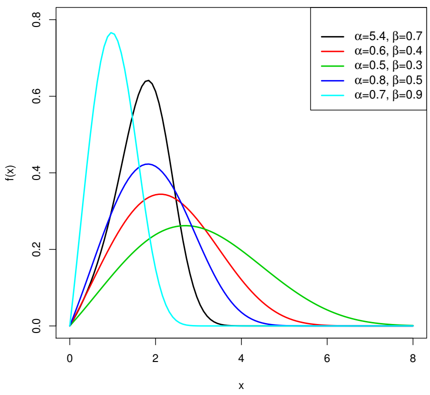
$$\begin{aligned} g_k(x) =& \frac{n(n-1)!}{(n-k)(n-k-1)!(k-1)!}\left[(\alpha-1)^{-1}(\alpha^{1-exp(\beta x-e^{\beta x}+1)}-1)\right]^{k-1} \\ &\times\Big[1-(\alpha-1)^{-1}(\alpha^{1-exp(\beta x-e^{\beta x}+1)}-1)]^{n-k} \\ &\times\frac{\beta(e^{\beta x}-1)exp(\beta x-e^{\beta x}+1)\log\alpha}{(\alpha-1)}\alpha^{1-exp(\beta x-e^{\beta x}+1)}. \end{aligned} \tag{10}$$

The minimum and maximum order statistics are obtained when $k = 1$ and $k = n$ respectively. When n is odd, then $n = 2m + 1$ and $k = m + 1$, then the distribution of the median is

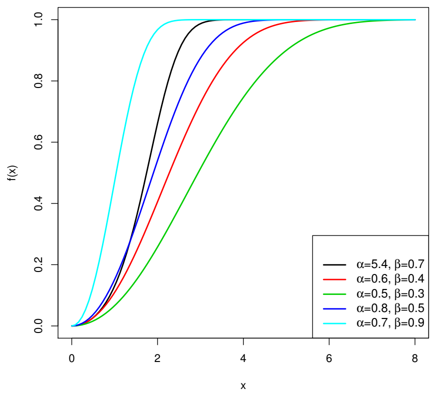
$$\begin{aligned} g_{m+1}(x) =& \frac{2m(2m+1)!}{(m)(m-1)!(m)!}\left[(\alpha-1)^{-1}(\alpha^{1-exp(\beta x-e^{\beta x}+1)}-1)\right]^{m} \\ &\times\Big[1-(\alpha-1)^{-1}(\alpha^{1-exp(\beta x-e^{\beta x}+1)}-1)]^{m} \\ &\times\frac{\beta(e^{\beta x}-1)exp(\beta x-e^{\beta x}+1)\log\alpha}{(\alpha-1)}\alpha^{1-exp(\beta x-e^{\beta x}+1)}. \end{aligned} \tag{11}$$

3.5. Maximum likelihood estimation

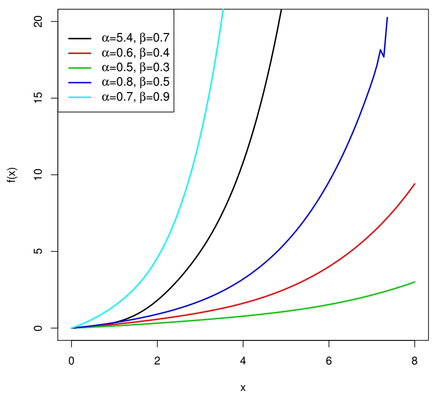
Let ℓ be the $\log$ −likelihood function. Then, the APOT likelihood is given as



(a) APOT pdf



(b) APOT cdf



(c) APOT hazard rate function

Fig. 1. The plots of *APOT* for different parameter values

$$\begin{aligned}\ell =& n\log\beta + \sum_{i=1}^{n}\log(exp(\beta x_i)-1) + \sum_{i=1}^{n}(\beta x_i - exp(\beta x_i)+1) - n\log(\alpha-1) \\ &+ n\log(\log(\alpha)) + \sum_{i=1}^{n}(1-exp(\beta x_i + 1 - exp(\beta x_i)))\log\alpha.\end{aligned} \tag{12}$$

Taking the partial derivative of Equation (12) with respect to the parameters and equating to zero, we have

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha \log \alpha} - \frac{n}{\alpha - 1} + \frac{1}{\alpha}\sum_{i=1}^{n}(1 - exp(\beta x_i + 1 - exp(\beta x_i))) = 0. \tag{13}$$

We also have

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\frac{z'_{i\beta}}{z_\beta} + \sum_{i=1}^{n}\frac{p'_{i\beta}}{p_\beta} + \sum_{i=1}^{n}\frac{a'_{i\beta}}{a_\beta} = 0, \tag{14}$$

where $z_i = exp(\beta x_i) - 1$, $p_i = \beta x - exp(\beta x_i) + 1$, and $a_i = 1 - exp(\beta x - exp(\beta x_i) + 1)$. Equations (13) and (14) are nonlinear. Thus, the unknown parameters can be obtained using the Newton-Raphson numerical algorithm in R, *Mathematica* and *Maple* .

3.6. APOT model Simulation study

The tractability and the performance of the APOT model was tested by a simulation. The simulation was carried out using the quantile function. Table 1 shows the simulation results. Data were generated using the quantile function of the APOT model, and each sample size was replicated 6000 times with 15, 25, 50, 100, 150, 200, 250, 300, 350, 400, 450 and 500 sample sizes. The model parameters were simulated with $\hat{\alpha} = 1.5, \hat{\beta} = 2.0$ respectively.

In Table 1, the performance of the proposed model was examined. The mean estimated value tends to the true values in all cases considered. More so, the mean squared errors decreases as the sample size increases.

Table 1. Monte Carlo simulation results for mean estimates (ME), biases and mean squared errors (MSE) for the APOT Model.

n	ME	Bias	MSE
05	1.7289, 1.7491	0.2289, 0.2491	0.1725, 0.3702
10	1.5868, 1.8260	0.0868, 0.3260	0.0712, 0.2674
15	1.5385, 1.8542	0.0385, 0.3542	0.0445 0.2443
25	1.5097, 1.8495	0.0097, 0.3495	0.0290 0.2024
50	1.4879, 1.8428	-0.0121, 0.3428	0.0162, 0.1398
100	1.4815, 1.8595	-0.0185, 0.3595	0.0086, 0.0981
150	1.4809, 1.8849	-0.0191, 0.3849	0.0061, 0.0860
200	1.4794, 1.8987	-0.0206, 0.3987	0.0049, 0.0770
250	1.4785, 1.9010	-0.0215, 0.4010	0.0040, 0.0688
300	1.4781, 1.9112	-0.0219, 0.4112	0.0036, 0.0616
350	1.4779, 1.9157	-0.0221, 0.4157	0.0032, 0.0596
400	1.4778, 1.9186	-0.0222, 0.4186	0.0029, 0.0569
450	1.4777, 1.9220	-0.0228, 0.4220	0.0027, 0.0558
500	1.4775, 1.9233	-0.0229 0.4233	0.0025, 0.0550

4. Real life analysis

In this section, two sets of real life data were used to show the performance of the APOT distribution. The results were obtained in this section with R PROGRAM. The goodness-of-fit model of the APOT model was compared with some existing models in classical distributions using the Akaike Information Criteria (AIC), p-value and the Kolmogorov-Smirnov (KS) statistic. The APOT distribution was compared with the Kumaraswamy Gompertz (KG), transmuted generalized Gompertz (TGG), Weibull Gompertz (WG), alpha power Gompertz (APG), Topp-Leone Gompertz (TLG), Gompertz (G), alpha power Weibull (APW), Gompertz Weibull (GW), alpha power inverted Weibull (APIW), Kumaraswamy Weibull (KW), transmuted Weibull (TW), Topp-Leone Weibull (TLW), Weibull (W), Topp-Leone Frechet (TLF), Weibull Frechet (WF), Kumaraswamy Frechet (KF), gamma extended Frechet (GF), Gompertz Frechet (GF) , beta Frechet (BF), Topp-Leone Frechet (TLF), transmuted power function (TPs), Lindley Poisson (LP), transmuted Pareto (TP), Bur XII negative binomial (BNB), exponential generalized Frechet (EGF), Weibull Marshall-Olkin (WMO-W), betanMarshall-Olkin Weibull (BMO-W), Mc-Donald Weibull (MC-W), generalized Kumaraswamy Weibull (GKW-W), and Frechet (F) distributions.

The first data consist of stress-rupture life of kevlar 49/epoxy strands that were subjected to constant sustained pressure at the 90 percent stress level until all strands had failed such that the failure times of the complete data were obtained. The data were studied by Korkmaz *et al.*(2018), Haq *et al.*(2016), Cooray and Ananda(2008), Sen *et al.*(2019) and Paranaiba *et al.*(2013). The data were given as:
1.8, 1.8, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.2, 4.69, 7.89, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.2, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.4, 0.42, 0.43, 0.52, 0.54, 0.56, 0.6, 0.6, 0.63, 0.65, 0.67, 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07,0.08, 0.09, 0.09, 0.1, 0.1, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.8, 0.8, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 0.83, 0.85, 0.9, 0.92, 0.95, 0.99, 1, 1.01, 1.02, 1.03, 1.05, 1.1, 1.1, 1.11, 1.15, 1.18, 1.2, 1.29, 1.31, 1.33, 1.34, 1.4, 1.43, 1.45, 1.5, 1.51, 1.52, 1.53.
The results of the test statistics are shown in Table 2. Figure 2 shows the empirical densities, and cdfs with the Stress-rupture life data set for some models.

Table 2: The performance rating with Stress-rupture life dataset with standard errors in parentheses

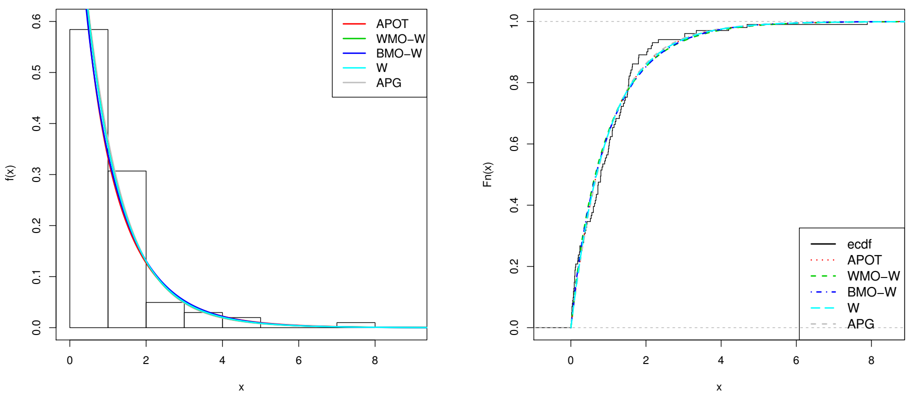
Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value
APOT	$\hat{\alpha} = 8.31(2.38)$ $\hat{\beta} = 2.11(0.13)$	158.6	158.7	163.8	160.7	0.23	1.93	0.92
WMO-W	$\hat{\gamma} = 6.52(2.47)$ $\hat{\alpha} = 0.52(0.04)$ $\hat{\beta} = 2.29(0.01)$	202.8	203.2	213.1	206.9	0.23	1.28	0.87

Table 2 – *Continued from previous page*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value
	$\hat{\lambda} = 1.33(0.01)$							
BMO-W	$\hat{\gamma} = 0.81(0.35)$ $\hat{\alpha} = 0.78(0.67)$ $\hat{\beta} = 4.88(5.31)$ $\hat{\lambda} = 2.26(1.06)$ $\hat{\sigma} = 0.76(0.29)$	209.3	209.9	222.2	214.5	0.20	1.15	0.72
W	$\hat{\beta} = 0.99(0.11)$ $\hat{\lambda} = 0.94(0.07)$	206.2	206.3	211.4	208.3	0.19	1.10	0.37
GKW-W	$\hat{\gamma} = 3.02(4.19)$ $\hat{\alpha} = 2.10(1.76)$ $\hat{\beta} = 0.93(0.09)$ $\hat{\lambda} = 0.36(1.79)$ $\hat{\sigma} = 0.36(0.22)$	209.0	209.7	222.0	214.3	0.15	0.93	0.74
MC-W	$\hat{\gamma} = 0.30(0.27)$ $\hat{\alpha} = 4.32(5.37)$ $\hat{\beta} = 4.37(5.66)$ $\hat{\lambda} = 0.66(0.66)$ $\hat{\sigma} = 0.67(0.37)$	211.0	211.6	223.9	216.2	0.20	1.15	0.46
GFr	$\hat{\alpha} = 0.33(0.77)$ $\hat{\beta} = 3.44(3.86)$ $\hat{a} = 0.35(0.23)$ $\hat{b} = 0.34(1.27)$	212.0	212.5	222.5	216.3	0.18	0.99	0.44
APG	$\hat{a} = 1.47(1.18)$ $\hat{\alpha} = 1.18(0.33)$ $\hat{b} = -0.09(0.11)$	212.2	212.5	220.0	215.4	0.19	1.10	0.40
GW	$\hat{\alpha} = 1.26(60.82)$ $\hat{\beta} = 0.01(0.53)$ $\hat{a} = 0.78(41.23)$ $\hat{b} = 0.92(0.12)$	214.0	214.4	224.4	218.2	0.19	1.10	0.38
TP	$\hat{\alpha} = 0.53(2.45)$ $\hat{\beta} = 48.2(0.00)$ $\hat{\lambda} = 0.90(1.47)$	310.5	311.1	316.2	312.7	0.11	0.72	0.46
LP	$\hat{\beta} = 0.16(0.04)$ $\hat{\lambda} = 2.36(1.12)$	319.9	320.2	323.7	321.4	0.06	0.31	0.16
BNB	$\hat{\gamma} = 93(3416)$ $\hat{\alpha} = 0.0(0.014)$ $\hat{\beta} = 135(0.70)$ $\hat{\lambda} = 1795(7248)$ $\hat{\sigma} = 0.82(0.21)$	311.8	312.8	321.0	315.1	0.08	0.43	0.57
	$\hat{\gamma} = 32.16(0.00)$ $\hat{\alpha} = 2.90(0.00)$							

Table 2 – *Continued from previous page*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value
EGF	$\hat{\beta} = 1049(44.0)$ $\hat{\lambda} = 0.30(0.00)$	317.4	318.3	325,0	320.3	0.22	1.17	0.04
TP	$\hat{\alpha} = 0.25(5.38)$ $\hat{\beta} = -1.00(1.03)$ $\hat{\lambda} = 0.00(0.00)$	373.2	374.2	379.4	375.8	0.70	3.81	0.00



(a) APOT histogram (b) APOT empirical cdfs

Fig. 2. The plots of empirical estimate for stress-rupture life data set

The second data as used in Afify *et al.*(2016), Eghwerido *et al.*(2019), Eghwerido *et al.*(2020), Nzei *et al.*(2020), Eghwerido *et al.*(2021a), Eghwerido *et al.*(2021b), Eghwerido *et al.*(2021) and Zelibe *et al.*(2019) consist of 63 workers at the UK National Physical Laboratory observations of strength of 1.5cm glass fibers in Korkmaz *et al.*(2018), Eghwerido and Agu(2021c) and Smith and Naylor(1987). The results of the test statistics are shown in Table 3.

Table 3: The performance rating with glass fibres dataset with standard errors in parentheses

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value
APOT	$\hat{\alpha} = 7.62(0.07)$ $\hat{\beta} = 1.20(0.08)$	28.51	28.71	32.79	30.19	0.04	0.86	0.45

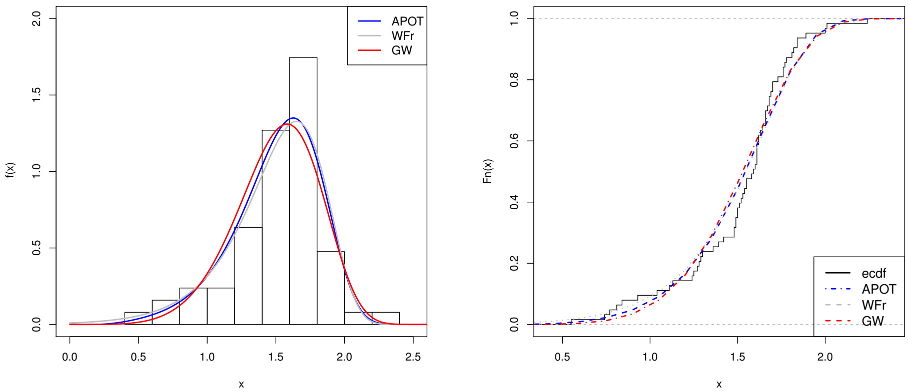
Table 3 – *Continued from previous page*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value
TLG	$\hat{\beta}=0.02(0.02)$ $\hat{\mu}=1.67(0.59)$ $\hat{\lambda}=2.82(0.53)$	34.30	34.71	40.73	36.83	0.17	0.93	0.21
TGG	$\hat{a}=0.85(0.20)$ $\hat{b}=0.02(0.20)$ $\hat{\alpha}=2.95(0.66)$ $\hat{\beta}=1.58(0.54)$	34.85	35.54	43.42	38.22	0.15	0.84	0.11
TG	$\hat{\alpha}=-0.75(0.34)$ $\hat{\beta}=0.06(0.04)$ $\hat{\lambda}=2.93(0.50)$	34.90	35.10	39.19	36.58	0.14	0.76	0.31
WG	$\hat{a}=0.04(0.09)$ $\hat{b}=3.23(1.31)$ $\hat{\alpha}=0.84(0.57)$ $\hat{\beta}=-0.01(0.37)$	36.84	37.53	45.41	40.21	0.19	1.02	0.17
KG	$\hat{a}=1.56(0.43)$ $\hat{b}=0.22(0.05)$ $\hat{\alpha}=0.10(0.02)$ $\hat{\beta}=3.12(0.02)$	37.48	38.17	46.05	40.85	0.18	1.10	0.10
WFr	$\hat{\alpha}=0.40(0.81)$ $\hat{\beta}=0.30(0.30)$ $\hat{a}=1.49(4.77)$ $\hat{b}=16.85(20.48)$	38.80	39.48	47.38	42.17	0.25	1.36	0.09
KFr	$\hat{\alpha}=2.12(4.56)$ $\hat{\beta}=0.74(0.07)$ $\hat{a}=5.51(7.98)$ $\hat{b}=857.35(153.94)$	47.63	48.31	56.18	52.84	0.31	0.57	0.07
EFr	$\hat{\alpha}=7.82(2.95)$ $\hat{\beta}=1.01(0.14)$ $\hat{\mu}=132.83(116.64)$	50.50	50.70	56.70	52.80	0.31	0.58	0.05
TMFr	$\hat{\alpha}=0.66(0.06)$ $\hat{\beta}=0.16(0.34)$ $\hat{a}=6.88(0.61)$ $\hat{b}=376.27(246.84)$	56.51	57.11	65.10	59.81	0.16	1.29	0.02
MFr	$\hat{\beta}=0.17(0.045)$ $\hat{\gamma}=6.48(0.56)$ $\hat{\mu}=161.612(91.50)$	57.11	57.51	63.51	59.61	0.22	2.80	0.02
KW	$\hat{\alpha}=0.55(0.01)$ $\hat{\beta}=0.23(0.01)$ $\hat{a}=0.74(0.01)$ $\hat{b}=7.10(0.01)$	35.413	36.11	43.99	38.79	0.16	0.87	0.61
	$\hat{\alpha}=-0.51(0.28)$							

Table 3 – *Continued from previous page*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A	p-value
TW	$\hat{\beta} = 0.66(0.04)$ $\hat{\lambda} = 5.17(0.68)$	36.69	37.38	45.26	40.06	0.22	1.13	0.55
APW	$\hat{\alpha} = 6.57(8.04)$ $\hat{\beta} = 0.16(0.10)$ $\hat{\lambda} = 4.74(0.82)$	38.19	38.59	44.62	40.72	0.18	0.97	0.32
GW	$\hat{\alpha} = 0.23(0.82)$ $\hat{\beta} = 0.01(0.05)$ $\hat{a} = 0.80(0.52)$ $\hat{b} = 5.62(0.51)$	38.38	39.07	46.95	41.75	0.24	1.29	0.11
APIW	$\hat{\alpha} = 61.03(48.15)$ $\hat{\beta} = 0.79(0.17)$ $\hat{\lambda} = 3.82(0.30)$	82.59	83.00	89.02	85.13	0.99	5.30	0.00

Figure 3 show the empirical densities and *cdf* s with the glass fiber data set for some models.



(a) APOT histogram (b) APOT empirical cdfs

Fig. 3. The plots of empirical estimate for glass data set

In classical statistical distribution, the model with the lowest Akaike Information Criteria (*AIC*) or the highest p-value or Log-likelihood value is regarded as the best model with goodness-of-fit. In the real life cases considered, the *APOT* model has the lowest AIC value in stress data and glass fibres data. Hence, it outperformed many existing classical models for the data used in literature Researched.

5. Conclusion

A two parameter *APOT* distribution has been introduced. The quantile function of *APOT* distribution function was derived using the Lambert W function. The parameter of the *APOT* model was obtained by maximum likelihood method. A Monte Carlo simulation study and a two real life data sets were used to validate the goodness-of-fit and tractability of the *APOT* model. The outcome of the Monte Carlo simulation study, and a real life application indicated that performance and relevance of the goodness-of-fit of the *APOT* model is flexible and tractable with real life data applications compared to some existing classical statistical models.
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