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Asset Liability Management for the Parliamentary Pension Scheme of Uganda by Stochastic Programming

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Abstract. We develop a model for asset liability management of pension funds, which is solved by stochastic programming techniques. Using data provided by the Parliamentary Pension Scheme of Uganda, we obtain the optimal investment policies. Randomly sampled scenario trees using the mean, and covariance structure of the return distribution are used for generating the coefficients of the stochastic program. Liabilities are modelled by remaining years of life expectancy and guaranteed period for monthly pension. We obtain the funding situation of the scheme at each stage under three different asset investment limits.

Key words: asset liability management; stochastic programming; scenario generation; finance.
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Résumé. (Abstract in French) Nous développons un modèle de gestion actif-passif des fonds de pension, qui est résolu par des techniques de programmation stochastiques. En utilisant les données fournies par le Régime de retraite parlementaire de l'Ouganda, nous obtenons les politiques d'investissement optimales. Des arbres de scénario échantillonnés de façon aléatoire à l'aide de la moyenne et de la structure de covariance de la distribution de retour sont utilisés pour générer les coefficients du programme stochastique. Les passifs sont modélisés en se limitant aux années restantes d'espérance de vie et à la période garantie de la pension mensuelle. Nous obtenons la situation de financement du régime à chaque étape sous trois limites d'investissement d'actifs différentes.

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1. Introduction

A pension is a term for single or periodic payments to a beneficiary, which replaces the income of an employee in case of reaching a certain age, or in the case of disability or death. A pension fund is considered to be an organisation, obliged with paying pensions and it has a task of making benefit payments to members who have ended their active working and earning careers. The payments are made to the retirees in accordance to a benefit formula that prescribes the flow of payments to which each member in the fund is entitled. The pension funds planning horizons stretch for several decades, while receiving contributions from active members and paying benefits to retirees. Hence the fund managers have a trade-off between long term gains and fulfilling short-term solvency requirements, while anticipating future policy adjustments. This can be achieved by using stochastic programming with dynamic portfolio allocations.

When modelling optimization problems, the deterministic approach is used, where parameters are known at the time of making the decision, or stochastic programming in which the parameters are uncertain at the time of making the decision. The goal of stochastic programming is to find optimal decision policies in problems involving uncertain data. In this terminology, stochastic is opposed to deterministic, whereas programming refers to the fact that various parts of the problem can be modelled as linear or non-linear mathematical program Birge and Louveaux (2011). We refer readers not familiar with stochastic programming to

Appendix 6.1 for detailed explanations.

Asset Liability Management (ALM) for pension funds is a risk management approach, which takes into account the assets, liabilities, and different policies and regulations. The management of a pension fund should find acceptable policies that guarantee with a large probability that the solvency of the fund is sufficient during the planning horizon, and at the same time, all benefit payouts can be made. Management of assets involves decisions on the investment portfolio while the liability consists of future pension payments Haneveld et al. (2010).

Stochastic programming models have been applied to ALM by Mulvey and Vladimirou (1992); Consigli and Dempster (1998); John et al. (2018), and for a Japanese insurance company by Carino et al. (1994), while Kouwenberg and Vorst (1998) tested scenario generation methods and developed a new stochastic programming ALM model. Other studies include, ALM for a Dutch pension fund using chance constraints by Dert (1995), multi-stage stochastic programming for a Dutch pension fund by Kouwenberg (2001), Bogentoft et al. (2001) studied ALM for Dutch pension fund using Conditional Value at Risk (CVaR) constraints, Haneveld et al. (2010) studied application of ALM with integrated chance constraints for Dutch pension funds, Hussin et al. (2014) studied a two stage stochastic programming model using integrated chance constraints for Employees Provident Fund Malaysia, Dupačová and Polívka (2009) studied ALM for Czech pension funds, Hilli et al. (2007) for a Finnish pension company, Mulvey et al. (2000) for an American pension fund Towers Perrin-Tillinghast, Geyer and Ziemba (2008) for Innovest Austrian pension fund, Bogentoft et al. (2001) studied formal optimal decision approaches for a multi-period asset liability management model for a pension fund, using CVaR as a risk measure, Boender (1997) described a support model to sustain management of pension funds in strategic planning of available asset and liability policy instruments, Yu et al. (2004) applied the grid method to generate scenarios and exercise a dynamic stochastic programming model for bond portfolio management, and Bai and Ma (2009) studied ALM for Chinese DB enterprise pension funds using CVaR constraints.

To the best of my knowledge, no research has been done in Uganda on asset liability management of pension funds by stochastic programming. Some studies on Uganda's social security system include, the status of social security in Uganda by Bukuluki et al. (2016), Kamukama (2015) proposed adoption of a twin peak mechanism in the financial sector, Nzabona et al. (2016) examined social, economic and demographic risk factors, Bogomolova et al. (2006) used PROST to analyse the future liabilities that the Ugandan public service pension scheme might accumulate under the current regulations.

The Ministry of Finance Planning and Economic Development through the Uganda Retirement Benefits Regulatory Authority (*URBRA*), set up the investment limits for all the different retirement benefits schemes in Uganda. However, the different retirement benefit schemes can set their strategic asset allocation limits which

Table 1: Asset investment limits

Assets	*URBRA* (%)		*PPS* (%)		Modified (%)	
	Lower	Upper	Lower	Upper	Lower	Upper
Government securities	0.0	70.0	43.0	60.0	0.0	30.0
Corporate bonds	0.0	15.0	2.5	12.5	2.0	15.0
Fixed deposits	0.0	30.0	7.5	17.5	5.0	40.0
Equities	0.0	85.0	20.0	44.0	5.0	40.0
Loans	0.0	10.0	5.0	10.0	5.0	15.0

should not violet the limits set by *URBRA*. The strategic asset allocation limits for the Parliamentary Pension Scheme (*PPS*) from their policy statement for the period 2018 to 2021 are also given in Table 1. They allow for small short term deviations due to challenges in management of emerging market conditions and exploitation of exceptional opportunities. All the asset allocation limits used in this research are given in Table 1.

The aim of this study is to develop a stochastic programming model for asset liability management of pension funds. As an application, we consider the financial planning problem of the *PPS* of Uganda. We find an optimal investment policy, optimal contribution rates and funding status for the *PPS*. The multi-stage stochastic programming asset liability management model is done for a horizon of 50 years from 2018 to 2068. We use data from the scheme's annual reports and bio-data information. Established abridged mortality tables are used for future expectation of life and survival probabilities.

The remainder of the paper is organised as follows. In Section 2 we formulate the stochastic programming model for asset liability management of pension funds. In Section 3 we present the scenario generation methods for economic factors, liabilities and benefit payments. In Section 4 we present the demographic evolution of the *PPS* scheme members, analyze results from application of our ALM model to the *PPS* and Section 5 gives the conclusion.

2. Stochastic programming ALM model

We formulate a stochastic programming model for the asset liability management of the *PPS*. The decisions are made for a planning horizon of 50 years, from 2018 to 2068. The different stages are indexed by $t = 0, \ldots, T$, with $t = 0$ as the start of the planning horizon and $t = T$ denotes the end of the planning horizon. The model is based on Kouwenberg (2001), the model by Dert (1995) includes chance constraints for solvency of pension funds and this complicates the numerical solutions. Following Carino et al. (1994); Kouwenberg (2001), we penalise deficits in the objective function to avoid computational complications. The model is presented in compact form so that the structure of the scenario tree is not described by a set of constraints but is implicitly incorporated in the model. This necessitates the change of notation from a set of scenarios $s \in \{1, \ldots, S\}$ to the

Table 2: Indices

Index	Description
t	Time, $t = 0, \ldots, T$
i	Asset class, $i = 1, \ldots, I$
n	Node, $n = 1, \ldots, N_t$

Table 3: Random parameters

Parameter	Description
B_{tn}	Benefit payment at node n of stage t
L_{tn}	Liabilities at node n of stage t
S_{tn}	Total salaries of members at node n of stage t
r_{itn}	Return on asset category i at node n of stage t

Table 4: Deterministic parameters

Parameter	Description
X_i^0	Initial amount held in asset i
M_0	Initial cash position
c_L	Minimum contribution rate
c_U	Maximum contribution rate
Δc^L	Lower bound for decrease in contribution rate
Δc^U	Upper bound for increase in contribution rate
F_{min}	Minimum funding ratio
F_T	Funding ratio required at end of planning horizon
w_L	Lower bound on proportion of asset mix
w_U	Upper bound on proportion of asset mix
γ_i^p	Transaction cost incurred in purchasing asset i
γ_i^s	Transaction cost incurred in selling asset i
λ	Risk aversion parameter

nodes of the scenario tree $n \in \{1, \ldots, N_t\}$. A scenario s corresponds to the path from the root node to the terminal node. The realizations of random variables at different stages are represented by the nodes of the scenario tree and N_t denotes the number of nodes of the scenario tree in stage t. If node $n \in \{1, \ldots, N_t\}$ at time t, then its predecessor at time $t-1$ is denoted by $\tilde{n}$.

The asset liability management model is formulated as a linear multi-stage stochastic program. Decisions x_t are taken in time stages $t = 1, \ldots, T$. Hence the asset portfolio is not optimized at the beginning of the horizon. The model is introduced in terms of the objective function and constraints. We define the following indices, variables, random parameters and deterministic parameters.

2.1. Objective

We adopt the objective function in Kouwenberg (2001), which minimizes the overall contribution rate and risk. Risk aversion is modelled by quadratic penalty on the deficits D_{tn}. To ensure solvency of the fund at the end of the planning horizon, the final contribution rate is set to c_{Tn} to achieve the target funding ratio of the pension

Table 5: Decision variables

Variable	Description
X^h_{itn}	Amount held in asset category i at node n of stage t
X^p_{itn}	Amount purchased of asset category i at node n of stage t
X^s_{itn}	Amount sold of asset category i at node n of stage t
A_{tn}	Asset value at node n of stage t
c_{tn}	Contribution rate at node n of stage t
c_{Tn}	Contribution rate at node n of the end of the horizon, required to make the funding ratio F_T
D_{tn}	Deficit relative to the minimum funding ratio at node n of stage t

fund F_T.

$$\min \sum_{t=0}^{T-1} \left(\sum_{n=1}^{N_t} \frac{c_{tn}}{N_t} \right) + \lambda \sum_{t=1}^{T} \left(\sum_{n=1}^{N_t} \frac{1}{N_t} \left(\frac{D_{tn}}{L_{tn}} \right)^2 \right) + \sum_{n=1}^{N_T} \frac{c_{Tn}}{N_T}. \tag{1}$$

In the objective function, λ is the risk aversion penalty parameter, the first term is the sum of average contribution rates for every stage, the second term is the risk aversion, using square of the ratio of deficit to liability and the third term is the average contribution rate at the end of the planning horizon, which measures scheme's condition at the end. The scheme sponsor wishes to minimize his overall contribution over the planning horizon while keeping the fund solvent.

2.2. Asset inventory constraints

These are the constraints that describe the dynamic change in asset investment portfolio at each stage. There is no rebalancing at the end of the horizon.

$$X^h_{i01} = X^0_i + X^p_{i01} - X^s_{i01} \quad \text{for} \quad i = 1, \ldots, I, \tag{2}$$

$$X^h_{itn} = (1 + r_{itn}) X^h_{i,t-1,\tilde{n}} + X^p_{itn} - X^s_{itn} \quad \text{for} \quad n = 1, \ldots, N_t, \ t = 1, \ldots, T-1, \ i = 1, \ldots, I. \tag{3}$$

Equation (2) describes the initial amount invested in each asset at the initial stage when $t = 0$.

2.3. Total asset value

At the end of each stage, the fund measures its total asset value to determine its solvency. The asset value at the end of a given period is the sum of the asset value at the beginning of the period and the returns on each asset during the period.

$$A_{tn} = \sum_{i=1}^{I} (1 + r_{itn}) X^h_{i,t-1,\tilde{n}} \quad \text{for} \quad n = 1, \ldots, N_t, \ t = 1, \ldots, T. \tag{4}$$

2.4. Cash balance constraints

These constraints ensure that the cash inflow into the scheme is equal to cash outflow from the scheme. Cash inflow is due to contributions from the members and the selling of assets. The cash outflow is due to benefit payments to the retirees and purchase of assets. We incorporate the transaction costs incurred in buying and selling of assets on the asset prices. A loan asset is included in the portfolio, and thus we do not need to consider separate variables for borrowing and lending. Ensuring that cash inflow is equal to cash outflow yields the following equations.

$$c_{01}S_{01} + M_0 + \sum_{i=1}^{I}(1-\gamma_i^s)\,X_{i01}^s = B_{01} + \sum_{i=1}^{I}(1+\gamma_i^p)\,X_{i01}^p, \tag{5}$$

$$c_{tn}S_{tn} + \sum_{i=1}^{I}(1-\gamma_i^s)\,X_{itn}^s = B_{tn} + \sum_{i=1}^{I}(1+\gamma_i^p)\,X_{itn}^p \;\; \text{for} \;\; n=1,\dots,N_t,\; t=1,\dots,T-1. \tag{6}$$

2.5. Goal constraints

The minimum funding ratio set by a pension fund becomes its goal. Deficits are registered whenever the funding ratio is less than $F_{\min}$. These deficits are penalized in the objective function. To guarantee that there are no deficits at the end of the planning horizon, we set the contribution rate c_{Tn} which will result in a desired funding ratio F_T at the end of the planning horizon.

$$A_{tn} \geq F_{\min}L_{tn} - D_{tn} \;\; \text{for} \;\; n=1,\dots,N_t,\; t=1,\dots,T, \tag{7}$$

$$A_{Tn} \geq F_T L_{Tn} - c_{Tn}S_{Tn} \;\; \text{for} \;\; n=1,\dots,N_T, \tag{8}$$

$$D_{tn} \geq 0 \;\; \text{for} \;\; t=1,\dots,T,\; n=1,\dots,N_t, \tag{9}$$

$$c_{Tn} \geq 0 \;\; \text{for} \;\; n=1,\dots,N_T. \tag{10}$$

2.6. Short sales constraints

$$X_{itn}^s \leq X_i^0 \;\; \text{for} \;\; i=1,\dots,I,\; t=1,\; n=1,\dots,N_t, \tag{11}$$

$$X_{itn}^s \leq X_{i,t-1,\tilde{n}}^h \;\; \text{for} \;\; i=1,\dots,I,\; n=1,\dots,N_t,\; t=2,\dots,T. \tag{12}$$

We do not consider short sales in this problem, hence amount of assets sold must be less than or equal to the amount of assets held in the previous time period.

2.7. Contribution rate constraints

The level of contribution as well as the change in contribution rates are bounded and specified by the pension fund.

$$c_L \leq c_{tn} \leq c_U \;\; \text{for} \;\; n=1,\dots,N_t,\; t=0,\dots,T-1, \tag{13}$$

$$\Delta c^L \leq c_{tn} - c_{t-1,\tilde{n}} \leq \Delta c^U \;\; \text{for}\; n=1,\dots,N_t,\; t=1,\dots,T-1. \tag{14}$$

2.8. Asset weight mix boundaries

The asset weight mix is bounded through the investment limits, the limits used in the model are given in Table 1.

$$w_L \sum_{i=1}^{I} X_{itn}^h \leq X_{itn}^h \leq w_U \sum_{i=1}^{I} X_{itn}^h \quad \text{for} \quad n = 1, \ldots, N_t, \; t = 0, \ldots, T-1, \; i = 1, \ldots, I. \quad (15)$$

3. Scenario generation

A stochastic programming model requires scenarios of the possible realizations of stochastic elements. The random elements of the model include salaries and returns for all asset classes. Creation of scenario inputs is similar to creation of means, variances, and correlations for a mean-variance model. They are ultimately an expression of the decision maker's probability beliefs. Data on the actual values of the stochastic parameters becomes available in stages, and the decisions at every stage depend on the observations at that particular time and not on the future realizations.

3.1. Economic scenario generation

The asset return scenarios provide information about future asset returns so that we can evaluate possible investment policies for the pension fund. Since ALM focusses on strategic long term decisions, a small set of asset classes is sufficient. Each asset scenario should contain a time series of salary increases, to transform the real expected values of the benefits and liabilities into nominal values.

According to *URBRA*, the investment field of pension funds in Uganda is limited to cash and call deposits, fixed deposits, government securities, equities, real estate and a very small portion not exceeding 5% is allowed for investment in other financial products with good liquidity. The *PPS* invests in five kinds of assets; government securities (treasury bonds and treasury bills), corporate bonds, equities, fixed deposits and loans. We need to generate 6 kinds of economic scenarios, these are the total salaries of the scheme members at the beginning of year t, S_{tn} and return rates of the five kinds of assets r_{itn}, for $i = 1, 2, 3, 4, 5$. The scenarios for liabilities at the end of year t, L_{tn} and benefit payments in year t, B_{tn} will be designed based on the economic scenario generation.

We need to forecast the future distribution and consider correlations among variables, in order to simulate the 6 kinds of economic scenario variables within the planning horizon of 50 years. To model asset returns, we generate the time series using a vector autoregressive model as applied in Boender (1997); Kouwenberg (2001).

$$h_t = \kappa + \Omega h_{t-1} + \epsilon_t, \quad \text{where} \quad \epsilon_t \sim N(0, \Sigma) \quad \text{for} \quad t = 1, \ldots, T, \quad (16)$$

$$h_{it} = \ln(1 + r_{it}) \quad \text{for} \quad t = 1, \ldots, T, \; i = 1, \ldots, I, \quad (17)$$

where I is the number of time series, r_{it} is the discrete rate of return of asset i in stage t. The returns on each asset are transformed to $\ln(1 + r_{it})$ to avoid heteroscedasticity, h_t is a $\{I \times 1\}$ vector of continuously compounded rates, κ is a $\{I \times 1\}$ vector of intercept terms, Ω is the $\{I \times I\}$ matrix of coefficients, $\epsilon_t = (\epsilon_{1t}, \ldots, \epsilon_{It})^T$ is a I dimensional vector of error terms, with $\mathbb{E}(\epsilon_t) = 0,\ \mathbb{E}\left(\epsilon_t \epsilon_t^T\right) = \Sigma$ and $\mathbb{E}\left(\epsilon_s \epsilon_t^T\right) = 0$ for $s \neq t$. The covariance matrix Σ is assumed to be non singular.

To obtain the simulated returns, we incorporate the following relation from Geyer and Ziemba (2008) to adjust simulated returns for the length of each planning period.

$$r_{it} = (1 + \mu_i)^{\tau_t} + p_{it}\sigma_i\sqrt{\tau_t} - 1. \tag{18}$$

In Equation (18), p_{it} is the rate of return produced by the vector autoregressive model, μ_i is the mean return of asset i, σ_i is the standard deviation in the return of asset i, and τ_t is the length of planning period t. We construct the scenario tree with a planning horizon of 50 years, where the length of the planning periods are $3, 5, 10, 10, 10$ and 12, respectively and a branching structure of 1–10–5–5–4–4–2 for $t = 0, 3, 8, 18, 28, 38, 50$ from 2018 to 2068. This tree has 13311 nodes and 8000 scenarios. We use the software MatLab to simulate scenarios, with simulations in node n basing on data in the predecessor node $\tilde{n}$ to obtain all economic scenarios data at all nodes.

3.2. Liabilities and benefit payments scenario generation

Liabilities are the future benefits to be paid to members when they retire, and the value of the liabilities is the present value of the expected benefit payments. From the Parliamentary Pensions Regulations from 2012, commuted benefit depends on the accumulated amount on the member's notional account, while the monthly pensions depend on both the accumulated amount on the member's notional account and the age at retirement.

Consider the pension fund of a member in year j, who has contributed a portion ι of its salary S_j in year j to the scheme for the last $j - \nu$ years, where ν is the year this member joined the scheme. Each year the entire value of the fund, including the previous returns, are re-invested and earn a rate of return ϑ. If the pensionable emolument of this member in the year he starts contributing to the fund is S_ν, and there is a stochastic rate of growth of the annual salary of $g_{\nu+k}$ for each year k, then the salary S_j after $j - \nu$ years is given by

$$S_j = S_\nu \prod_{k=0}^{j-\nu} (1 + g_{\nu+k}). \tag{19}$$

Since the members' contributions are remitted monthly, the interest on funds collected from members in a given year are considered to earn half-year interest while the funds from the previous years earn full yearly interest. After putting all this into consideration, we obtain the value of fund $\mathrm{AF}_{t,\nu}$ for this member in year t as

$$\mathrm{AF}_{t,\nu} = \iota S_\nu \left(1+\frac{1}{2}\vartheta\right) \sum_{i=0}^{t-\nu-1} \left(\prod_{k=0}^{i} (1+g_{\nu+k})\,(1+\vartheta)^{t-\nu-i-1} \right), \tag{20}$$

where $i = 0, 1, \ldots, t-\nu-1$, and we assume no return earned in the last year of contribution.

The monthly pension for the *PPS* is given for life to a retiree, and is guaranteed for a period of $\tau = 15$ years. It is given by

$$MB = \frac{\mathrm{AF}_{t,\nu}}{\mathrm{CAF}} \times 75\% \times \frac{1}{12}, \tag{21}$$

CAF is the expected present value of a conversion of life annuity of 1 per annum, payable monthly at the time of retirement of a member, based on appropriate terms of interest and expense factors. The values of CAF for retirees at different ages used by the *PPS* are given in Table 18 of Appendix 6.3. Since the model developed will not treat men and women separately, we use historical data on composition of the scheme to obtain the values of CAF to use as a weighted average of the values given in Table 18. The women are given a weight of 35% to obtain weighted CAF values in Table 19 of Appendix 6.3.

We convert the monthly pension to annual benefit in Equation (22), which we use in calculations that follow.

$$AB = MB \times 12. \tag{22}$$

The commuted benefit at the time of retirement is

$$CB = \mathrm{AF}_{t,\nu} \times 25\%. \tag{23}$$

The leavers benefit in year t is given by

$$LB_t = NL_t \times \overline{AF}_{L,t}, \tag{24}$$

where NL_t is the number of leavers in year t and $\overline{AF}_{L,t}$ is the average value of the leavers accumulated funds in year t. The death benefit in year t is given by

$$DB_t = ND_t \times \overline{AF}_{D,t}, \tag{25}$$

where ND_t is the number of members dying in year t, and $\overline{AF}_{D,t}$ is the average value of accumulated funds for members dying in the year.

Using Equations (22)–(25), the total benefit payouts B_t in year t is given by

$$B_t = NR_t \times CB + NO_t \times AB + LB_t + DB_t, \tag{26}$$

where NR_t is the total number of members retiring in year t and NO_t is the total number of old age pensioners in the same year. Basing on data provided from the life tables and economic scenarios data at each node, we calculate scenario data at each node for benefit payments B_{tn}.

The total expected commuted benefit in year t for members of age j is given by

$$CB_{tj} = P_t^{r_{\text{age}}-j} \times n_j \times \overline{AF}_{tj} \times 25\%, \tag{27}$$

where $P_t^{r_{\text{age}}-j}$ is the probability that a member aged j years in year t lives for $r_{\text{age}} - j$ more years until the retirement age of r_{age} years, n_j is the number of members aged j years in year t and $\overline{AF}_{tj}$ is the average value of fund for members of age j in year t. The probabilities are given in Table 16 in Appendix 6.2.

The total expected yearly benefits in year t for members aged j years with a guaranteed period of τ years after retirement is

$$AB_{tj} = P_t^{r_{\text{age}}-j} \times n_j \times \overline{AF}_{tj} \times 75\% \times \left(\tau + P_t^{r_{\text{age}}+\tau-j} \times \mathbb{E}P_{(r_{\text{age}}+\tau)_{tj}}\right), \tag{28}$$

where $P_t^{r_{\text{age}}+\tau-j}$ is the probability that a member aged j years in year t lives for $r_{\text{age}} + \tau - j$ more years after retirement age of r_{age} years, and $\mathbb{E}P_{(r_{\text{age}}+\tau)_{tj}}$ is the expected remaining life expectancy in year t for a member aged j years, when he reaches the age of $r_{\text{age}} + \tau$ years, these are shown in Table 17 of Appendix 6.2.

Total expected benefit in year t for members aged j years is

$$B_{tj} = CB_{tj} + AB_{tj}.$$

The liability at time t is the discounted present value of expected total benefit. The total liability in year t is hence given by

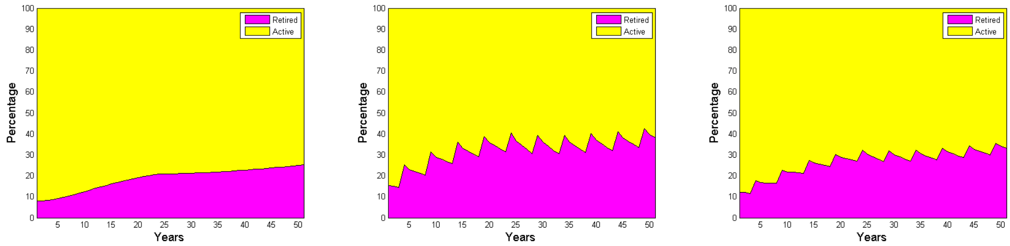
$$L_t = \sum_{j=j_0}^{r_{\text{age}}-1} \frac{B_{tj}}{(1+r)^{r_{\text{age}}-j}}, \tag{29}$$

where j_0 is the minimum age of the active members, and r is a discounting factor. Basing on data provided from the life tables and economic scenarios data at each node, we calculate scenario data at each node for liabilities L_{tn}.

4. Numerical results

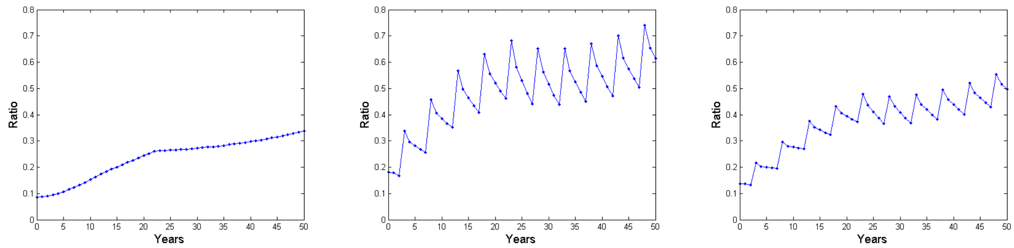
The future demographic status of the fund members in the different categories is modelled by a Markov model which uses state transition probabilities. The relative composition of active and retired members and the resulting dependence ratios increase over the horizon, as shown in Figures 1 and 2, respectively. This is due to an increase in number of retirees as more members retire and as mortality reduces over the horizon.

There is a gradual growth in staff total population, as shown in Figure 3. The population of active members experiences a very slow growth, caused by a small



(a) Staff (b) MPs (c) Combined

Fig. 1: Percentage composition of active and retired members



(a) Staff (b) MPs (c) Combined

Fig. 2: *PPS* dependence ratios

difference between number of new members and that of retiring members. The population of retired and old age members increases gradually for the first 22 years and then slows down for the remainder of the horizon. After 22 years, the initial surviving retirees and those from succeeding years are in advanced ages. Even though mortality reduces, at advanced ages there is a higher risk of death. The population of new members remains stable on the horizon, as the scheme matures there is little expansion in the work force. Leaving members stabilize to between 4 and 5 per year on the horizon, since only a handful of members leave the high paying jobs as civil servants. The number of retiring members each year grows fast for the first ten years and then more gradually for the rest of the horizon, because of the stabilization in aggregate age states after ten years.

There is a moderate growth in population of Members of Parliament (MPs) as shown in Figure 4. The active members' population experiences a gradual growth, caused by creation of new elective positions. The MPs populations are affected by dynamic political cycles, caused by regular elections after every 5 years. There is a moderate growth in pensioners' population during election years, since those who loose their seats either retire or leave the scheme. Those retiring or leaving in non-election years are insignificant, hence pensioners' population gradually reduces due to deaths of some pensioners.

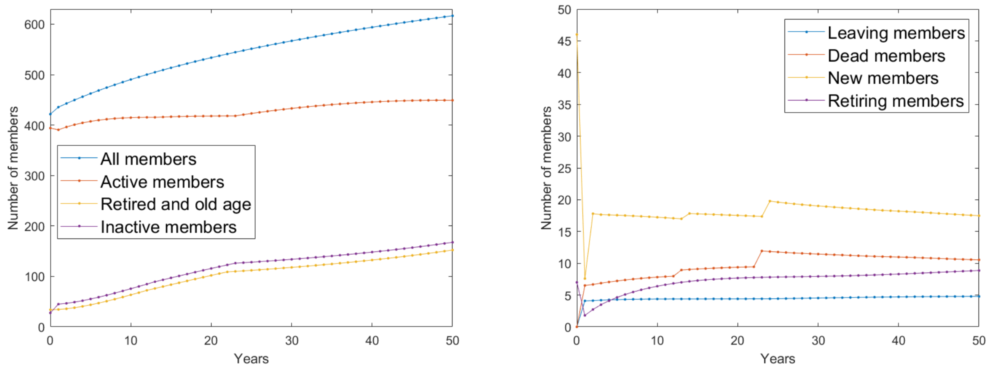


Fig. 3: Evolution of fund population dynamics for staff (note the different scales)

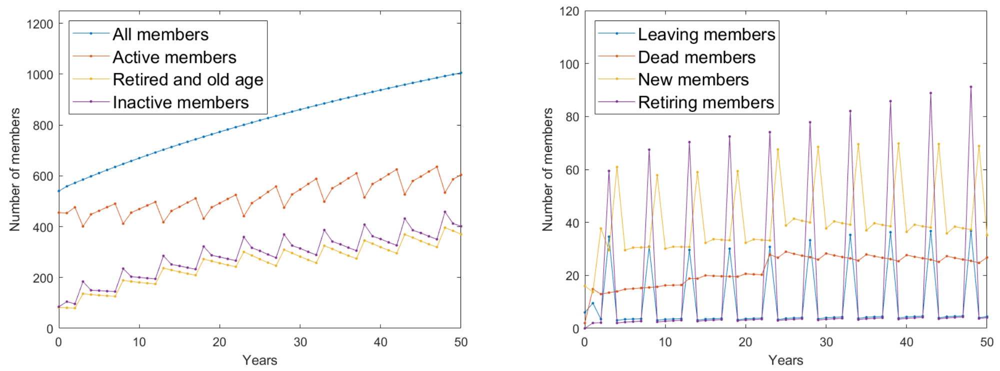


Fig. 4: Evolution of fund population dynamics for MPs (note the different scales)

The assets considered in this study are government securities (Gs), corporate bonds (Cb), fixed deposits (Fd), equities (Eq) and loans (Lo). Each scenario includes salary (Sa) growth, which is based on salary growth from historical data. We use data from the scheme's annual reports about the total asset returns and general salary increase from 2010 to 2018 to estimate the coefficients of the VAR model. The descriptive statistics of the time series are given in Table 6, and Table 7 gives the correlation matrix.

In specifying the VAR model, we do not use lagged terms in modelling returns of government securities, corporate bonds, equities and loans as shown in Table 8. This is done to avoid having unstable and predictable returns. We model return on fixed deposits and rate of salary increase by a first order autoregressive model, since the two have memory of the previous value. We estimate the parameters of the VAR model using iterative least squares minimization as discussed in Drijver

Table 6: Statistics, time series 2010–2018

	Mean	St. Dev
Sa	0.0564	0.0436
Gs	0.1306	0.0800
Cb	0.0851	0.0351
Fd	0.1184	0.0655
Eq	0.1234	0.1561
Lo	0.0848	0.0045

Table 7: Correlations, annually 2010–2018

	Sa	Gs	Cb	Fd	Eq	Lo
Sa	1					
Gs	-0.4905	1				
Cb	-0.1864	0.3497	1			
Fd	0.0096	-0.1323	-0.1894	1		
Eq	0.0313	-0.5732	-0.4113	-0.0195	1	
Lo	-0.8186	0.3212	0.2998	-0.4002	0.2340	1

Table 8: Coefficients of the VAR model

							R^2
$\ln(1+\text{Sa}_t) =$	0.0489	+	$0.0801\ln(1+\text{Sa}_{t-1})$	+	ϵ_{1t}	$\sigma_{1t} = 0.0442$	0.0060
$\ln(1+\text{Gs}_t) =$	0.1233	+	ϵ_{2t}			$\sigma_{2t} = 0.0753$	0.0005
$\ln(1+\text{Cb}_t) =$	0.0839	+	ϵ_{3t}			$\sigma_{3t} = 0.0334$	0.0020
$\ln(1+\text{Fd}_t) =$	0.1098	+	$0.0159\ln(1+\text{Fd}_{t-1})$	+	ϵ_{4t}	$\sigma_{4t} = 0.0611$	0.0002
$\ln(1+\text{Eq}_t) =$	0.1092	+	ϵ_{5t}			$\sigma_{5t} = 0.1595$	0.0001
$\ln(1+\text{Lo}_t) =$	0.0833	+	ϵ_{6t}			$\sigma_{6t} = 0.0040$	0.0604

(2005). The estimated correlation matrix of the residuals is shown in Table 9. Monte Carlo simulation and Cholesky decomposition are used to generate the scenario tree for the stochastic programming model. Cholesky decomposition is used to preserve the covariance structure of return rates. Future returns are obtained by sampling from the error distributions of the equations estimated in Table 8. The simulated returns are then used in Equation (18) which accounts for the duration of each planning period, thus giving the simulated future returns at each node.

The R^2 value measures the percentage of variation in the values of the dependent variable that can be explained by the variation in the independent variable. The last column of Table 8 gives the R^2 values, a 0.6% R^2 value for Sa means that 0.6% of variation in Sa can be explained by variation in its lagged term and the remaining 99.4% is due to random variability. The rest of the R^2 values show that variability in all the returns is mainly due to random effects.

In Section 2, we developed a stochastic programming model for ALM of pension funds. The stochastic program is based on a scenario tree, which describes the

Table 9: Residual correlations of the VAR model

	Sa	Gs	Cb	Fd	Eq	Lo
Sa	1					
Gs	0.4993	1				
Cb	0.6365	0.4109	1			
Fd	0.0527	0.0684	0.0866	1		
Eq	-0.3775	-0.5774	-0.4443	-0.4551	1	
Lo	0.3895	0.3662	0.1933	0.2536	0.2611	1

Table 10: Parameters

M_0	c_L	c_U	Δc^L	Δc^U	$F_{\min}$	F_T	γ_i^p/γ_i^s	λ	S_{01}
UGX 5.81 bn	0.30	0.75	-0.05	0.05	0.80	1.00	0.005	4	UGX 80.90 bn

Table 11: Objective and terminal contribution

Limits	*URBRA*	*PPS*	Modified
c_{Tn}	0.0497	4.0447	10.6933
Objective value	3.0940	7.7766	14.6919

return distributions and evolution of the liabilities. In this section, we present results of the solution to the model. The stochastic programming model has been solved with a randomly sampled scenario tree as input. The size of the model formulated as a compact linear programming problem is 210007 constraints, 252906 variables and 1 objective. The stochastic programming model is solved with AMPL and Cplex. (Here, and in the following, all monetary values are given in billion (bn), Uganda Shillings, UGX.)

The model parameters are displayed in Table 10. The initial contribution rate and initial funding ratio obtained from historical data are 0.45 and 1.06 respectively, initial cash position is UGX 5.81 bn, initial total annual salary is UGX 80.90 bn and the initial asset value is UGX 204.55 bn. The minimum funding ratio is 0.80. Whenever the funding ratio is less than this value, deficits are given a quadratic penalty in the objective with a risk aversion parameter of 4. The target funding ratio is set at 1.00. At the end of the planning horizon, the contribution rate c_{Tn} is paid to lift the funding ratio to the target value of 1.00. The upper and lower bounds on contribution rate are set at 0.30 and 0.75. The decrease and increase in contribution rate are bounded by -0.05 and 0.05. Based on historical data, transaction costs of 0.005 are incurred in buying and selling of assets.

The initial asset mix consists of 62.22% government securities, 0.27% corporate bonds, 9.08% fixed deposits, 22.99% equities and 5.44% loans as shown in Figure 5.
We present the optimal solutions under each of the asset allocation limits given in Table 1. The optimal objective values and terminal contribution rates are given in Table 11.

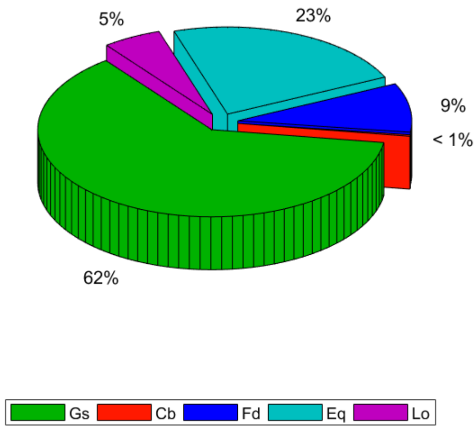


Fig. 5: Initial asset mix

Table 12 gives the information about the optimal solution of the stochastic programming model, using *URBRA* asset allocation limits. The optimal objective value is 3.094 and the optimal investment strategy is shown in Table 12. The allocation to government securities remains constant at its upper limit of 0.7 from stage 2 to stage 5, and then slightly reduces to 0.6928 in stage 6. This results from the government securities having the highest returns with relatively low risk. The remaining 0.3 of the portfolio is shared among fixed deposits and equities from stage 2 to stage 5. In stage 6, leading to the end of the horizon, the financial status of the fund is good. Hence a very small portion of 0.0042 and 0.004 is allocated to corporate bonds and loans respectively, which have much lower return but also very low risk. The share for equity reduces from 0.2411 in stage 2 to 0.1436 in stage 6. Although equities give higher returns than fixed deposits, they are more risky. Hence the the allocation to fixed deposits increases from 0.0589 in stage 2 to 0.1554 in stage 6, in a way that minimizes risk of underfunding towards the end of the horizon. There is no allocation to corporate bonds and loans from stage 2 to stage 5, because their returns are very low compared to the other assets.

Table 13 gives the information about the optimal solution of the stochastic programming model, using *PPS* asset allocation limits. The optimal objective value is 7.7766 and the optimal investment strategy is shown in Table 13. The allocation to government securities remains constant at its upper limit of 0.6 from stage 2 to the end, as earlier explained. The allocations to corporate bonds and loans are constant at their lower limits of 0.025 and 0.05 respectively, because their returns are very low compared to the other assets. The remaining 0.325 of the portfolio is shared among fixed deposits and equities. The share for equity reduces from 0.2463

Table 12: Optimal investment strategy with *URBRA* limits

Variables	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6
c_{tn}	0.4500	0.5	0.5306	0.5028	0.4669	0.4303
X^h_{1tn}	123.65	251.62	463.76	1445.19	4479.13	13744.80
X^h_{2tn}	0.53	0.00	0.00	0.00	0.00	83.60
X^h_{3tn}	18.05	21.16	66.09	275.90	973.54	3082.17
X^h_{4tn}	45.69	86.68	132.67	343.47	946.09	2849.03
X^h_{5tn}	10.82	0.00	0.00	0.00	0.00	79.98
X^p_{1tn}	0.00	12.81	8.55	0.00	0.00	748.32
X^p_{2tn}	0.00	0.00	0.00	0.00	0.00	83.60
X^p_{3tn}	0.00	4.14	32.91	91.80	230.03	679.82
X^p_{4tn}	0.00	14.51	7.90	18.22	93.98	602.92
X^p_{5tn}	0.00	0.00	0.00	0.00	0.00	79.98
X^s_{1tn}	0.00	0.00	0.00	51.30	185.59	1464.28
X^s_{2tn}	0.00	0.00	0.00	0.00	0.00	0.00
X^s_{3tn}	0.00	7.82	2.76	4.43	44.08	377.12
X^s_{4tn}	0.00	1.13	24.83	57.58	136.92	479.64
X^s_{5tn}	0.00	0.00	0.00	0.00	0.00	0.00
w_1	0.6222	0.7000	0.7000	0.7000	0.7000	0.6928
w_2	0.0027	0.0000	0.0000	0.0000	0.0000	0.0042
w_3	0.0908	0.0589	0.0998	0.1336	0.1521	0.1554
w_4	0.2299	0.2411	0.2002	0.1664	0.1479	0.1436
w_5	0.0544	0.0000	0.0000	0.0000	0.0000	0.0040
A_{tn}	204.55	336.94	640.74	2067.84	6441.35	19965.90
D_{tn}	0.00	0.00	123.09	371.45	825.87	781.22

in stage 2 to 0.2327 in stage 6, while the allocation to fixed deposits increases from 0.0787 in stage 2 to 0.0923 in stage 6, as earlier explained.

Table 14 gives the information about the optimal solution of the stochastic programming model, using modified asset allocation limits. The optimal objective value is 14.6919 and the optimal investment strategy is shown in Table 14. The allocation to government securities remains constant at its upper limit of 0.3 from stage 2 to the end. The allocation to corporate bonds and loans are constant at their lower limits of 0.02 and 0.05 respectively, as earlier explained. The remaining 0.63 of the portfolio is shared among fixed deposits and equities. The share for equity reduces from 0.3941 in stage 2 to 0.3522 in stage 6, while the the allocation to fixed deposits increases from 0.2359 in stage 2 to 0.2778 in stage 6, as earlier explained.

The average contribution rate represents the cost of the pension scheme. The risk term is the second downside moment of the funding ratio. The variation in cost and risk terms of the objective at all the stages is shown in Table 15. Costs under *PPS* and modified limits are the same in the first six stages and only differ in the final stage. There is increase in cost by 0.05 in the subsequent stages, which is the maximum increase allowed in the model. This is due to deficits which should be reduced by extra contributions by the sponsor. On the other hand, under the *URBRA* limits, the cost reduces from 0.5 in stage 2 to 0.0497 in the final stage. The government securities which give the highest return are given a big upper bound of 0.7. This allows for high growth rate in asset value, so that the sponsor can reduce its contribution and the fund remains solvent. In the final stage where

Table 13: Optimal investment strategy with *PPS* limits

Variables	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6
c_{tn}	0.45	0.50	0.55	0.60	0.65	0.70
X^h_{1tn}	123.65	213.88	390.10	1194.79	3646.88	11132.80
X^h_{2tn}	0.53	8.91	16.25	49.78	151.95	463.87
X^h_{3tn}	18.05	28.05	52.17	165.17	525.22	1712.69
X^h_{4tn}	45.69	87.80	159.13	482.01	1450.17	4317.59
X^h_{5tn}	10.82	17.82	32.51	99.57	303.91	927.74
X^p_{1tn}	0.00	9.09	3.22	0.00	0.00	0.00
X^p_{2tn}	0.00	1.31	2.87	13.27	40.10	122.51
X^p_{3tn}	0.00	1.63	5.13	18.71	63.59	257.38
X^p_{4tn}	0.00	7.87	7.98	27.00	80.06	240.60
X^p_{5tn}	0.00	2.70	5.91	27.19	82.24	251.14
X^s_{1tn}	0.00	0.00	0.07	64.00	209.63	641.01
X^s_{2tn}	0.00	0.00	0.00	0.00	0.00	0.00
X^s_{3tn}	0.00	0.00	0.61	2.41	9.91	44.20
X^s_{4tn}	0.00	0.00	0.13	2.66	14.96	91.92
X^s_{5tn}	0.00	0.00	0.00	0.00	0.00	0.00
w_1	0.6222	0.6000	0.6000	0.6000	0.6000	0.6000
w_2	0.0027	0.0250	0.0250	0.0250	0.0250	0.0250
w_3	0.0908	0.0787	0.0802	0.0829	0.0864	0.0923
w_4	0.2299	0.2463	0.2448	0.2421	0.2386	0.2327
w_5	0.0544	0.0500	0.0500	0.0500	0.0500	0.0500
A_{tn}	204.55	333.87	625.86	1974.21	6046.65	18460.20
D_{tn}	0.00	0.00	137.97	465.08	1220.58	2286.73

there is no limitation in contribution rate increase, huge contribution rates are required to clear all the accumulated deficits under *PPS* and modified limits.

In stage 1, there is no risk as the fund begins with no deficits. In stage 2, the risk is very low as small deficits begin to emerge. For the rest of the horizon, there is relatively high risk due to increase in deficits under *PPS* and modified limits. Under *URBRA* limits, risk reduces from stage 2 to stage 6 due to high growth rate in asset value.

The variation in contribution rates is as shown in Figure 6, and was explained from the costs in Table 15. In the final stage where there are no restrictions on contribution rate, the sponsor makes the necessary terminal contribution required to attain the target funding ratio $F_T = 1$. The high growth rate in asset value under *URBRA* limits enables the sponsor to reduce its contribution rate on the horizon. The deficits under each of the investment limits are given in Figure 7. The deficits increase from the first stage to stage 6 under all asset allocation limits. Bigger deficits appear under modified limits, where there is a small upper bound on government securities, this is followed by deficits under *PPS* limits. There are also restrictions to ensure that part of the asset is allocated to corporate bonds and loans under *PPS* and modified limits, this results into high deficits. The lowest deficits are observed under *URBRA* limits, where 0.7 of the asset is always invested in government securities, which yield the highest returns at relatively low risk. Deficits are calculated at the beginning of each planning period, before contributions for that period are received. In order to clear deficits in the final stage, we

Table 14: Optimal investment strategy with modified limits

Variables	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6
c_{tn}	0.45	0.50	0.55	0.60	0.65	0.70
X^h_{1tn}	123.65	105.49	189.48	560.49	1651.94	4870.36
X^h_{2tn}	0.53	7.03	12.63	37.37	110.13	324.69
X^h_{3tn}	18.05	82.95	153.53	464.27	1434.17	4510.21
X^h_{4tn}	45.69	138.59	244.36	712.75	2034.90	5717.55
X^h_{5tn}	10.82	17.58	31.58	93.41	275.32	811.73
X^p_{1tn}	0.00	3.32	0.40	0.00	0.00	0.00
X^p_{2tn}	0.00	0.97	2.07	8.99	26.18	77.29
X^p_{3tn}	0.00	3.98	13.46	34.42	131.36	525.96
X^p_{4tn}	0.00	11.84	9.53	25.39	67.72	236.35
X^p_{5tn}	0.00	2.49	5.34	23.11	67.35	198.77
X^s_{1tn}	0.00	0.00	1.78	50.92	157.19	462.87
X^s_{2tn}	0.00	0.00	0.00	0.00	0.00	0.00
X^s_{3tn}	0.00	0.00	0.84	8.24	22.65	110.35
X^s_{4tn}	0.00	0.00	3.94	15.69	81.55	372.33
X^s_{5tn}	0.00	0.00	0.00	0.00	0.00	0.00
w_1	0.6222	0.3000	0.3000	0.3000	0.3000	0.3000
w_2	0.0027	0.0200	0.0200	0.0200	0.0200	0.0200
w_3	0.0908	0.2359	0.2431	0.2485	0.2605	0.2778
w_4	0.2299	0.3941	0.3869	0.3815	0.3695	0.3522
w_5	0.0544	0.0500	0.0500	0.0500	0.0500	0.0500
A_{tn}	204.55	329.04	607.34	1851.24	5475.24	16141.70
D_{tn}	0.00	0.00	156.49	588.05	1791.98	4605.23

Table 15: Variation in cost and risk

Term	Limits	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6	Stage 7
Costs: $\sum_{n=1}^{N_t} \frac{c_{tn}}{N_t}$	*URBRA*	4.50×10^{-1}	5.00×10^{-1}	5.31×10^{-1}	5.03×10^{-1}	4.67×10^{-1}	4.30×10^{-1}	4.97×10^{-2}
	PPS	4.50×10^{-1}	5.00×10^{-1}	5.50×10^{-1}	6.00×10^{-1}	6.50×10^{-1}	7.00×10^{-1}	4.04×10^{0}
	Modified	4.50×10^{-1}	5.00×10^{-1}	5.50×10^{-1}	6.00×10^{-1}	6.50×10^{-1}	7.00×10^{-1}	1.07×10^{1}
Risk: $\sum_{n=1}^{N_t} \frac{1}{N_t} \left(\frac{D_{tn}}{L_{tn}}\right)^2$	*URBRA*	–	1.21×10^{-13}	1.67×10^{-2}	1.49×10^{-2}	8.32×10^{-3}	9.65×10^{-4}	6.08×10^{-11}
	PPS	–	1.17×10^{-11}	2.10×10^{-2}	2.34×10^{-2}	1.81×10^{-2}	7.96×10^{-3}	8.20×10^{-10}
	Modified	–	1.86×10^{-11}	2.70×10^{-2}	3.74×10^{-2}	3.91×10^{-2}	3.18×10^{-2}	1.81×10^{-3}

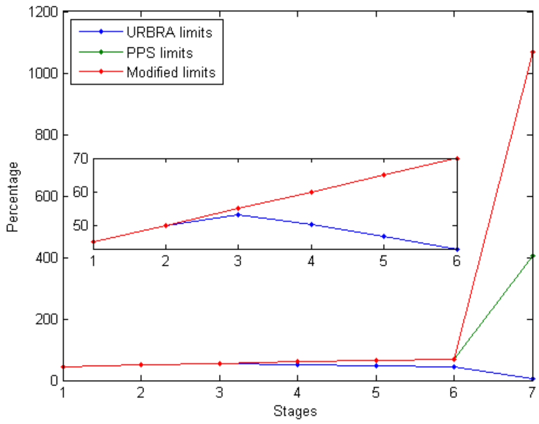


Fig. 6: Average contribution rates

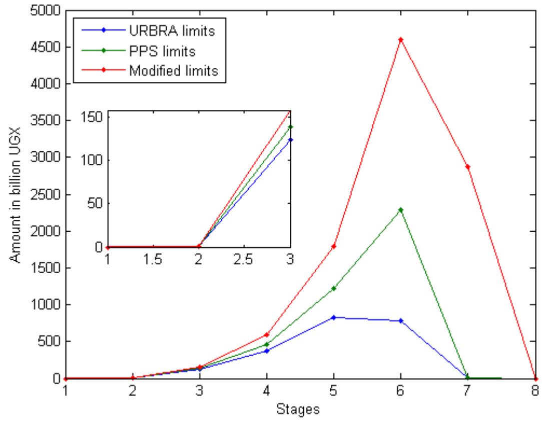


Fig. 7: Average deficits

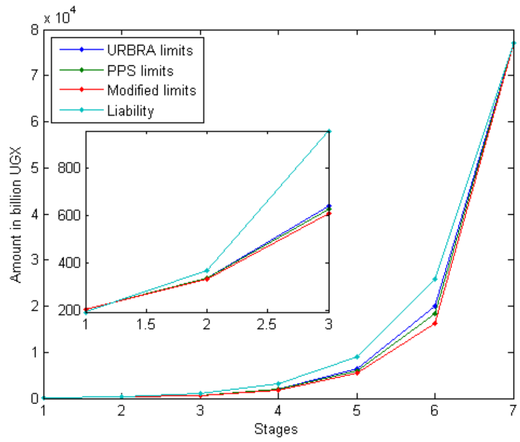


Fig. 8: Average asset and liability values

include additional stage 8. This is used to clear the deficits in stage 7 after final contributions are received.

The variation in average values of assets and liabilities at each stage is shown in Figure 8, in the subsequent stages, the sponsor pays extra contributions to reduce the gap between assets and liabilities. In the final stage, the value of assets and liabilities are equal due to unrestricted contributions made to ensure parity.
The average funding ratios at each stage are shown in Figure 9, and are obtained from the equation

$$\bar{F}_t = \frac{\bar{A}_t}{\bar{L}_t}, \tag{30}$$

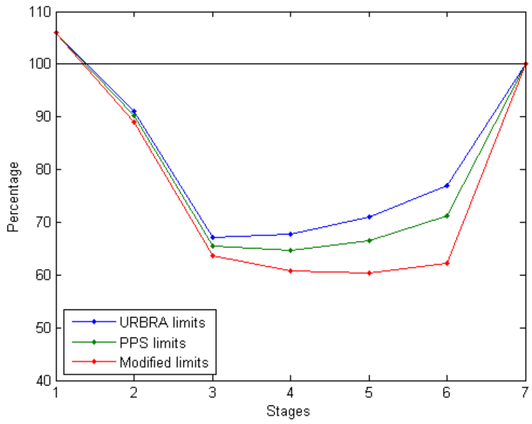


Fig. 9: Average funding ratios

where $\bar{A}_t$ and $\bar{L}_t$ are the average values for assets and liabilities, respectively at stage t. The funding ratios have a fast decline from stage 1 to stage 3, followed by moderate increase under *URBRA* limits, gradual increase under *PPS* limits and gradual decline under modified limits. From stage 6 to stage 7, there is a sharp rise in funding ratios to end at the target value $F_T = 1$ due to unrestricted contribution rates needed to balance assets and liabilities at the end of the final stage.

5. Conclusion

In this paper, we have developed a stochastic programming model for asset liability management of pension funds. We applied the model to the financial planning problem of the *PPS* of Uganda. The model was solved by stochastic programming techniques, to find optimal portfolio allocations and associated costs and risk. The model takes into account the funding situation of the fund at each stage. Randomly sampled scenario trees using the mean, and covariance structure of the return distribution were used for generating the coefficients of the stochastic program. Scenario trees were generated by Monte Carlo simulation. Liabilities were modelled by remaining years of life expectancy and guaranteed period for monthly pension. We calculated the average cost and risk of the stochastic programming policy under three separate asset investment limits, and studied the variation in optimal values of contribution rates, risk, deficits, assets and funding ratios.

Our results suggest that in order to reduce the financial burden of the fund on the sponsor, it is necessary to make reforms regarding benefits indexation, and guaranteed period for pension payment.

Data availability

Data was obtained from the *PPS* and mortality data was obtained from the source cited.

Conflict of interest.

The authors declare that there is no conflict of interests regarding the publication of this paper.

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APPENDICES.

6. Appendices

6.1. Stochastic programming

The Committee on Stochastic Programming (COSP) (2011) describes, stochastic programming as a framework for modelling optimization problems that involve uncertainty. Stochastic programming models allow for progressive revelation of information in time and multiple decision stages, where each decision is adapted to the available information Kouwenberg and Vorst (1998). It is a general framework for modelling optimization problems that involve uncertainty. We explain some important concepts in stochastic programming. Stages are the time periods at which decisions are made, horizon is the number of stages. We assume that at any stage, there is a finite number states of the system where the states are described by the state variables. In stochastic programming the state variables are affected by uncertainty. Starting from the initial state of the system, the goal is to maximize or minimize an objective function of an immediate return for all stages and states Kall et al. (1994).

In stochastic programming, we classify the models depending on the way information is revealed. In anticipative models, feasibility is expressed in terms of probabilistic or chance constraints. Anticipative models select a policy that leads to some desirable characteristics of the constraint, and objective functional under the realizations of the random vector. In adaptive models, observations related to uncertainty become available before decisions are made. The recourse formulation combines the discussed two models in a common mathematical framework, which seeks a policy that not only anticipates future observations but also takes into account temporarily available information to make recourse decisions. Applications of stochastic programming models in asset liability management has been done in Carino et al. (1994); Dert (1995); Zenios (1995); Consigli and Dempster (1998); Mulvey et al. (2000); Kouwenberg (2001); Bogentoft et al. (2001); Drijver (2005); Kouwenberg and Zenios (2008); Dupačová and Polívka (2009); Bai and Ma (2009); Haneveld et al. (2010); Hussin et al. (2014); John et al. (2018).

6.1.1. Two-stage stochastic programming

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be the probability space on which the random vector $\vec{\xi}_t$ is defined. From Ermoliev and Wets (1988), the general formulation for a stochastic program is

$$\begin{aligned} &\min_x \mathbb{E}\psi_0\left(x, \vec{\xi}\right), \\ &\text{subject to } \mathbb{E}\psi_i\left(x, \vec{\xi}\right) \le 0, i = 1, 2, \dots, m, \\ &x \in \chi \subseteq \mathbb{R}^d, \end{aligned} \tag{31}$$

where $\mathbb{E}$ is the expectation operator, ψ_i denotes a mapping $\psi_i \colon \chi \times \Xi \to \mathbb{R}$, $i = 0, 1, \dots, m$ and Ξ is the support of $\vec{\xi}$. Formulation (31) covers a wide range of stochastic programs, including chance constrained and recourse models. We make an assumption that the probability distribution $\mathbb{P}$ of the random variable $\vec{\xi}$ is known and independent of x. Applications of stochastic programming under incomplete information on the distribution of $\mathbb{P}$ is well studied in Dupačová (1994). To ensure that Formulation (31) is well defined, the following assumptions are made:

(i) the expectation of both the objective function and constraints are finite for all $x \in \chi$,
(ii) the feasible region in Formulation (31) is non empty,
(iii) the minimizer of Formulation (31) is achieved on its feasible region.

A two stage stochastic program with recourse can be formulated as

$$\begin{aligned} &\min_{x_1} \mathbb{E}\phi_1\left(x, \vec{\xi}_2\right), \\ &\text{subject to } x_1 \in \chi_1, \end{aligned} \tag{32}$$

where

$$\begin{aligned} &\phi_1\left(x, \xi_2\right) = \min_{x_2} \phi_2\left(x_1, x_2, \xi_2\right), \\ &\text{subject to } x_2 \in \chi_2\left(x_1, \xi_2\right). \end{aligned} \tag{33}$$

Model Formulations (32) and (33) capture the type of dynamics that arises in many real world decision making processes. Specifically the first stage decision x_1 must be made before the realization of the random vector $\vec{\xi}$ is known. After observing the realization of the random vector $\vec{\xi}$, an adaptive or recourse decision x_2 is then made. The associated cost of the decisions x_1 and x_2 is then made. The associated cost of the decisions x_1 and x_2 under realization ξ_2 is given by $\phi_2\left(x_1, x_2, \xi_2\right)$. The requirement that the decision x_1 is made with only distributional knowledge of the random vector $\vec{\xi}$ is known as nonanticipativity in stochastic programming literature. The two stage stochastic program with recourse is a special case of Formulation (31) in which the constraints involving $\psi_i,\ i = 1, 2, \ldots, m$ are not considered, $\chi = \chi_1$ and $\psi_0 = \phi_1$, defined by the second stage program in Formulation (33).

6.1.2. Multi-stage stochastic programming

In multi-stage stochastic programming, we take the decisions x_t at time stages $t = 1, 2, \ldots, T$, where each decision x_t is followed by a random realization. Suppose there are $T \geq 2$ discrete stages and the uncertainty is expressed by the random variable $\xi_1, \xi_2, \ldots, \xi_T$, which is realised gradually over the T stages. The decision process $x_1, x_2, \ldots, x_T$ is adapted to the realizations of the random variables with the form,

$$\text{decision}\left(x_1\right) \rightsquigarrow \text{realization}\left(\xi_1\right) \rightsquigarrow \text{decision}\left(x_2\right) \rightsquigarrow \ldots \text{decision}\left(x_T\right) \rightsquigarrow \text{realization}\left(\xi_T\right).$$

The sequence of realizations ξ_t for each t is viewed as a stochastic process according to Shapiro et al. (2009). Let $\vec{\xi}_t = (\xi_1, \xi_2, \ldots, \xi_t)$ be the history of the process up to time t. The decision x_t at each stage t depends of the history $\vec{\xi}_t$ of the process up to time t, and not on the future realization. This nonanticipativity property is fundamental in stochastic programming. The generic form of a T stage stochastic programming model can be written in nested form as

$$\min_{x_1 \in \chi_1} f_1(x_1) + \mathbb{E}\left[\min_{x_2 \in \chi_2(x_1,\xi_2)} f_2(x_2,\xi_2) + \mathbb{E}\left[\ldots + \mathbb{E}\left[\min_{x_T \in \chi_T(x_{T-1},\xi_T)} f_T(x_T,\xi_T)\right]\right]\right], \quad (34)$$

where $\mathbb{E}$ is the expectation operator, the function $f_1 : \mathbb{R}^1 \longrightarrow \mathbb{R}$ is continuous and deterministic and the set $\chi_1 \subset \mathbb{R}^1$ is deterministic, $x_t \in \mathbb{R}^{n_t}, t = 1, 2, \ldots, T$, are decision variables and $f_t : \mathbb{R}^{n_t} \times \mathbb{R}^{m_t} \longrightarrow \mathbb{R}$ are continuous functions at stages $t = 2, \ldots, T$. The multi-stage problem is linear if the objective functions and the constraint functions are linear. The commonly used formulation in stochastic optimization models is

$$\begin{aligned}
&\min_{x_1,x_2,\ldots,x_T} \mathbb{E}\left[f_1(x_1) + f_2\left(x_2\left(\vec{\xi}_2\right),\xi_2\right) + \ldots + f_T\left(x_T\left(\vec{\xi}_T\right),\xi_T\right)\right],\\
&\text{subject to } x_1 \in \chi_1, \qquad (35)\\
&x_t\left(\vec{\xi}_T\right) \in \chi_t\left(x_{t-1}\left(\vec{\xi}_{t-1}\right),\xi_t\right),\ t = 2,\ldots,T.
\end{aligned}$$

The decision variable $x_t = x_t\left(\vec{\xi}_t\right),\ t = 1, 2, \ldots, T$, is considered as a function of the data process $\vec{\xi}_t$ up to time t.

6.1.3. Scenario tree

In stochastic programming, the uncertainty of parameter values are described by a scenario tree. The nodes in the scenario tree represent states of the world at a particular point in time. In stochastic programming, decisions are made at the nodes. The arcs represent realizations of the uncertain variables. The scenario tree branches off for each possible value of a random vector $\vec{\xi}_t = (\xi_1, \xi_2, \ldots, \xi_t)$ in each stage $t = 1, 2, \ldots, T$. This requires a finite discrete distribution, hence there is a limitation on the number of possible values of the random parameters. The performance of stochastic programming can be improved by selecting an appropriate scenario generation method Kouwenberg (2001). A path through the tree is called a scenario and consists of realizations of all random variables in all time periods. An example of a scenario tree is illustrated in Figure 10.

Scenario trees have probability valuations on both the nodes and on the arcs. A scenario tree has the root node, intermediate nodes and the terminal nodes.

N_1 is the root node,
$N_t,\ t = 2, \ldots, T-1$ are intermediate nodes,
N_T are terminal nodes.

The stages correspond to the time periods at which decisions are made and the horizon refers to the number of stages. It is assumed that at any stage, finitely many states of the system exist, and the states are described by state variables. The state

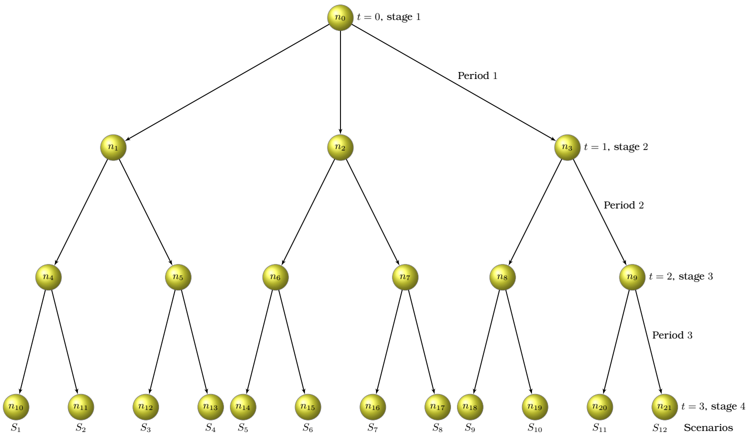

Fig. 10: Scenario tree with four stages

variables are affected by uncertainty in stochastic programming. Given the initial state of the system, the overall objective is to maximize or minimize some objective function of an immediate return for all stages and states. The stages correspond to instances of time when some information is revealed and decisions are made.

6.2. Mortality high income countries 2015–2070

These are given in Tables 16 and 17.

Table 16: Probabilities of dying

Age	15/20	20/25	25/30	30/35	35/40	40/45	45/50	50/55	55/60	60/65	65/70
20	0.00306	0.00273	0.00258	0.00237	0.00221	0.00208	0.00196	0.00184	0.00172	0.00161	0.00151
25	0.00359	0.00350	0.00337	0.00321	0.00303	0.00286	0.00275	0.00266	0.00253	0.00240	0.00226
30	0.00412	0.00412	0.00399	0.00381	0.00365	0.00344	0.00326	0.00314	0.00303	0.00289	0.00273
35	0.00527	0.00515	0.00500	0.00471	0.00445	0.00421	0.00394	0.00371	0.00354	0.00338	0.00320
40	0.00747	0.00725	0.00692	0.00650	0.00610	0.00570	0.00535	0.00499	0.00467	0.00442	0.00418
45	0.01160	0.01103	0.01048	0.00976	0.00922	0.00860	0.00801	0.00751	0.00700	0.00653	0.00616
50	0.01845	0.01715	0.01613	0.01508	0.01418	0.01337	0.01249	0.01165	0.01092	0.01018	0.00950
55	0.02864	0.02644	0.02454	0.02281	0.02157	0.02022	0.01908	0.01781	0.01659	0.01556	0.01452
60	0.04244	0.03965	0.03663	0.03362	0.03168	0.02987	0.02798	0.02637	0.02456	0.02285	0.02145
65	0.05905	0.05599	0.05194	0.04741	0.04410	0.04138	0.03882	0.03630	0.03406	0.03180	0.02941
70	0.09327	0.08797	0.08227	0.07580	0.07069	0.06610	0.06182	0.05813	0.05456	0.05143	0.04762
75	0.15648	0.14885	0.14046	0.13087	0.12326	0.11626	0.10979	0.10399	0.09835	0.09319	0.08745
80	0.25361	0.24350	0.23211	0.21894	0.20846	0.19853	0.18947	0.18097	0.17259	0.16456	0.15663

Table 17: Expectation of life

Age	15/20	20/25	25/30	30/35	35/40	40/45	45/50	50/55	55/60	60/65	65/70
20	61.52	62.07	62.67	63.39	63.97	64.54	65.08	65.61	66.14	66.67	67.17
25	56.70	57.23	57.83	58.53	59.11	59.67	60.20	60.72	61.25	61.77	62.27
30	51.89	52.42	53.02	53.72	54.28	54.83	55.36	55.88	56.40	56.91	57.41
35	47.10	47.63	48.22	48.91	49.47	50.01	50.53	51.04	51.56	52.07	52.56
40	42.33	42.86	43.45	44.13	44.68	45.21	45.72	46.23	46.73	47.24	47.72
45	37.63	38.15	38.73	39.40	39.93	40.46	40.96	41.44	41.94	42.43	42.91
50	33.04	33.55	34.11	34.76	35.28	35.78	36.26	36.74	37.22	37.69	38.16
55	28.61	29.09	29.63	30.25	30.75	31.23	31.69	32.14	32.60	33.05	33.50
60	24.38	24.80	25.31	25.90	26.37	26.82	27.25	27.67	28.10	28.54	28.95
65	20.34	20.72	21.17	21.71	22.14	22.57	22.96	23.35	23.74	24.14	24.53
70	16.45	16.79	17.19	17.66	18.05	18.43	18.78	19.13	19.49	19.85	20.19
75	12.87	13.15	13.48	13.88	14.21	14.54	14.84	15.14	15.46	15.78	16.06
80	9.77	9.99	10.25	10.57	10.83	11.10	11.34	11.59	11.85	12.12	12.34

6.3. The CAF rates

The annuity rates for converting members' accumulated funds into pension of UGX 1000 per annum at retirement are given in Tables 18 and 19.

Table 18: CAF rates for males and females

Age	Male	Female	Age	Male	Female
45	11,202	11,553	73	6,277	7,197
46	11,101	11,473	74	6,046	6,970
47	10,994	11,388	75	5,815	6,740
48	10,882	11,298	76	5,585	6,509
49	10,764	11,202	77	5,357	6,277
50	10,640	11,101	78	5,132	6,046
51	10,511	10,994	79	4,911	5,815
52	10,375	10,882	80	4,693	5,585
53	10,234	10,764	81	4,480	5,357
54	10,087	10,640	82	4,272	5,132
55	9,934	10,511	83	4,069	4,911
56	9,774	10,375	84	3,873	4,693
57	9,609	10,234	85	3,683	4,480
58	9,438	10,087	86	3,500	4,272
59	9,260	9,934	87	3,324	4,069
60	9,077	9,774	88	3,155	3,873
61	8,888	9,609	89	2,993	3,683
62	8,693	9,438	90	2,838	3,500
63	8,493	9,260	91	2,690	3,324
64	8,287	9,077	92	2,548	3,155
65	8,077	8,888	93	2,411	2,993
66	7,863	8,693	94	2,275	2,838
67	7,644	8,493	95	2,138	2,690
68	7,422	8,287	96	1,989	2,548
69	7,197	8,077	97	1,815	2,411
70	6,970	7,863	98	1,581	2,275
71	6,740	7,644	99	1,224	2,138
72	6,509	7,422	100	600	1,989

Table 19: Weighted CAF rates

Age	Rate	Age	Rate
45	11,325	73	6,599
46	11,231	74	6,369
47	11,132	75	6,139
48	11,028	76	5,908
49	10,917	77	5,679
50	10,801	78	5,452
51	10,680	79	5,227
52	10,552	80	5,005
53	10,420	81	4,787
54	10,281	82	4,573
55	10,136	83	4,364
56	9,984	84	4,160
57	9,828	85	3,962
58	9,665	86	3,770
59	9,496	87	3,585
60	9,321	88	3,406
61	9,140	89	3,235
62	8,954	90	3,070
63	8,761	91	2,912
64	8,564	92	2,760
65	8,361	93	2,615
66	8,154	94	2,472
67	7,941	95	2,331
68	7,725	96	2,185
69	7,505	97	2,024
70	7,283	98	1,824
71	7,056	99	1,544
72	6,829	100	1,086