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Adaptive Realized Hyperbolic GARCH Process: Stability and Estimation

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Abstract. In this paper, we propose an Adaptive Realized Hyperbolic *GARCH* (A-Realized *HYGARCH*) process to model the long memory of high-frequency time series with possible structural breaks. The structural change is modeled by allowing the intercept to follow the smooth and flexible function form introduced by Gallant (1984). In addition, stability conditions of the process are investigated.
A Monte Carlo study is considered in order to illustrate the performance of the A-Realized *HYGARCH* process compared to the Realized *HYGARCH* with or without structural change.

Résumé. Dans cet article, nous proposons un modèle hyperbolique *GARCH* réalisé adaptatif (A-Realized *HYGARCH*) pour modéliser la longue mémoire des séries chronologiques á haute fréquence avec d'éventuelles changements de régimes. Le changement de régime est modélisé, en permettant l'intercepte de suivre une forme de fonction lisse et flexible introduite par Gallant (1984). De plus, les conditions de stabilité pour ce modèle sont établies dans ce papier. Une étude de Monte Carlo est considérée afin d'illustrer les performances du modèle (A-Realized *HYGARCH*) comparé au modèle *HYGARCH* Realisé sur des données avec ou sans changement structurel.

Key words: Realized *HYGARCH* model; high-frequency data; long memory; realized measures; structural changes .
AMS 2010 **Mathematics Subject Classification Objects :** $60F15$; $62P20$.

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1. Introduction

The volatility forecast of asset returns is very important for option pricing as well as for risk management. The Autoregressive Conditional Heteroskedasticity (*ARCH*) models introduced by Engle (1982) and generalized by Bollerslev (1986) are widely used to study the properties of volatility for economic and financial data.

However, there are several shortcomings with using the Generalized *ARCH* (*GARCH*) models for risk management or forecasting volatility. The major issue is the persistence of variance that evolves over time and that the *GARCH* model cannot handle . It has been extensively observed and studied in various fields of economic and finance over the last decades (See Aggarwal *et al.* (1999), Fan *et al.* (2008), Granger and Hyung (204)).

The long memory exists in the studies of the volatility of high-frequency for financial time series Richard *et al.* (1996), Michael *et al.* (1993), Ding *et al.* (1993), Granger and Ding (1996). A widely accepted definition of long memory is

$$Var(S_T) = O(T^{2d+1}),$$

where

$$S_T = \sum_{t=1}^{T} y_t,$$

y_t is a sequence of financial series and T is the number of observations, d is the long-memory parameter Diebold and Atsushi (2001).

To overcome this problem, many models are introduced in the literature. Among others, we can cite the Fractionally Integrated *GARCH* (*FIGARCH*) model proposed by Richard *et al.* (1996), the Seasonal-*HYGARCH* model by Diongue and Guegan (2007), the New *HYGARCH* by Li *et al.* (2015) and the Hyperbolic *GARCH* (*HYGARCH*) proposed by Davidson (2004).

Moreover, as stated by Hansen *et al.* (2012) and discussed by Andersen *et al.* (2003), a single return offers only a weak signal about the current level of volatility. Thus, the implication is that *GARCH* models are poorly suited for situations where the volatility changes rapidly to a new level. The reason is that the *GARCH* model is slow at catching up, and it will take many periods for the conditional variance to reach its new level.

Therefore, by incorporating the realized measures in the *GARCH* model, one can alleviates this problem. In addition, with the advent of high-frequency data, several measures have been developing in the literature, such as the Realized Variance and Realized Kernel, among many others (see Andersen and Bollerslev (1998), Barndorff-Nielsen and Shephard (2002) and Barndorff-Nielsen *et. al.* (2008)).

All of these measures provide more information on the current level of volatility compare to the square of returns. This aspect makes the realized measures very important in modeling and forecasting future volatility. Therefore, by introducing the *GARCH-X* model, Engle (2002) incorporated the realized measures in the *GARCH* model. Hansen *et al.* (2012) introduced the Realized *GARCH* model by combining a *GARCH* structure for returns with an integrated model for realized measures of volatility. From this latter, several models have emerged. For example, Hansen and Zhuo (2016) introduced the Realized *EGARCH* to capture the leverage effects. To highlight the property of long memory observed on the realized measure, Vander Elst(2015), introduced the *FloGARCH* model (Fractionally integrated realized volatility *GARCH*) and more recently, Sall *et al.* (2021) proposed the Realized *HYGARCH* model for modeling risk.

Nevertheless, as stated by Si and Yang (2018), although these models bring some improvements, they have the same main weakness as the original *GARCH* model. This limit is the assumption that the conditional volatility has only one regime over the entire period. Furthermore, many studies show that structural change is common in financial datasets (see Beltratti and Morana (2006), Engle and Rangel (2008)). Diebold and Atsushi (2001) argue that the existence of structural change or stochastic regime-switching is not only related to the long memory, and they are generally easily confused. Therefore, a more appropriate volatility model would consider the long memory and the structural change simultaneously, (see, eg. Richard and Morana (2009), Si and Yang (2018).)

The aim of this work is to investigate an Adaptive Realized *HYGARCH* (A-Realized *HYGARCH*) model. This paper starts from the proposition that both long memory and structural breaks are likely to be present in the volatility processes of many economic and financial time series. It is designed for modeling the long memory of high-frequency financial time series with structural changes. This model incorporates the structure of the new *HYGARCH* model of Li *et al.* (2015), further considering the time-varying deterministic component in the flexible, functional form provided by Gallant (1984).

This paper is organized as follows. Section 2 is dedicated to our proposed solution, the adaptive Realized *HYGARCH* model with structural breaks, while in section 3, we study the stability of the model. Section 4 is reserved for the simulation of the experiment we did to evaluate our model. Section 5 concludes this paper and gives some future works.

2. Adaptive Realized Hyperbolic *GARCH*

In this section, we present the Adaptive Realized *HYGARCH* model which takes into account structural changes and long memory of high-frequency data. Indeed, the incorporation of structural change in the long memory model is not a new idea in the literature. Richard and Morana (2009) presented the Adaptive Fractional Integrated *GARCH* (*A-FIGARCH*) model, which is designed for both long memory and regime change in financial time series. Si and Yang (2018) developed also the adaptive Hyperbolic Exponential *GARCH* model. Following this methodology, we propose in this paper the Adaptive Realized *HYGARCH* model which contains two components: a long memory part and a deterministic time varied function. The adaptive Realized *HYGARCH* (p, q, d, k) model can be expressed as:

$$r_t = h_t^{1/2} z_t, \tag{1}$$

$$\log h_t = \omega_t + \delta \left[1 - \frac{1-\gamma(L)}{1-\beta(L)}(1-L)^d\right] \log x_t, \tag{2}$$

$$\log x_t = \xi + \phi \log h_t + \tau(z_t) + u_t, \tag{3}$$

where

$$\omega_t = \omega_0 + \sum_{j=1}^{k} \left[a_j \sin(2\pi j t/T) + b_j \cos(2\pi j t/T)\right],$$

with

$$d \geq 0 \quad and \quad 0 \leq \delta \leq 1.$$

r_t is the return of the time series, x_t a realized measure of volatility, $(z_t)_t$ are independently identically distributed (*i.i.d*) with mean zero and variance one, h_t is the conditional variance and $(u_t)_t$ are also *i.i.d* with mean zero and variance σ_u^2. Here $(z_t)_t$ and $(u_t)_t$ are mutually independent. We label Equation (1) as return equation, Equation (2) as the *GARCH* model and Equation (3) as the measurement statement.

L denotes the lag or backshift operator

$$\beta(L) = \beta_1 L + \beta_2 L^2 + \cdots + \beta_p L^p$$

and

$$\gamma(L) = \gamma_1 L + \gamma_2 L^2 + \cdots + \gamma_q L^q.$$

The polynomial

$$\tau(z) = \tau_1 z + \tau_2 \left(z^2 - 1\right)$$

is called the leverage function and facilitate a modeling of the dependence between return shocks and volatility shocks.
The main difference between the A-Realized *HYGARCH* model and the conventional Realized *HYGARCH* model is the inclusion of the time-varying intercept (ω_t). The A-Realized *HYGARCH* model can be reduced to the standard Realized *HYGARCH* model by setting

$$\omega_t = \omega_0(1-\beta(1))^{-1}$$

if all the roots of the polynomials

$$1-\beta(L)$$

lie outside the unit circle.

In the rest of this study, we consider the A-Realized *HYGARCH*$(1,d,1,k)$ model. The *GARCH* equation is given by:

$$\log h_t = \omega_t + \delta\left[1-\frac{1-\gamma L}{1-\beta L}(1-L)^d\right]\log x_t. \tag{4}$$

The fractional differencing operator has the following representation:

$$(1-L)^d = \sum_{k=0}^{\infty}\frac{\Gamma(d+1)(-L)^k}{\Gamma(k+1)\Gamma(d-k+1)}, \tag{5}$$

where $\Gamma(.)$ denotes the Gamma function.

3. Stability

The stability of the model is one of the main property for any new model. Here, the stability refers to the behavior of the second moment of the model. In this section, we show that the second moment of the A-Realized *HYGARCH* model is asymptotically bounded under some conditions. The second moment of the model is calculated as

$$\mathbb{E}(r_t^2) = \mathbb{E}(h_t z_t^2) = \mathbb{E}(h_t).$$

We have

$$\log h_t = \delta \log h_{1,t},$$

where

$$\log h_{1,t} = \beta \log h_{1,t-1} - \frac{\beta}{\delta}\omega_{t-1} + \frac{1}{\delta}\omega_t + (\beta - \gamma + \pi_1)\log x_{t-1} + \sum_{j=0}^{\infty}(\pi_{j+2} - \gamma\pi_{j+1})L^j \log x_{t-2} \quad (6)$$

and

$$\log h_t = \delta\beta \log h_{1,t-1} - \beta\omega_{t-1} + \omega_t + \delta(\beta - \gamma + \pi_1)\log x_{t-1} + \delta\sum_{j=0}^{\infty}(\pi_{j+2} - \gamma\pi_{j+1})L^j \log x_{t-2}. \quad (7)$$

Lemma 1. *If n_j and m_j are the non negative numbers with $j \in \{1,2,\cdots,k\}$ such that*

$$\sum_{j=1}^{k}(n_j + m_j) \leq \min(1, \omega_0),$$

then

$$0 \leq \omega_t = \omega_0 + \sum_{j=1}^{k}[a_j \sin(2\pi jt/T) + b_j \cos(2\pi jt/T)] \leq 1 + \omega_0 + 1 = c_0.$$

Proof. For the proof of Lemma 1, one can refer to Uwilingiyimana *et al.* (2020).

We also have:

Lemma 2. *Let $(V, ||.||_\infty)$ be a normed space such that,*

$$V = \left\{(y_t)_{t\in\mathbb{Z}} | \sup_{t\in\mathbb{Z}} \mathbb{E}|y_t| \leq \infty\right\}$$

and L be a linear operator on V defined by:

$$L : V \to V$$
$$y \to Ly = (Ly_t)_{t\in\mathbb{Z}} = (y_{t-1})_{t\in\mathbb{Z}},$$

with

$$||L||_\infty = \sup_{t\in\mathbb{Z}} \mathbb{E}|y_t|,$$

then the delayed operator satisfies

$$||L^i||_\infty = 1 \quad \forall i \in \mathbb{N}.$$

proof. For the proof of Lemma 2, one can refer to Uwilingiyimana *et al.* (2020).

Theorem 1. *The conditional variance* $\log h_t$ *of A-Realized HYGARCH model satisfies this follow inequality*

$$\mathbb{E}(\log h_t) \leq |\delta\beta| + f_0 + \delta\phi|\beta - \gamma + \pi_1|\mathbb{E}(\log h_{t-1}) + \delta\phi\sum_{j=0}^{\infty}|\pi_{j+2} - \gamma\pi_{j+1}|\mathbb{E}(\log h_{t-2})$$

where

$$f_0 = c_0(1 - |\beta|) + \xi\delta|\beta - \gamma + \pi_1| + \xi\delta\sum_{j=0}^{\infty}|\pi_{j+2} - \gamma\pi_{j+1}|$$

Proof. Using the equation (7), the expectation of $\log h_t$ is given by:

$$\begin{aligned}\mathbb{E}(\log h_t) = \delta\beta\mathbb{E}(\log h_{1,t-1}) - \beta\mathbb{E}(\omega_{t-1}) + \mathbb{E}(\omega_t) + \delta(\beta - \gamma + \pi_1)\mathbb{E}(\log x_{t-1}) \\ + \delta\sum_{j=0}^{\infty}(\pi_{j+2} - \gamma\pi_{j+1})L^j\mathbb{E}(\log x_{t-2}).\end{aligned} \tag{8}$$

Since

$$\mathbb{E}(\log x_t) = \xi + \phi\mathbb{E}(\log h_t),$$

then by using Lemma 1 and Lemma 2, an upper bound of (8) is calculated as follows:

$$\begin{cases} \mathbb{E}(\omega_t) \leq c_0, \\ \delta\beta\mathbb{E}(\log h_{1,t-1}) \leq |\delta\beta|\mathbb{E}(\log h_{1,t-1}), \\ \delta(\beta - \gamma + \pi_1)\mathbb{E}(\log x_t\omega_t) \leq \delta|(\beta - \gamma + \pi_1)|(\xi + \phi\mathbb{E}(\log h_t)), \\ \delta\sum_{j=0}^{\infty}(\pi_{j+2} - \gamma\pi_{j+1})L^j\mathbb{E}(\log x_t) \leq \delta\sum_{j=0}^{\infty}|(\pi_{j+2} - \gamma\pi_{j+1})|(\xi + \phi\mathbb{E}(\log h_t)). \end{cases}$$

By substituting the above results in (8), we get

$$\mathbb{E}(\log h_t) \leq |\delta\beta| + f_0 + \delta\phi|\beta - \gamma + \pi_1|\mathbb{E}(\log h_{t-1}) + \delta\phi\sum_{j=0}^{\infty}|\pi_{j+2} - \gamma\pi_{j+1}|\mathbb{E}(\log h_{t-2}).$$

□

Now, considering the A-Realized Hyperbolic *GARCH* process, we have

$$\mathbb{E}(\log h_t) \le |\delta\beta|\mathbb{E}(\log h_{1,t-1}) + f_0 + \delta\phi|\beta-\gamma+\pi_1|\mathbb{E}(\log h_{t-1}) + \delta\phi\sum_{j=0}^{\infty}|\pi_{j+2}-\gamma\pi_{j+1}|\mathbb{E}(\log h_{t-2}), \quad (9)$$

and

$$\mathbb{E}(\log h_{1,t}) \le |\beta|\mathbb{E}(\log h_{1,t-1}) + f_1 + \phi|\beta-\gamma+\pi_1|\mathbb{E}(\log h_{t-1}) + \phi\sum_{j=0}^{\infty}|\pi_{j+2}-\gamma\pi_{j+1}|\mathbb{E}(\log h_{t-2}), \quad (10)$$

where $f_1 = f_0/\delta$. Note that inequalities (9) and (10) can be rewritten in matrix form as

$$H_t \le M + BH_{t-1}, \quad (11)$$

with some initial condition H_{-1}. Iterating inequality (11), we get

$$H_t \le M\sum_{i=0}^{t-1}B^i + B^tH_0 = D_t. \quad (12)$$

The matrices H_t, M and B are defined as follows:

$$H_t = \begin{bmatrix} \mathbb{E}(\log h_t) \\ \mathbb{E}(\log h_{1,t}) \\ \mathbb{E}(\log h_{t-1}) \end{bmatrix}; \; M = \begin{bmatrix} f_0 \\ f_1 \\ 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} \phi\delta|\delta(\beta-\gamma+\pi_1)| & |\beta\delta| & \phi\delta\sum_{j=0}^{\infty}|(\pi_{j+2}-\gamma\pi_{j+1}))| \\ \phi|(\beta-\gamma+\pi_1)| & |\beta| & \phi\sum_{j=0}^{\infty}|(\pi_{j+2}-\gamma\pi_{j+1}))| \\ 1 & 0 & 0 \end{bmatrix}.$$

Lemma 3. *Let δ, ϕ, β and γ be the parameters of the A-Realized HYGARCH model. If*

$$\begin{cases} \delta\phi|\beta-\gamma+\pi_1| + |\beta| + \phi\delta\sum_{j=0}^{\infty}|\pi_{j+2}-\gamma\pi_{j+1}| - 1 \le 0 \\ \delta\phi|\beta-\gamma+\pi_1| + |\beta| \le 2, \end{cases}$$

then the spectral radius of B, $\rho(B) < 1$.

Proof. Let show that the spectrum $\Lambda(B)$ is not an empty set and its maximum eigenvalue is strictly less than one.

$$B = \begin{bmatrix} \phi\delta|\delta(\beta-\gamma+\pi_1)| & |\beta\delta| & \phi\delta\sum_{j=0}^{\infty}|(\pi_{j+2}-\gamma\pi_{j+1}))| \\ \phi|(\beta-\gamma+\pi_1)| & |\beta| & \phi\sum_{j=0}^{\infty}|(\pi_{j+2}-\gamma\pi_{j+1}))| \\ 1 & 0 & 0 \end{bmatrix}$$

For sake of simplicity, let us rewrite the matrix B as

$$B = \begin{bmatrix} a & b & c \\ \frac{a}{\delta} & \frac{b}{\delta} & \frac{c}{\delta} \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of B is given by

$$\mathbb{P}_B(\lambda) = \lambda(-\lambda^2 + (a + \frac{b}{\delta})\lambda + c).$$

By solving the equation $\mathbb{P}_B(\lambda) = 0$, the eigenvalues of the matrix B are

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = \frac{1}{2}\left[(a+\frac{b}{\delta}) - \sqrt{(a+\frac{b}{\delta})^2+4c}\right] \\ \lambda_3 = \frac{1}{2}\left[(a+\frac{b}{\delta}) + \sqrt{(a+\frac{b}{\delta})^2+4c}\right] \end{cases}$$

So

$$max\{\lambda_1, \lambda_2, \lambda_3\} = \lambda_3$$

that is,

$$\rho(B) = \frac{1}{2}\left[\left(a+\frac{b}{\delta}\right) + \sqrt{\left(a+\frac{b}{\delta}\right)^2 + 4c}\right].$$

The spectral radius of B is less than one if and only if the following condition are satisfied

$$\begin{cases} a + \frac{b}{\delta} + c - 1 \leq 0, \\ a + \frac{b}{\delta} \leq 2. \end{cases} \tag{13}$$

We just have to replace a, b and c by their expressions in (13), where

$$a = \phi\delta|(\beta - \gamma + \pi_1),$$

$$b = |\beta\delta|$$

and

$$c = \phi\delta \sum_{j=0}^{\infty} |(\pi_{j+2} - \gamma\pi_{j+1}))|.$$

Thus, (13) is rewritten as follows:

$$\begin{cases} \delta\phi|\beta - \gamma + \pi_1| + |\beta| + \phi\delta \sum_{j=0}^{\infty} |\pi_{j+2} - \gamma\pi_{j+1}| - 1 \leq 0 \\ \delta\phi|\beta - \gamma + \pi_1| + |\beta| \leq 2 \end{cases} \blacksquare$$

We also have:

Theorem 2. *Let δ, ϕ, β and γ be the parameters of the A-Realized HYGARCH model. If*

$$\begin{cases} \delta\phi|\beta - \gamma + \pi_1| + |\beta| + \phi\delta \sum_{j=0}^{\infty} |\pi_{j+2} - \gamma\pi_{j+1}| - 1 \leq 0 \\ \delta\phi|\beta - \gamma + \pi_1| + |\beta| \leq 2, \end{cases}$$

then the process $\{r_t\}$ followings an A-Realized Hyperbolic GARCH model defined in relations (1),(2) and (3) is asymptotically stable with finite variance.

proof. From (11) and (12), we recall that

$$H_t \leq M \sum_{i=0}^{t-1} B^i + B^t H_0 = D_t \quad t \leq 0.$$

According to the convergence matrix (see, Lancasterand Tismenetsky (1985)), the necessary and sufficient condition for the convergence of D_t when $t \to \infty$ is $\rho(B) < 1$, by Lemma 3, suppose that the spectral radius is strictly less than one. Now we show that if $(I - B)$ exists, its inverse exists and

$$\sum_{i=0}^{t-1} B^i = (I - B)^{-1}$$

as

$$\lim_{t\to\infty} B^t H_0 = 0.$$

The eigenvalues of $(I-B)$ are $(1-\lambda(B))$, where $\lambda(B)$ are the eigenvalues of matrix B.

The set of eigenvalues of $(I - B)$ is not empty, hence matrix $(1 - \lambda(B))$ is invertible.

Let

$$S_n = I + B + B^2 + \cdots + B^{n-1} = \sum_{i=0}^{n-1} B^i$$

and so

$$BS_n = B + B^2 + \cdots + B^n.$$

Hence,

$$(I - B)S_n = I - B^n.$$

By using the fact that

$$\lim_{n\to\infty} B^n = 0,$$

we can prove that

$$\lim_{n\to\infty} (I - B^n) = I.$$

We get

$$(I - B) \lim_{n\to\infty} S_n = I,$$

that is

$$\lim_{n\to\infty} S_n = (I - B)^{-1}.$$

More precisely

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} B^i = (I-B)^{-1}\lim_{t\to\infty} B^t = 0$$

as

$$\lim_{t\to\infty} B^t = 0,$$

under Lemma 3. We conclude that

$$\lim_{t\to\infty} H_t \leq (I-B)^{-1}M. \blacksquare$$

4. Estimation

In this section, we report the Monte Carlo simulation evidence on the estimation of our Adaptive Realized Hyperbolic *GARCH* model for **Data Generating Processes (*DGP*)**. For all models used in this section, we assume that z_t and u_t follow respectively the student $\mathcal{T}$ distribution with 3 degrees of freedom and the normal distribution $\mathcal{N}(0,\sigma_u)$. We consider $p=q=1$ and $\omega=0.1$, $\gamma=0.1$, $\beta=0.4$, $d=0.25, 0.35, 0.45.$, $\delta=0.9$, $\epsilon=0$, $\phi=1$, $\tau_1=-0.08$, $\tau_2=0.06$, and $\sigma_u^2=0.4$. The three values of the long memory parameter d are those proposed by Si and Yang (2018), as *low memory* ($d=0.25$), *moderate memory* ($d=0.35$) and *high memory* ($d=0.45$).

To obtain the ***DGP*** samples from Realized Hyperbolic *GARCH* with structural change, we fellow the **Step** 1, **Step** 2 and **Step** 3 below. Notice that, **step** 3 acts as the core part of this simulation study, and it must be repeated for each model and each replication. **Step** 1 is also repeated for each replication while **Step** 2 only needs to be performed once for each model. Following Si and Yang (2018), we consider 500 Monte Carlo replications.

1. **Step** 1: Set $z_t \backsim \mathcal{T}(0,1,\nu)$ and $u_t \backsim \mathcal{N}(0,\sigma_u)$. We get an *i.i.d* sample $\{z_t\}_{t=m}^{T}$ and $\{u_t\}_{t=m}^{T}$, where m represents the number of extra burn in the data generated.

2. **Step** 2: Choose appropriate designs for the intercept term in each model. In this step, we consider three different designs:
   - *Design* 1*:* (m_1) assumes a constant intercept $\omega=\omega_t=0.1$, and corresponds to the standard experiment setting where no structural breaks are allowed in the conditional variance.
   - *Design* 2*:* (m_2) adopts the permanent break structure used by Richard and Morana (2009) and has one step change in the intercept. The intercept jumps from 0.1 to 0.5 without bouncing back in the future. Hence,

$$\omega_t = \begin{cases} 0.1, & t=1,...\frac{T}{2} \\ 0.5, & t=\frac{T}{2}+1,...,T \end{cases}$$

Table 1. Simulation results for estimation of A-Realized *HYGARCH* model without structural change.

A-Realized *HYGARCH* (1,d,1,0)			
d	Bias	*RMSE*	*SE*
0.25	0.0539	0.0932	0.0761
0.35	0.0409	0.0868	0.0765
0.45	0.0363	0.0760	0.0667

- *Design* 3: (m_3) has two step changes. With the intercept jumping from 0.1 to 0.5 at the first break point and bouncing back to 0.3 at the second break point. Hence,

$$\omega_t = \begin{cases} 0.1, & t = 1, ... \frac{T}{3} \\ 0.5, & t = \frac{T}{3} + 1, ..., \frac{2T}{3} \\ 0.3, & t = \frac{2T}{3} + 1, ..., T. \end{cases}$$

3. **Step** 3: The sample $\{r_t\}_{t=1}^{T}$ and $\{x_t\}_{t=1}^{T}$ are obtained by using the specification Realized *ARCH*(∞).

The log-likelihood function is applied on the models used in this paper can be described as follows:

$$l(r, x; \theta_t) = \underbrace{-\sum_{t=1}^{n} \left[A(\nu) + \log(\pi(\nu-2)) + 0.5 \log(h_t) + \frac{\nu+1}{2} \log\left(1 + \frac{r_t^2}{h_t(\nu-2)}\right) \right]}_{l(r|x;\theta_t)} \underbrace{-\frac{1}{2} \sum_{t=1}^{n} \left[\log 2\pi + \log\left(\theta_u^2\right) + \frac{u_t^2}{\theta_u^2} \right]}_{=l(x;\theta_t)}, \tag{14}$$

where

$$u_t = \log x_t - \epsilon - \phi \log h_t - \tau_1 z_t - \tau_2(z_t^2 - 1)$$

and

$$A(\nu) = \log(\Gamma(\frac{\nu}{2})) - \log(\Gamma(\frac{\nu+1}{2})).$$

The parameter vector to be estimated is $\theta = (\omega', \gamma, \beta, d, \delta, \nu, \epsilon, \phi, \tau_1, \tau_2, \sigma_u^2, a', b')$. We maximize equation (14) with the help to statistical packages in R software to estimate the vector θ.

Table 2. Simulation results for estimation of A-Realized *HYGARCH* models without structural change.

A-Realized *HYGARCH* (1,d,1,k)			
d	Bias	*RMSE*	*SE*
	k=1		
0.25	0.0816	0.1184	0.0858
0.35	0.0239	0.0795	0.0758
0.45	0.0082	0.0793	0.0789
	k=2		
0.25	0.0563	0.0989	0.0813
0.35	0.0090	0.0811	0.0806
0.45	0.0059	0.0904	0.0903
	k=3		
0.25	0.0302	0.0877	0.0823
0.35	0.0028	0.0914	0.0914
0.45	-0.0276	0.1029	0.0992
	k=4		
0.25	0.0390	0.0787	0.0684
0.35	-0.0132	0.0769	0.0758
0.45	-0.0411	0.1026	0.0940

Table 3. Simulation results for estimation of A-Realized *HYGARCH*$(1, d, 1, k)$ models with various structural change designs

		A-Realized *HY-GARCH*(1,0.25,1,k)			A-Realized *HY-GARCH*(1,0.35,1,k)			A-Realized *HY-GARCH*(1,0.45,1,k)		
		BIAS	*RMSE*	*SE*	*BIAS*	*RMSE*	SE	*BIAS*	*RMSE*	*SE*
k=0	m2	0.0886	0.0959	0.0366	0.0105	0.0336	0.0319	-0.058	0.0684	0.0362
	m3	0.2353	0.2466	0.0737	0.1474	0.1590	0.0596	0.1127	0.1291	0.0630
k=1	m2	0.1134	0.1203	0.0402	0.0320	0.0487	0.0367	0.0329	0.0452	0.0419
	m3	0.1245	0.1366	0.0562	0.0753	0.0890	0.0474	0.0340	0.0605	0.0500
k=2	m2	0.1244	0.1358	0.0544	0.0307	0.0568	0.0478	-0.046	0.0643	0.0443
	m3	0.0895	0.1047	0.0542	0.0494	0.0719	0.0522	0.0108	0.0539	0.0528
k=3	m2	0.1481	0.1613	0.0640	0.0666	0.0954	0.0682	-0.03	0.0647	0.0569
	m3	0.0566	0.0801	0.0567	0.0190	0.0671	0.0644	0.0011	0.0750	0.0750
k=4	m2	0.1416	0.1559	0.0653	0.0703	0.1053	0.0784	-0.035	0.0599	0.0023
	m3	0.0529	0.0835	0.0646	-0.004	0.0731	0.0730	-0.019	0.0882	0.0860

Note 1: The Table 1 reports simulation results for the bias, root mean square error (*RMSE*) and standard error (*SE*) for estimation of the fractional differencing parameter d from simulations with sample size $T = 3000$. All the results are based on 500 replications.
Note 2: As for Table 1, the Table 2 compute the estimated models used in Gallant (1984) 's k^{th} order flexible functional form, with $k = 1, 2, 3, 4$, for the adaptive component.
Note 3: The Table 3 reports simulation results for the bias, root mean square error (*RMSE*) and the standard error (*SE*) for estimation of the fractional differencing parameter d from a sample size of $T = 3000$ observations. All the results are based

on 500 replications. The simulations are for two different experiments of: a single break point (m_2) and two break points (m_3).

Table 1 summarizes estimation results of the A-Realized *HYGARCH* models with $k = 0$ equivalent to the ordinary A-Realized *HYGARCH* models for the Realized *HYGARCH DGP* with *Design* 1. We remark that the estimated long memory parameter d has a very small bias. This result is consistent with the three values assumed for d. The Realized *HYGARCH*$(1, d, 1)$ *DGP* with $d = 0.45$ has the lowest estimation bias.

Table 2 summarizes estimation results for an A-Realized *HYGARCH* models with $k = (1, 2, 3, 4)$. There is an important result obtained by comparing Table 1 and Table 2. More than half of the model shows a reduction in bias after adopting an adaptive structure. As d increases, the reduction in the degree of bias tends to increase. The estimated long memory parameter, obtained from the A-Realized *HYGARCH* and the Realized *HYGARCH* model estimation, has, in both, approximately the same degree of small sample *RMSE*. This result suggests that the intercept used (which follows a flexible function form with more than one pair of trigonometric components) can adjust for some uncertainties in the estimation of the long memory parameter d (see Richard and Morana (2009)).

Table 3 reports estimation results for estimates of A-Realized *HYGARCH*$(1, d, 1, k)$ models. From Table 3, it can be seen that most A-Realized *HYGARCH*$(1, d, 1, k)$ models appear to have smaller estimation bias for the m_3 structural change design than the m_2 design. For the two cases, from the high persistence case $(d = 0.45)$, the degree of bias in the estimates of d is very small for both estimators.

However, the bias is always smaller using the A-Realized *HYGARCH* model than the pure Realized *HYGARCH* model. Furthermore, the *RMSE* of the estimated of d is generally lower from the A-Realized *HYGARCH* estimation compared to the pure Realized *HYGARCH* one. Finally, we can say that the A-Realized *HYGARCH* model performs, generally, better than the standard Realized *HYGARCH* model in the sense of *RMSE* and *SE* criteria. Indeed, the former is robust across the three values used in the designs contrary to the latter. Furthermore, the improvement increases as the degree of persistence increases.

In general, the A-Realized *HYGARCH* model consistently outperforms the Realized *HYGARCH* model across different simulation designs with and without structural change. This fact suggests the usefulness of the A-Realized *HYGARCH* model in practice. The findings of this research are consistent with those from Richard and Morana (2009).

5. Conclusion and future works

In this article, we have developed the adaptive Realized *HYGARCH* process. It is much more flexible in modeling long-memory behavior and structural change often encountered in financial data. Under some assumptions, the model is shown to be stable. The quasi-maximum likelihood procedure is used to estimate the parameter of this model. Finite sample behaviors of this method were studied using Monte Carlo simulations. It indicates that the A-Realized *HYGARCH* model outperforms the Realized *HYGARCH* model with and without structural change.

Since the results and the estimation methodology are encouraging, it will be interesting to examine the Adaptive Realized *HYGARCH* model's empirical application in financial data.

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