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The Marshall-Olkin-Gumbel extended Weibull distribution: Properties and Applications

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Abstract. . We introduce a new lifetime distribution called Marshall-Olkin extended Gumbel-Weibull. Some properties of distribution such as moments, TL-moments, quantile function, entropy, and order statistics are studied. The fexibility of the distribution to model unimodal, monotone shapes as well as unimodal, bimodal, monotone failure rates are presented. The estimators of the parameters of the distribution were obtained using the maximum likelihood estimation method. The performance of the maximum likelihood estimates of the Marshall-Olkin extended Gumbel-Weibulll parameters was observed through simulation studies.Two real life applications to illustrate the potentials of the new distribution are presented, and comparison with other distribution having the same baseline is done using goodness-of-test statistics.

Key words: Marshall-Olkin-Gumbel-G; moments; reliability functions; TL-moments.
AMS 2010 **Mathematics Subject Classification Objects :** 60*E*05; 62*E*10; 62*F*10; 62*F*12.

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Résumé (Abstract in French) Nous introduisons une nouvelle distribution pour modéliser des durée de vie appelée extension Marshall-Olkin de la loi de Gumbel-Weibull. Certaines propriétés de la distribution telles que les moments, les moments TL, la fonction quantile, l'entropie et les statistiques d'ordre sont étudiées. La fexibilité de la distribution pour modéliser des formes unimodales et monotones ainsi les taux de défaillance unimodaux, bimodaux et monotones sont présentés. Les estimateurs des paramètres de la distribution ont été obtenus à l'aide de la méthode d'estimation du maximum de vraisemblance. La performance des estimations du maximum de vraisemblance des paramètres de Gumbel-Weibulll étendus de Marshall-Olkin a été observée au moyen d'études de simulation. Deux applications réelles pour illustrer les potentiels de la nouvelle distribution sont présentées, et la comparaison avec d'autres distributions ayant la même base de référence est effectuée à l'aide de tests d'ajustage.

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1. Introduction

The **Weibull** distribution having exponential, Gompertz, and Rayleigh distributions as special cases, has been intensely used in reliability in many fields of study (Johnson *et al.*(1994), Murthy *et al.*(2004), Xie and Lai (1995)). A random variable X follows Weibull distribution if its distribution function (*cdf*) is given by

$$G(x) = 1 - \exp\left[-\left(\frac{x}{\theta}\right)^{\alpha}\right]; \quad x > 0, \alpha, \theta > 0.$$

where θ is the scale parameter and α the shape parameter of the distribution.

The failure rate function is very important in lifetime distribution Lai *et al.*(2001). Nadarajah(2009) pointed out that any lifetime distribution should have a failure rate capable of describing the important features such as regions of infant mortality, no change, and wear-out characterizing life phenomena. The Weibull distribution has decreasing failure rate function at $\theta < 1$, at $\theta = 1$ has constant failure rate function and increasing failure rate at $\theta > 1$ Johnson *et al.*(1994).

Extensions and modifications of Weibull distribution have been made to generate a more flexible distribution and a bathtub shape failure rate which describes life phenomena also (Lai *et al.*(2001) and Murthy *et al.*(2004)). Some of the extensions are exponetiated-Weibull due to Mudholkar *et al.*(1996) and Mudholkar *et al.*(1995), Marshall-Olkin extended Weibull of Ghitany *et al.*(2005), Beta-Weibull due to Lee *et al.*(2007), Kumaraswamy Weibull due to Cordeiro *et al.*(2010), transmuted Weibull due to Aryal and Tsokos(2011), Gumbel-Weibull due to Al-Aqtash *et al.*(2014), Odd log-logistic exponentiated Weibull due to

Afify*et al.* (2017), generalized odd log-logistic Weibull due to Cordeiro *et al.*(2017), and lindley-Weibull due to Cordeiro *et al.*(2018).

Ugwuowo and Nwezza(2018) introduced a new class of Marshall-Olkin extended Gumbel (*MOG-G*) family of distributions having distribution function (*cdf*) and density function (*pdf*) respectively given in Equations (1) and (2)

$$G(x)=\frac{exp\left[-\vartheta\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\right]}{1-\bar{\phi}\left\{1-exp\left[-\vartheta\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\right]\right\}}. \tag{1}$$

$$g(x)=\frac{\phi\vartheta f(x)}{\sigma F(x)\left(1-F(x)\right)}\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\times \frac{exp\left[-\vartheta\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\right]}{\left(1-\bar{\phi}\left\{1-exp\left[-\vartheta\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\right]\right\}\right)^{2}}; \tag{2}$$

where $\vartheta = exp\left(\frac{\mu}{\sigma}\right)$, $\sigma,\phi>0$ and $-\infty<\mu<\infty$. Furthermore, $F(x)$ is the *cdf* of any baseline distribution.

We consider a random variable X which denotes the failure rate of a device with q independent components such that $X_i; i=1,2,\cdots,q$ are independent and identically distributed with $X_i\in\mathbb{R}_+$. The probability that the device fails at time x is given by

$$\begin{aligned}\mathbb{P}(X\le x) &= \prod_{i=1}^{q}\mathbb{P}(X_i\le x)\\ &= \mathbb{P}(X_1\le x).\mathbb{P}(X_2\le x)\cdots\mathbb{P}(X_q\le x)\\ &= [\mathbb{P}(X\le x)]^q\\ G(x) &= [F(x)]^q. \end{aligned} \tag{3}$$

Mudholkar *et al.*(1996) introduced the exponentiated-Weibull distribution which has unimodal, bathtub shape failure rate, and allows for broader class monotone failure rates. A random variable w has exponentiated-Weibull (*exp-W*) distribution with power parameter q if its *cdf* can be defined as

$$H_q(w)=\left[1-exp\left(-\left(\frac{w}{\theta}\right)^{\alpha}\right)\right]^q. \tag{4}$$

Equation (4) corresponds to the formulation of the framework of expressed in Equation (3) where $F(x)=1-exp\left[-\left(\frac{w}{\theta}\right)^{\alpha}\right]$.

We propose a new five-parameter distribution called the Marshall-Olkin-Gumbel extended Weibull (*MOG-W*) and present in detail some of its statistical properties.

The main motivations of this paper are to generate a new flexible distribution with bathtub shape failure rate function which is a very important feature of distributions for reliability studies and modeling lifetime data, have bimodal *pdf* shape amongst other shape, and have a heavier tail with capacity of providing better fit to real-life data set.

The rest of the paper is structured as follows: the derivation of the new distribution is presented in Section 2. In Section 3, the linear representation of the *MOG-W* distribution is obtained. The quantile function of the distribution is derived in Section 4. In Section 5, we obtain the moment of the distribution. Section 6 considered the entropy. The density function of the order statistics of the distribution is derived in Section 7. The maximum likelihood estimates (*MLEs*) of the parameters of the distribution are obtained in Section 8. In Section 9, we verify the asymptotic performance of the *MLEs* by means of simulations. In section 10, we illustrate the potentials of *MOG-W* to provide better fit than other distributions having Weibull distribution as baseline distribution by means of applications of two real-life data sets. Finally, we provided concluding remarks in Section 11.

2. The *MOG-W* distribution

The *cdf* of the *MOG-W* is given in *Eq* (5) and it is obtanied by substituting for $F(x) = 1 - exp\left[-\left(\frac{x}{\theta}\right)^{\alpha}\right]$ in Equation (1).

$$G(x) = \frac{exp\left[-\vartheta\left(exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right)^{-\frac{1}{\sigma}}\right]}{1-\bar{\phi}\left\{1-exp\left[-\vartheta\left(exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right)^{-\frac{1}{\sigma}}\right]\right\}}. \tag{5}$$

The corresponding *pdf* of Equation (5) is given by

$$g(x) = \frac{\phi\vartheta\alpha x^{\alpha-1}exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]}{\sigma\theta^{\alpha}\left\{exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right\}^{\frac{1}{\sigma}+1}} \times \frac{exp\left[-\vartheta\left(exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right)^{-\frac{1}{\sigma}}\right]}{\left(1-\bar{\phi}\left\{1-exp\left[-\vartheta\left(exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right)^{-\frac{1}{\sigma}}\right]\right\}\right)^{2}}; \tag{6}$$

where $\bar{\phi} = 1 - \phi$.

Figure 2 illustrates the flexibility of *MOG-W* to model data sets of shapes which include monotone increasing and decreasing, symmetric, bimodal, and skewed.

3. Linear representation

In this section, we represent the *cdf* and *pdf* of *MOG-W* as weighted sum of *exp-W* distribution. Applying the general binomial series expansion in Equation (1), the *cdf* of *MOG-W* distribution can be re-written as

$$G(x) = \sum_{q=0}^{\infty} t_q H_q(x); \tag{7}$$

where

$$t_q = (-1)^q \sum_{i,j,k,m=0}^{\infty} \sum_{n=q}^{\infty} \frac{(-1)^{n+j+k+m}}{k!} \bar{\phi}^i \left[\vartheta(1+j)\right]^k \binom{i}{j} \binom{\frac{k}{\sigma}}{m} \times \binom{m-\frac{k}{\sigma}}{n} \binom{n}{q}$$

and

$$H_q(x) = \left\{1 - exp\left[-\left(\frac{x}{\theta}\right)^{\alpha}\right]\right\}^q.$$

By differentiating Equation (7), we can re-write Equation (6) as

$$g(x) = \sum_{q=0}^{\infty} t_q^* h_{q+1}(x); \tag{8}$$

where

$$t_q^* = (-1)^q \frac{\phi\vartheta}{\sigma} \sum_{i,j,k,m=0}^{\infty} \sum_{n=q}^{\infty} \frac{(-1)^{n+j+k+m}}{k!(q+1)} \bar{\phi}^i \left[\vartheta(1+j)\right]^k \binom{i+1}{i} \binom{i}{j} \times \binom{\frac{1}{\sigma}-\frac{k}{\sigma}-1}{m} \binom{m-\frac{k}{\sigma}-\frac{1}{\sigma}-1}{n} \binom{n}{q}$$

and

$$h_{q+1}(x) = (q+1)\frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1} exp\left[-\left(\frac{x}{\theta}\right)^{\alpha}\right] \left\{1 - exp\left[-\left(\frac{x}{\theta}\right)^{\alpha}\right]\right\}^q.$$

4. Statistical properties

Theorem 1. *If a random variable Y follows Marshall-Olkin extended Gumbel distribution then*

$$X = \theta\left[\log\left(exp(Y)+1\right)\right]^{\frac{1}{\alpha}}$$

follows MOG-W distribution.

Proof. Given that Y follows Marshall-Olkin extended Gumbel distribution, the *pdf* of Y is given by

$$h(y) = \frac{\phi\vartheta exp\left(-\frac{y}{\sigma}\right) exp\left[-\vartheta exp\left(-\frac{y}{\sigma}\right)\right]}{\sigma\left(1-\bar{\phi}\left\{1-exp\left[-\vartheta exp\left(-\frac{y}{\sigma}\right)\right]\right\}\right)^2}. \tag{9}$$

By transformation, the *pdf* of *MOG-W* is defined by

$$g(x) = h(y)\left|\frac{dy}{dx}\right|; \tag{10}$$

where

$$\frac{dy}{dx} = \frac{\alpha x^{\alpha-1} exp\left[\left(\frac{x}{\theta}\right)^\alpha\right]}{\theta^\alpha\left\{exp\left[\left(\frac{x}{\theta}\right)^\alpha\right]-1\right\}}.$$

By Substituting appropriately $\frac{dy}{dx}$ and Equation (9) with corresponding value of y in Equation (10), simplifying further the *pdf* of *MOG-W* is obtained as

$$g(x) = \frac{\phi\vartheta\alpha x^{\alpha-1} exp\left[\left(\frac{x}{\theta}\right)^\alpha\right] exp\left[-\vartheta\left(exp\left[\left(\frac{x}{\theta}\right)^\alpha\right]-1\right)^{-\frac{1}{\sigma}}\right]}{\sigma\theta^\alpha\left\{exp\left[\left(\frac{x}{\theta}\right)^\alpha\right]-1\right\}^{\frac{1}{\sigma}+1}\left(1-\bar{\phi}\left\{1-exp\left[-\vartheta\left(exp\left[\left(\frac{x}{\theta}\right)^\alpha\right]-1\right)^{-\frac{1}{\sigma}}\right]\right\}\right)^2}.$$

□

4.1. Quantile function

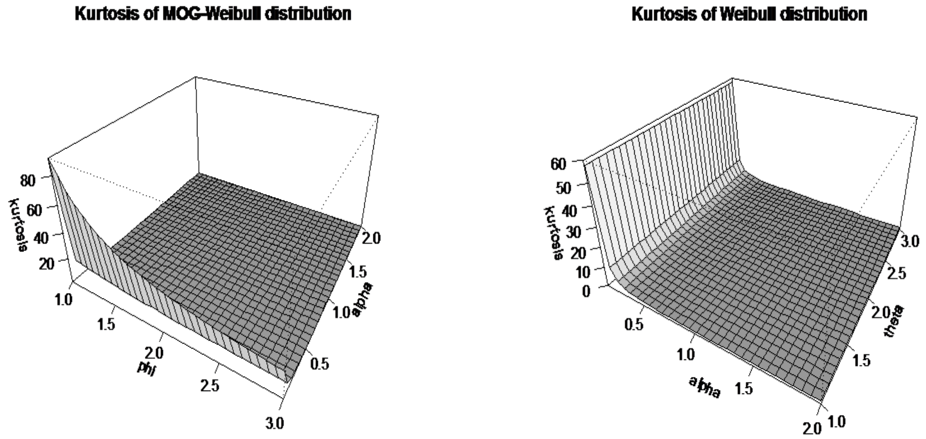
Using the probability integral transformation, the qunatile function $[Q(.)]$ of *MOG-W* distribution is obtained as

$$Q(u) = \theta\left[\log\left(1+exp(\mu)\left\{\log\left[\left(\frac{u-u\bar{\phi}}{1-u\bar{\phi}}\right)^{-1}\right]\right\}^{-\sigma}\right)\right]^{\frac{1}{\alpha}}; \; u \in (0,1). \tag{11}$$

Substituting the value $u = 0.5$ in Equation (11), the median of *MOG-W* denoted by M is obtained as

$$M = \theta\left[\log\left(1+exp(\mu)\left\{\log\left[\left(\frac{1-\bar{\phi}}{2-\bar{\phi}}\right)^{-1}\right]\right\}^{-\sigma}\right)\right]^{\frac{1}{\alpha}}. \tag{12}$$

The p^{th} quantile can also be obtained by substituting different values of u in the interval $(0,1)$. Moors(1988) introduced an alternative measure of kurtosis which exists for *pdfs* that its moments do not exist. The Moore's measure of kurtosis is defined as



(a) (b)

Fig. 1: Plots of Kurtosis: a) *MOG-W* distribution b) Weibull distribution

$$K = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}.$$

The respective kurtosis plots of *MOG-W* and Weibull distribution for parameter values $\phi = 1$ to 3, $\mu = 0.1$, $\sigma = 3$, $\alpha = 0.1$ to 2, $\theta = 2$ and $\alpha = 0.1$ to 2, $\theta = 1$ to 3 are shown in Figure 1. By Figure 1, the *MOG-W* produces a heavier tail with set of parameter values which has the same value with that of Weibull distribution.

Corollary 1. *Suppose a random variable X has distribution function as defined in Equation (6), then the n^{th} moment about the origin, say $\mu_n^{'}$, is given by*

$$\mu_n^{'} = \frac{\phi\vartheta\theta^n}{\sigma}\sum_{k=0}^{\infty}\Psi_k\Gamma\left(\frac{n}{\alpha}+1\right);$$

where

$$\Psi_k = \frac{(-1)^k}{k!}\left[\vartheta(1+j)\right]^k\left(\frac{k}{\sigma}+\frac{1}{\sigma}+m\right)^{-\left(\frac{n}{\alpha}+1\right)}\sum_{i,j,m=0}^{\infty}(-1)^j\bar{\phi}^i\binom{i+1}{i}\times \binom{i}{j}\binom{\frac{k}{\sigma}+\frac{1}{\sigma}+m}{m}$$

and $\Gamma(.)$ *is gamma function such that*

$$\Gamma(w)=\int_0^\infty t^{w-1}exp(-t)dt.$$

Proof. The n^{th} moment of *MOG-W* random variable X is given by

$$\mu_n^{'} = \int_0^\infty x^n g(x)dx. \tag{13}$$

By the value of X in Theorem 1, and Equation (9); Equation (13) becomes

$$\mu_n^{'}=\int_0^\infty\left\{\theta\left[\log\left(exp(y)+1\right)\right]^{\frac{1}{\alpha}}\right\}^n\frac{\phi\vartheta exp\left(-\frac{y}{\sigma}\right)exp\left[-\vartheta exp\left(-\frac{y}{\sigma}\right)\right]}{\sigma\left(1-\bar{\phi}\left\{1-exp\left[-\vartheta exp\left(-\frac{y}{\sigma}\right)\right]\right\}\right)^2}dy \tag{14}$$

and applying the general binomial series expansion in Equation (14) yields

$$\mu_n^{'}=\frac{\phi\vartheta\theta^n}{\sigma}\sum_{k=0}^{\infty}\Psi_k\int_0^\infty\left[\log\left(exp(y)+1\right)\right]^{\frac{n}{\alpha}}exp\left[-(1+k)\frac{y}{\sigma}\right]dy; \tag{15}$$

where

$$\Psi_k=\frac{(-1)^k}{k!}\left[\vartheta(1+j)\right]^k\sum_{i,j=0}^{\infty}(-1)^j\bar{\phi}^i\binom{i+1}{i}\binom{i}{j}.$$

Letting $z=\log\left(exp(y)+1\right)$, then $dy=\frac{exp(z)}{exp(z)-1}dz$; and substituting in Equation (15) yields

$$\begin{aligned}\mu_n^{'} &= \frac{\phi\vartheta\theta^n}{\sigma}\sum_{k=0}^{\infty}\Psi_k\int_0^\infty z^{\frac{n}{\alpha}}exp\left[-\left(\frac{k}{\sigma}+\frac{1}{\sigma}\right)z\right]\left(1-exp(-z)\right)^{\frac{k}{\sigma}+\frac{1}{\sigma}+1}dz\\ &= \frac{\phi\vartheta\theta^n}{\sigma}\sum_{k=0}^{\infty}\Psi_k^*\int_0^\infty z^{\frac{n}{\alpha}}exp\left[-\left(\frac{k}{\sigma}+\frac{1}{\sigma}+1\right)z\right]dz. \end{aligned}\tag{16}$$

By change of variable $w=\left(\frac{k}{\sigma}+\frac{1}{\sigma}+1\right)z$ in Equation (16) and simplifying yields

$$\mu_n^{'}=\frac{\phi\vartheta\theta^n}{\sigma}\sum_{k=0}^{\infty}\Psi_k^*\Gamma\left(\frac{n}{\alpha}+1\right);$$

where

$$\Psi_k^* = \frac{(-1)^k}{k!}\left[\vartheta(1+j)\right]^k \sum_{i,j,m=0}^{\infty} (-1)^j \bar{\phi}^i \binom{i+1}{i}\binom{i}{j}\binom{\frac{k}{\sigma}+\frac{1}{\sigma}+m}{m}.$$

□

The incomplete moment plays an important role in determining the shape of distributions and measures of inequalities such as Lorenz curve, Gini and Pietra measures Butler and McDonald(1989). For a random variable X with *pdf* $f(x)$, the n^{th} incomplete moment is given by

$$I(y,n) = \int_0^y x^n f(x)dx.$$

If X has *pdf* as defined in Equation (6), then the n^{th} incomplete moment is defined as

$$\begin{aligned} I(y,n) &= \int_0^y x^n \frac{\phi\vartheta\alpha x^{\alpha-1} exp\left[\left(\frac{x}{\theta}\right)^\alpha\right]}{\sigma\theta^\alpha \left\{exp\left[\left(\frac{x}{\theta}\right)^\alpha\right]-1\right\}^{\frac{1}{\sigma}+1}} \times \\ &\quad \frac{exp\left[-\vartheta\left(exp\left[\left(\frac{x}{\theta}\right)^\alpha\right]-1\right)^{-\frac{1}{\sigma}}\right]}{\left(1-\bar{\phi}\left\{1-exp\left[-\vartheta\left(exp\left[\left(\frac{x}{\theta}\right)^\alpha\right]-1\right)^{-\frac{1}{\sigma}}\right]\right\}\right)^2}dx \\ &= \sum_{m=0}^{\infty} W_m \int_0^y x^{n+\alpha-1} exp\left[-\left(\frac{k}{\sigma}+\frac{1}{\sigma}+m\right)\left(\frac{x}{\theta}\right)^\alpha\right]dx; \end{aligned} \tag{17}$$

where

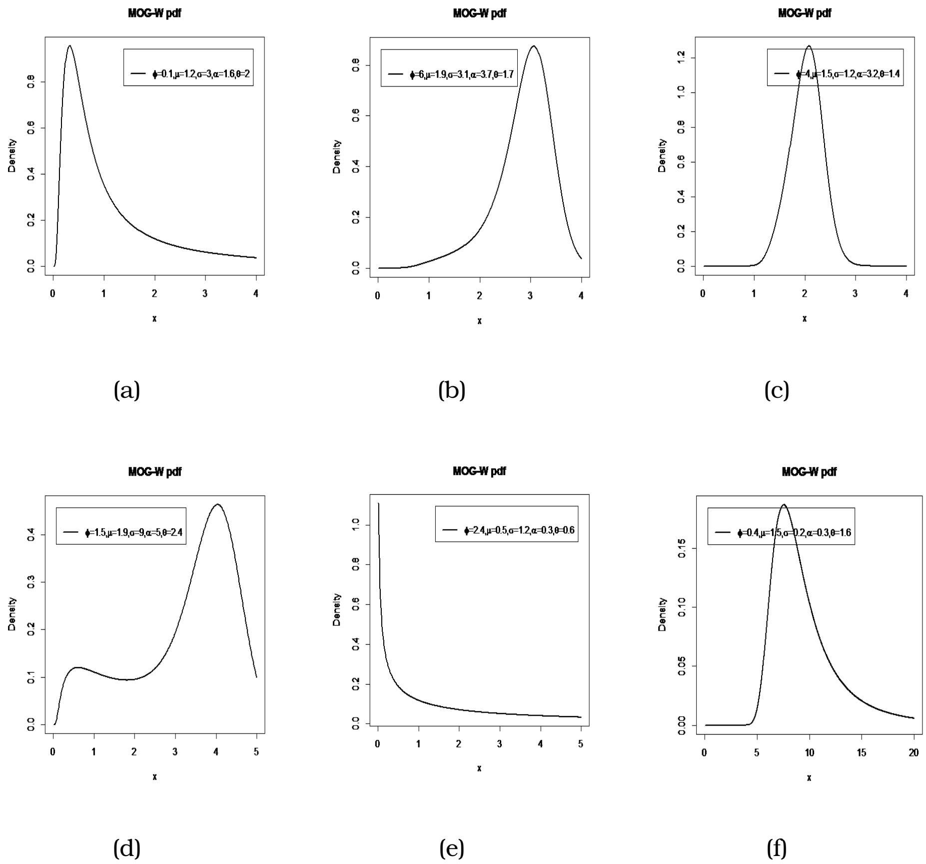
$$W_m = \frac{\phi\vartheta\alpha}{\sigma\theta^\alpha}\sum_{i,j,k=m}^{\infty}\frac{(-1)^{j+k}}{k!}\left[\vartheta(1+j)\right]^k \bar{\phi}^i \binom{i+1}{i}\binom{i}{j}\binom{\frac{k}{\sigma}+\frac{1}{\sigma}+m}{m}.$$

By change of variable $h = \left(\frac{k}{\sigma}+\frac{1}{\sigma}+m\right)\left(\frac{x}{\theta}\right)^\alpha$ in Equation (17) integral; simplifying, we obtain the n^{th} incomplete moment for *MOG-W* random variable as

$$I(y,n) = \sum_{m=0}^{\infty} W_m^* \left[\Gamma\left(\frac{n}{\alpha}+1\right) - \Gamma\left(\frac{n}{\alpha}+1, y\right)\right];$$

where

$$\begin{aligned} W_m^* &= \frac{\phi\vartheta\theta^n}{\sigma\left(\frac{k}{\sigma}+\frac{1}{\sigma}+m\right)^{\frac{n}{\alpha}+1}}\sum_{i,j,k=m}^{\infty}\frac{(-1)^{j+k}}{k!}\left[\vartheta(1+j)\right]^k \bar{\phi}^i \binom{i+1}{i}\binom{i}{j} \times \\ &\quad \binom{\frac{k}{\sigma}+\frac{1}{\sigma}+m}{m}. \end{aligned}$$



(a) (b) (c)

(d) (e) (f)

Fig. 2: Some possible shapes of *MOG-W* density function: a) Right-Skewed b) left-Skewed c) Symmetric d) Bimodal e) Monotone decreasing and f) Unimodal

4.2. Mean and median deviations

The mean and median deviations are measures used to evaluate the dispersion and spread in a population from the center. The mean and median deviations are, respectively, defined by $D(\mu)$ is defined as $D(\mu) = 2\mu_1^{'}G(\mu_1^{'}) - 2I(\mu_1^{'})$ and $D(M) = \mu_1^{'} - 2I(M)$. The $D(\mu)$ and $D(M)$ of a random variable X from *MOG-W* population are respectively given by

$$D(\mu) = 2\mu_1^{'}G(\mu_1^{'}) - 2\sum_{m=0}^{\infty} B_m \left[\Gamma\left(\frac{1}{\alpha}+1\right) - \Gamma\left(\frac{1}{\alpha}+1, y\right)\right]$$

and

$$D(M) = \mu_1^{'} - 2\mu_1^{'} G(\mu_1^{'}) - 2\sum_{m=0}^{\infty} B_m \left[\Gamma\left(\frac{1}{\alpha}+1\right) - \Gamma\left(\frac{1}{\alpha}+1, y\right)\right];$$

where

$$B_m = \frac{\phi\vartheta\theta}{\sigma\left(\frac{k}{\sigma}+\frac{1}{\sigma}+m\right)^{\frac{1}{\alpha}+1}} \sum_{i,j,k=m}^{\infty} \frac{(-1)^{j+k}}{k!} \left[\vartheta(1+j)\right]^k \bar{\phi}^i \binom{i+1}{i}\binom{i}{j} \times \binom{\frac{k}{\sigma}+\frac{1}{\sigma}+m}{m},$$

M, and $I(.)$ are as defined in *Eqs.*(12) and (17) respectively.

4.3. Trimmed L-moments

The Trimmed L-moments provide more robust generalization method of moments. Elamir and Scheult(2003) introduced the trimmed-L moments and define the r^{th} population trimmed-L moment as

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left(Y_{r+t_1-k:r+t_1+t_2}\right);\ r = 1, 2, \cdots$$

However, a special case for $t_1 = t_2 = t$ is given by

$$\lambda_r^t = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left(Y_{r+t-k:r+2t}\right);\ r = 1, 2, \cdots \tag{18}$$

where

$$E\left(Y_{r+t-k:r+2t}\right) = \frac{(n+2t)!}{(n+t-k-1)!(t+k)!}\int_{\forall x} x g(x) G(x)^{n-t-k-1}\left(1-G(x)\right)^{t+k} dx.$$

Substituting the values of $g(x)$ and $G(x)$ of *Eqs.*(6) and (5) respectively in Equation (18) and simplifying, we obtain the r^{th} *TL*-population moment for *MOG-W* random variable X as

$$\lambda_r^t = \Psi \sum_{j,m,q=0}^{\infty}\sum_{s=q}^{\infty} \frac{(-1)^{m+s}}{s!}\bar{\phi}^j \binom{n-t-k+i+1}{j}\binom{j}{m}\binom{\frac{1}{\sigma}+\frac{s}{\sigma}+1}{q};$$

where

$$\Psi = \frac{1}{r}\sum_{k=0}^{r-1}\sum_{i=0}^{t+k}(-1)^{k+i}\binom{n-1}{k}\binom{t+k}{i}.$$

The population measures of location, scale, skewness and kurtosis can be obtained for$t = 1$ and $r = 1, 2, 3, 4$ respectively.

5. Reliability functions

In this section, we consider functions used in characterizing lifetime distribution.

5.1. Failure rate function

Let X be a random variable with *cdf* $F(x)$ and associated *pdf* $f(x)$, the failure or hazard rate function (hrf) is defined by $\frac{f(x)}{S(x)}$; where $S(x) = 1 - F(x)$ is the survival function. Then, we obtained the hazard rate function (hrf) of *MOG-W* distribution as

$$hrf = \frac{\phi\vartheta\alpha x^{\alpha-1}exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]\left\{exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right\}^{-\left(\frac{1}{\sigma}+1\right)}}{\sigma\theta^{\alpha}\left\{1-\bar{\phi}\left[1-exp\left(-\vartheta\left\{exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}\right)\right]\right\}}\times \frac{exp\left(-\vartheta\left\{exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}\right)}{\left\{\phi-\phi\left[exp\left(-\vartheta\left\{exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}\right)\right]\right\}}.$$

Figure 3 shows graphical shapes of *MOG-W* failure rate for selected parameter values. The shapes show the flexibility and capacity of the *MOG-W* to model failure rate data sets with bathtub shape, in addition to monotonic increasing and decreasing, and unimodal shapes.

5.2. Mean residual life

The mean residual life function $m(t) = E(X - t|X > t)$ considers the expectation of the remaining life of a component after time t Guess and Proschan(1971). For a random variable X having *cdf* $F(x)$, the mean residual life function is defined by

$$E(X - t|X > t) = \frac{1}{1 - F(t)}\int_{t}^{\infty}[1 - F(x)]\,dx. \tag{19}$$

Let X be a random variable having *cdf* of Equation (5), substituting Equation (5) in Equation (19) and simplifying, we obtain

$$m(t) = \phi^{-1}\left[\sum_{i,s,v=0}^{\infty}\frac{(-1)^s}{s!}(i\vartheta)^s\binom{\frac{s}{\sigma}+v-1}{v}exp\left[-\left(\frac{s}{\sigma}+v\right)\left(\frac{t}{\theta}\right)^{\alpha}\right]-\bar{\phi}\right]\times$$
$$\phi\sum_{i,j,s,v=0}^{\infty}\frac{(-1)^{j+s}}{s!}\bar{\phi}^i\binom{i+1}{j}\binom{\frac{s}{\sigma}+v-1}{v}(j\vartheta)^s\times$$
$$\int_t^{\infty}exp\left[-\left(\frac{s}{\sigma}+v\right)\left(\frac{x}{\theta}\right)^{\alpha}\right]dx. \tag{20}$$

Letting $y=\left(\frac{s}{\sigma}+v\right)\left(\frac{x}{\theta}\right)^{\alpha}$, $dx=\frac{\theta y^{\frac{1}{\alpha}-1}}{\alpha\left(\frac{s}{\sigma}+v\right)^{\frac{1}{\alpha}}}dy$ then, by change of variable the integral in Equation (20) becomes

$$\frac{\theta}{\alpha\left(\frac{s}{\sigma}+v\right)^{\frac{1}{\alpha}}}\Gamma\left(\frac{1}{\alpha},t\right). \tag{21}$$

Combining *Eqs.*(20) and (21), the mean residual life function of *MOG-W* becomes

$$m(t) = \left[\sum_{i,s,v=0}^{\infty}\frac{(-1)^s}{s!}(i\vartheta)^s\binom{\frac{s}{\sigma}+v-1}{v}exp\left[-w\right]-\bar{\phi}\right]\frac{\theta}{\alpha\left(\frac{s}{\sigma}+v\right)^{\frac{1}{\alpha}}}\times$$
$$\sum_{i,j,s,v=0}^{\infty}\frac{(-1)^{j+s}}{s!}\bar{\phi}^i\binom{i+1}{j}\binom{\frac{s}{\sigma}+v-1}{v}(j\vartheta)^s\Gamma\left(\frac{1}{\alpha},t\right) \tag{22}$$

where $w=\left(\frac{s}{\sigma}+v\right)\left(\frac{t}{\theta}\right)^{\alpha}$.

6. Entropy

Entropy measure determines variation or uncertainty of a random variable. The two popular entropies are Rényi and Shannon. According to Rényi (1961), the Rényi entropy generalizes the Shannon entropy and for a random variable with $f(x)$, the Rényi entropy is defined by

$$I_{R(\gamma)}=\frac{1}{1-\gamma}\log\int_{\forall x}f^{\gamma}(x)dx;\quad \gamma>0,\gamma\neq 1. \tag{23}$$

Suppose $f(x)$ in Equation (23) is as defined in Equation (6), we have that

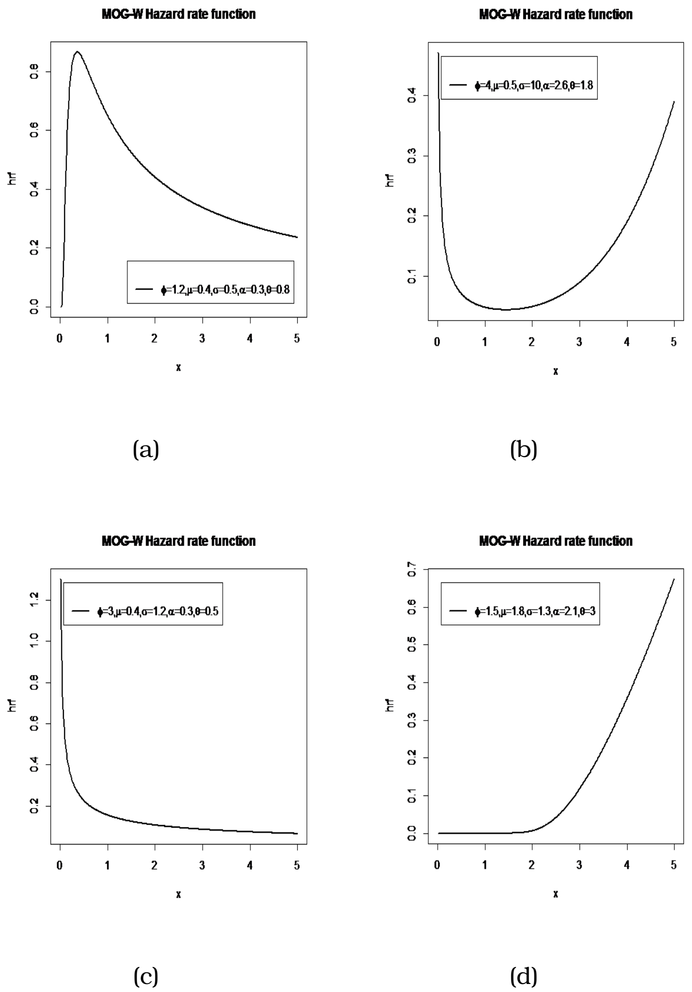


Fig. 3: Some possible shapes of *MOG-W* hazard rate function: a) Right-Skewed b) Bathtub c) Monotone decreasing and d) Monotone increasing

$$
\begin{aligned}
g^{\gamma}(x) &= \left(\frac{\phi\vartheta\alpha}{\sigma\theta^{\alpha}}\right)^{\gamma} \frac{x^{\gamma(\alpha-1)} exp\left[\gamma\left(\frac{x}{\theta}\right)^{\alpha}\right]}{\left\{exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right\}^{\gamma\left(\frac{1}{\sigma}+1\right)}} \times \\
&\frac{exp\left(-\gamma\vartheta\left\{exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}\right)}{\left\{1-\bar{\phi}\left[1-exp\left(-\vartheta\left\{exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}\right)\right]\right\}^{2\gamma}} \\
&= \left(\frac{\phi\vartheta\alpha}{\sigma\theta^{\alpha}}\right)^{\gamma} x^{\gamma(\alpha-1)} exp\left[\gamma\left(\frac{x}{\theta}\right)^{\alpha}\right] \sum_{i,j,k=0}^{\infty} \frac{(-1)^{j+k}}{k!}\bar{\phi}^{i} \times \\
&\binom{2\gamma+i-1}{i}\binom{i}{j}\left[\vartheta(\gamma+j)\right]^{k}\left(exp\left[\left(\frac{x}{\theta}\right)^{\alpha}\right]-1\right)^{-\left(\frac{k}{\sigma}+\frac{\gamma}{\sigma}+\gamma\right)} \\
&= \left(\frac{\phi\vartheta\alpha}{\sigma\theta^{\alpha}}\right)^{\gamma} \sum_{i,j,k,m=0}^{\infty} \frac{(-1)^{j+k}}{k!}\bar{\phi}^{i}\binom{2\gamma+i-1}{i}\binom{i}{j} \times \\
&\binom{\frac{k}{\sigma}+\frac{\gamma}{\sigma}+\gamma+m-1}{m}\left[\vartheta(\gamma+j)\right]^{k} x^{\gamma(\alpha-1)} \times \\
&exp\left[-\left(\frac{k}{\sigma}+\frac{\gamma}{\sigma}+m\right)\left(\frac{x}{\theta}\right)^{\alpha}\right]. \qquad (24)
\end{aligned}
$$

By *Eqs.*(23) and (24), the Rényi entropy for *MOG-W* random variable is defined by

$$\begin{aligned} I_R(\gamma) &= \frac{1}{1-\gamma} \log \Bigg\{ \left(\frac{\phi\vartheta\alpha}{\sigma\theta^{\alpha}}\right)^{\gamma} \sum_{i,j,k,m=0}^{\infty} \frac{(-1)^{j+k}}{k!} \bar{\phi}^{i} \binom{2\gamma+i-1}{i} \times \\ & \binom{i}{j} \binom{\frac{k}{\sigma}+\frac{\gamma}{\sigma}+\gamma+m-1}{m} \left[\vartheta(\gamma+j)\right]^{k} \times \\ & \int_0^{\infty} x^{\gamma(\alpha-1)} exp\left[-\left(\frac{k}{\sigma}+\frac{\gamma}{\sigma}+m\right)\left(\frac{x}{\theta}\right)^{\alpha}\right] \Bigg\} dx. \end{aligned} \tag{25}$$

The solution to the integral in Equation (25) is

$$\frac{\theta^{\gamma(\alpha-1)+1}}{\alpha\left(\frac{k}{\sigma}+\frac{\gamma}{\sigma}+m\right)^{\gamma(\alpha-1)+\frac{1}{\alpha}}} \Gamma\left(\gamma-\frac{\gamma}{\alpha}+\frac{1}{\alpha}\right). \tag{26}$$

Combining *Eqs.*(25) and (26), the Rényi entropy of *MOG-W* is given by

$$I_R(\gamma) = \frac{\gamma}{1-\gamma}\log(\phi\vartheta\alpha) - \frac{\gamma}{1-\gamma}\log(\sigma\theta) + \frac{1}{1-\gamma}\log\left(\sum_{m=0}^{\infty} d_m \Gamma\left(\gamma-\frac{\gamma}{\alpha}+\frac{1}{\alpha}\right)\right);$$

where

$$\begin{aligned} d_m &= \frac{\theta}{\alpha\left(\frac{k}{\sigma}+\frac{\gamma}{\sigma}+m\right)^{\gamma(\alpha-1)+\frac{1}{\alpha}}} \sum_{i,j,k=m}^{\infty} \frac{(-1)^{j+k}}{k!} \bar{\phi}^{i} \binom{2\gamma+i-1}{i} \binom{i}{j} \times \\ & \binom{\frac{k}{\sigma}+\frac{\gamma}{\sigma}+\gamma+m-1}{m} \left[\vartheta(\gamma+j)\right]^{k}. \end{aligned}$$

7. Order statistics

Let $X_1 < X_2 < \cdots < X_n$ denote the ordered sample of size n from *MOG-W* population. The *pdf* of the i^{th} order statistics of the sample is given by

$$g(x_{i:n}) = \frac{n!}{(i-1)!(n-i)!} g(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} G(x)^{i+j-1}. \tag{27}$$

Substituting the values of $G(x)$ and $g(x)$ in *Eqs.*(5) and Equation (6) respectively in Equation (27) and simplifying yields

$$
\begin{aligned}
g(x_{i:n}) &= \frac{n!}{(i-1)!(n-i)!}\frac{\phi\vartheta\alpha}{\sigma\theta^{\alpha}}\sum_{j=0}^{n-i}\binom{n-i}{j}\sum_{q=0}^{\infty}w_q x^{\alpha-1}\times \\
&\quad exp\left[-\left(\frac{m}{\sigma}+\frac{1}{\sigma}+q\right)\left(\frac{x}{\theta}\right)^{\alpha}\right];
\end{aligned}
$$

where

$$
w_q=\sum_{k,l,m=q}^{\infty}\frac{(-1)^{l+m}}{m!}\bar{\phi}^k\binom{i+j+k}{k}\binom{k}{l}\binom{\frac{m}{\sigma}+\frac{1}{\sigma}+q}{q}\left[\vartheta(i+j+l)\right]^m.
$$

8. Parameter estimation

We consider the estimation of the unknown parameters of *MOG-W* using the maximum likelihood approach. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from *MOG-W* population. The log-likelihood function is given by

$$
\begin{aligned}
\ell(\Theta) &= n\log(\phi)+n\log(\alpha)-n\log(\sigma)-n\alpha\log(\theta)+\frac{n\mu}{\sigma}+(\alpha-1)\log(x_i) \\
&\quad +\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\alpha}-exp\left(\frac{\mu}{\sigma}\right)\sum_{i=1}^{n}\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}} \\
&\quad -\left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\log\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]-1\right\} \\
&\quad -2\sum_{i=1}^{n}\log\left\{1-\bar{\phi}\left[1-exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}\right)\right]\right\}; \qquad (28)
\end{aligned}
$$

where $\Theta=(\phi,\mu,\sigma,\theta,\alpha)$ is parameter vector of *MOG-W* distribution. The score functions associated with Equation (28) are given by

$$
\frac{\partial\ell(\Theta)}{\partial\phi}=\frac{n}{\phi}-2\sum_{i=1}^{n}\frac{1-exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}\right)}{1-\bar{\phi}\left[1-exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}\right)\right]}
$$

$$
\begin{aligned}
\frac{\partial\ell(\Theta)}{\partial\mu} &= \frac{n}{\sigma}-\frac{\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}}{\sigma} \\
&\quad +2\sum_{i=1}^{n}\frac{\bar{\phi}\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}-exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}\right)}{\sigma\left\{1-\bar{\phi}\left[1-exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]-1\right\}^{-\frac{1}{\sigma}}\right)\right]\right\}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell(\Theta)}{\partial \sigma} &= \frac{n\mu}{\sigma^2} - \frac{n}{\sigma} + \vartheta \sum_{i=1}^{n} \frac{\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}}}{\sigma^2} \left[\mu - \log\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}\right] \\
&- \sum_{i=1}^{n} \frac{\log\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}}{\sigma^2} \\
&+ 2\bar{\phi}\vartheta \sum_{i=1}^{n} \frac{\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}} exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}}\right)}{\sigma\left\{1 - \bar{\phi}\left[1 - exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}}\right)\right]\right\}} \times \\
&\left[\mu - \log\left\{\sum_{i=1}^{n}\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell(\Theta)}{\partial \alpha} &= \frac{n}{\alpha} - n\log(\theta) + \sum_{i=1}^{n}\log(x_i) + \frac{\vartheta}{\sigma}\sum_{i=1}^{n}\log\left(\frac{x_i}{\theta}\right)\left(\frac{x_i}{\theta}\right)^{\alpha} \times \\
& exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}-1} \\
& - \left(\frac{1}{\sigma} + 1\right)\sum_{i=1}^{n}\frac{\log\left(\frac{x_i}{\theta}\right)\left(\frac{x_i}{\theta}\right)^{\alpha} exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]}{\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}} \\
& - 2\frac{\vartheta}{\sigma}\sum_{i=1}^{n}\frac{\log\left(\frac{x_i}{\theta}\right)\left(\frac{x_i}{\theta}\right)^{\alpha} exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}}}{\left\{1 - \bar{\phi}\left[1 - exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}}\right)\right]\right\}} \times \\
& exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell(\Theta)}{\partial \theta} &= -\frac{n\alpha}{\theta} - \frac{\alpha}{x_i}\left(\frac{\alpha}{x_i}\right)^{\alpha+1} - \frac{\vartheta}{\sigma}\sum_{i=1}^{n}\frac{\alpha}{x_i}\left(\frac{\alpha}{x_i}\right)^{\alpha+1} exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] \times \\
& \left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}-1} + \left(\frac{1}{\sigma} + 1\right)\sum_{i=1}^{n}\frac{\frac{\alpha}{x_i}\left(\frac{\alpha}{x_i}\right)^{\alpha+1} exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]}{\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}} \\
& + 2\frac{\bar{\phi}\vartheta}{\sigma}\sum_{i=1}^{n}\frac{\frac{\alpha}{x_i}\left(\frac{\alpha}{x_i}\right)^{\alpha+1} exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right]\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}-1}}{\left\{1 - \bar{\phi}\left[1 - exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}}\right)\right]\right\}} \times \\
& exp\left(-\vartheta\left\{exp\left[\left(\frac{x_i}{\theta}\right)^{\alpha}\right] - 1\right\}^{-\frac{1}{\sigma}}\right).
\end{aligned}
$$

The maximum likelihood estimators, say $\hat{\Theta} = (\hat{\phi}, \hat{\mu}, \hat{\sigma}, \hat{\theta}, \hat{\alpha})$ for the unknown parameters of *MOG-W* are values of the parameters that satisfy the resultant equations emanated from equating the score functions to zero. However, there are no closed-form solutions for the estimators but the maximum likelihood estimates $\hat{\Theta}_E$ can be obtain using some non-iterative optimization techniques such as *BFGS* which is implementable in *R* statistical software, Maple etc.

The Fisher's Information Matrix $I(\Theta) = (\Phi_{ij})\ i,j = 1,2,3,4,5$ is obtained by taking the negative expectation of the second partial derivative of $\ell(\Theta)$ evalueted at maximum likelihood estimates, where $\Phi_{ij} = -E\left[\frac{\partial^2 \ell(\Theta)}{\partial\Theta_i \partial\Theta_j}\right]$. Under some regularity conditions, it is well established that $\sqrt{n}\left(\hat{\Theta} - \Theta\right)$ is asymptotically multivariate normal $N_5(0, \Sigma)$ **?**, where Σ is the variance-covariance of estimated parameters is obtained by taking the inverse of FIM $I^{-1}(\hat{\Theta}_E)$. The asymptotic confidence interval for *MOG-W* parameters can be obtained respectively by $\Theta = \pm Z_{\frac{\eta}{2}}\sqrt{\Sigma_{ii}}$.

9. Asymptotic Consistency

We investigated the performance of *MOG-W* $\hat{\Theta}_E$ asymptotically using simulation study. The performance is based on average of mean estimate($\bar{\hat{\Theta}}_E$), bias and mean square error (*mse*). The sample data used were simulated from *MOG-W* distribution using quantile function for selected parameter values $\phi = 3.5, \mu = 1.3, \sigma = 3, \theta = 1.2, \alpha = 1.5$ and $\phi = 2, \mu = 0.5, \sigma = 1.5, \theta = 1.2, \alpha = 1.3$. Samples of size $n = 50, 125, 200, 500, 1,000$ were generated for $N = 10,000$ times. The $\hat{\Theta}_E$ was obtained for $i = 1, 2, \cdots, N$ respectively, and $\bar{\hat{\Theta}}_E$, average bias and mse were computed respectively using

$$\bar{\hat{\Theta}}_E(n) = \frac{1}{N}\sum_{i=1}^{N}\hat{\Theta}_i,$$

$$bias(n) = \frac{1}{N}\sum_{1}^{N}\left(\hat{\Theta}_i - \Theta\right),$$

and

$$mse(n) = \frac{1}{N}\sum_{1}^{N}\left(\hat{\Theta}_i - \Theta\right)^2.$$

Table 1: Summary of the Means, Average bias and *MSE* from $10,000$ simulations of *MOG-W* distribution for parameter values $\phi = 3.5, \mu = 1.3, \sigma = 3, \theta = 1.2, \alpha = 1.5$

Parameter	Sample size	Mean value	Bias	Mse
ϕ	50	4.7983	1.2983	36.3031
	125	4.2919	0.7919	12.8314
	200	4.0639	0.5639	7.9929
	500	3.8323	0.3327	5.4197
	1000	3.6901	0.1900	4.3885
μ	50	0.7046	-0.5954	13.7144
	125	1.2069	-0.9305	4.4382
	200	1.3247	0.0248	3.2515
	500	1.4060	0.1060	1.9290
	1000	1.4434	0.1434	4.3885
σ	50	3.3652	0.3652	3.3498
	125	3.1672	0.1672	1.3822
	200	3.1548	0.1548	1.0374
	500	3.133	0.1328	0.6344
	1000	3.1287	0.1287	0.4629
θ	50	2.0329	0.8329	1.8792
	125	1.6727	0.4727	0.3739
	200	1.6154	0.4154	0.2566
	500	1.5659	0.3659	0.1689
	1000	1.5491	0.3491	0.1421
α	50	1.8443	0.3444	1.0489
	125	1.4655	-0.0345	0.2941
	200	1.3794	-0.1207	0.1919
	500	1.2998	-0.2002	0.1090
	1000	1.2694	-0.2306	0.0882

From Table 1 and Table 2, the values show that as the sample size increases; the mean, average bias and *MSEs* reduce. This implies that the parameters of *MOG-W* are stable under the maximum likelihood estimation.

Table 2: Summary of the Means, Average bias and *MSE* from $10,000$ simulations of *MOG-W* distribution for parameter values $\phi = 2, \mu = 0.5, \sigma = 1.5, \theta = 1.2, \alpha = 1.3$

Parameter	Sample size	Mean value	Bias	*MSE*
ϕ	50	4.3529	2.3529	5.5428
	125	3.1349	1.1349	1.2892
	200	2.8539	0.8539	0.7298
	500	2.4594	0.4594	0.2112
	1000	2.3045	0.3045	0.0928
μ	50	-0.5369	-2.0369	4.1517
	125	-0.2474	-0.7474	0.5592
	200	0.0305	-0.4695	0.2207
	500	0.2875	-0.2125	0.0453
	1000	0.3740	-0.1259	0.0159
σ	50	2.1947	0.6947	0.4828
	125	1.7179	0.2179	0.0475
	200	1.6171	0.1171	0.0137
	500	1.5432	0.0432	0.0019
	1000	1.5194	0.0194	0.0004
θ	50	2.5411	1.3411	1.7991
	125	1.7164	0.5164	0.2667
	200	1.5421	0.3421	0.1170
	500	1.3975	0.1975	0.0390
	1000	1.3513	0.1513	0.0229
α	50	1.9647	0.6647	0.4419
	125	1.5979	0.2979	0.0889
	200	1.4811	0.1811	0.0328
	500	1.3444	0.0444	0.0019
	1000	1.2863	-0.0136	0.0002

10. Applications

We considered two real lifetime data sets applications of *MOG-W* distribution to illustrate its potentials by means of comparison with some distributions having Weibull distribution as its baseline distribution and also Weibull distribution. The first data set refers to data set taken from Gross and Clark(1975) and reported by Alizadeh *et al.*(2017). The data set is as follows 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The second data set is reported by Alzaatreh et al. [4]. The data set is as follows 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27,1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50,1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61,1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69,2.00, 2.01, 2.24. The distributions compared with *MOG-W* distribution are Gumbel-Weibull (*GW*) due to Al-Aqtash *et al.*(2014)

Table 3: Summary of the *MLEs* (Standard error in parentheses) for the first data set .

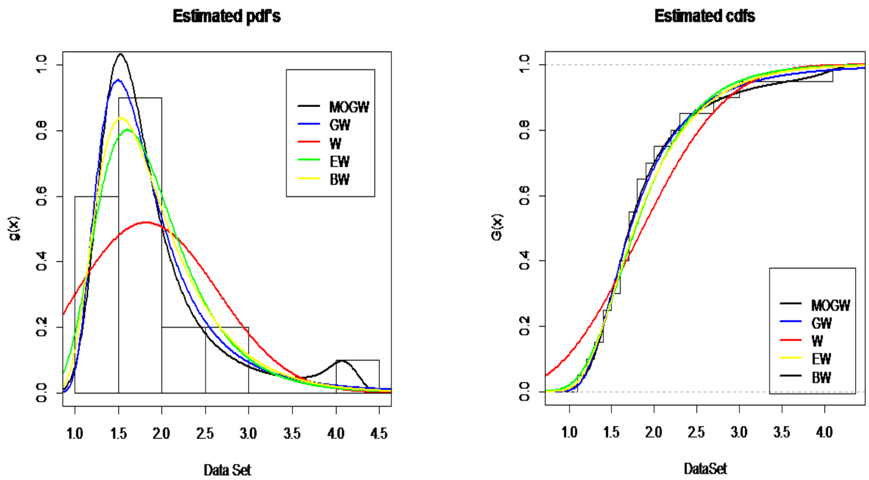
Distributions	Estimates				
MOG-W$(\phi,\mu,\sigma,\theta,\alpha)$	0.03837	-2.2697	13.4666	23.7064	3.7045
	(0.1356)	(27.1861)	(11.0861)	(0.0999)	(0.093)
GW$(\mu,\sigma,\theta,\alpha)$	0.7862	0.1149	0.2704	0.9022	
	(9.7622)	(0.4789)	(1.3141)	(20.6561)	
BW(a,b,θ,α)	36.4831	0.3565	1.365	0.4917	
	(45.9266)	(0.4453)	(0.7038)	(0.2564)	
MOW(ϕ,θ,α)	0.0069	5.7434	4.1748		
	(0.009)	(1.0198)	(0.9837)		
KW(a,b,θ,α)	32.6395	0.344	1.4676	0.5631	
	(37.545)	(0.3988)	(0.7122)	(0.2542)	
EW(a,θ,α)	34.7050	0.9710	0.4356		
	(39.4086)	(0.2849)	(0.3002)		
Weibull(θ,α)	2.787	2.1299			
	(0.4273)	(0.1820)			
AW(a,b,c,d)	9.5362	-4.5023	0.4695	2.7868	
	(118.5081)	(26.7699)	(0.0401)	(0.4273)	

with $\theta=\frac{1}{\theta^*}$, Beta-Weibull (*BW*) due to Lee *et al.*(2007), Kumaraswmy-Weibull (*KW*) distribution of Cordeiro *et al.*(2010), Marshall-Olkin extended Weibull (*MOW*) of Ghitany *et al.*(2005), Exponentiated-Weibull (*EW*) due to Mudholkar *et al.*(1996), and additive Weibull (*AW*) due to Xie and Lai (1995). The comparison is based on Cramer-von Mise (W), and Anderson-Darling (A) goodness-of-fit statistics. These two goodness-of-fit statistics are widely used to compare non-nested distributions and to determine how closely a specific *cdf* fits the empirical distribution of a given data set Tahmasebi and Jafari(2015). The lowest of these statistics indicate the model of best fit among the competing models Chen and Balakrishnan(1995).

Table 4: Cramer-von Mises (W^*) and Anderson-Darling (A^*) statistics for the first data set .

	MOG-W	*GW*	*BW*	*MOW*	*KW*	*EW*	Weibull	*AW*
W^*	0.0246	0.0285	0.0465	0.0579	0.496	0.0554	0.1857	0.1857
A^*	0.1448	0.1616	0.2709	0.3463	0.2896	0.3252	1.0929	1.0929

Tables 3 and 5 show the parameter estimates (standard errors in parentheses) of the first and second data sets respectively while Tables 4 and 6 indicate the estimated Cramer-von Mise (W^*) and Anderson-Darling (A^*) of first and second data sets respectively. The values of the goodness-of-fit statistics W^* and A^* in Tables 4 and 6 indicate that *MOG-W* has the least values among other competing distributions which imply that the *MOG-W* provided a better fit for the two data sets. The plots of fitted *pdfs* and empirical *cdf* with *cdfs* of *MOG-W* distribution and some competing distributions are shown in Figures 4 and 5.



(a) (b)

Fig. 4: Estimated plots for the first data set of a) some competing density functions b) empirical *cdf* and some competing distribution functions

Table 5: Summary of the *MLEs* (Standard error in parentheses) for the second data set .

Distributions	Estimates				
MOG-W$(\phi, \mu, \sigma, \theta, \alpha)$	20.5862	-0.2487	1.6951	3.0224	0.8783
	(40.4027)	(2.5646)	(1.1285)	(0.9765)	(0.2641)
GW$(\mu, \sigma, \theta, \alpha)$	4.9357	4.8538	4.4144	0.9659	
	(2.6415)	(1.8000)	(0.8342)	(0.1493)	
BW(a, b, θ, α)	0.8443	16.6744	5.7802	2.6400	
	(0.4101)	(412.6377)	(1.7462)	(11.3483)	
MOW(ϕ, θ, α)	24.4258	2.7041	0.9626		
	(36.6294)	(0.9198)	(0.2951)		
KW(a, b, θ, α)	0.6984	0.2049	5.9236	1.2248	
	(0.2100)	(0.0309)	(0.0027)	(0.0027)	
EW(a, θ, α)	0.8958	5.5561	1.5923		
	(0.3868)	(1.4241)	(0.1168)		
Weibull(θ, α)	5.2348	1.5643			
	(0.5628)	(0.0439)			
AW(a, b, c, d)	9.4086	-7.8667	0.6393	5.2344	
	(74.7204)	(43.3958)	(0.0179)	(0.5627)	

Table 6: Cramer-von Mises (W^*) and Anderson-Darling (A^*) statistics for the second data set.

	MOG-W	*GW*	*BW*	*MOW*	*KW*	*EW*	Weibull	*AW*
W^*	0.1872	0.2056	0.3131	2.0456	0.2757	0.3060	0.3030	0.3131
A^*	1.1267	1.2318	1.7238	10.0668	1.5698	1.6950	1.6826	1.7238

11. Conclusion

In this paper, we introduced a new distribution called Marshall-Olkin extended Gumbel-Weibull for modeling lifetime data set. Some fundamental properties of the new distribution including the failure rate function, quantile function, moment about the origin, mode, order statistics were studied. We also studied the kurtosis of the *MOG-W* distribution which revealed that the *MOG-W* distribution has a heavier-tailed distribution than its baseline distribution. The maximum likelihood method was considered for the estimation of the unknown parameters of the distribution. The asymptotic performance of *MLEs* was studied extensively through simulations on varied selected parameter values. The simulation studies indicated that the mean, average bias and *MSEs* decrease as the sample size increases which implies that the estimate of the *MOG-W* parameters were stable under the maximum likelihood method. The potential and flexibility of the *MOG-W* distribution were illustrated in two real-life data set applications which indicated that the *MOG-W* provided better fit among the competing distributions.

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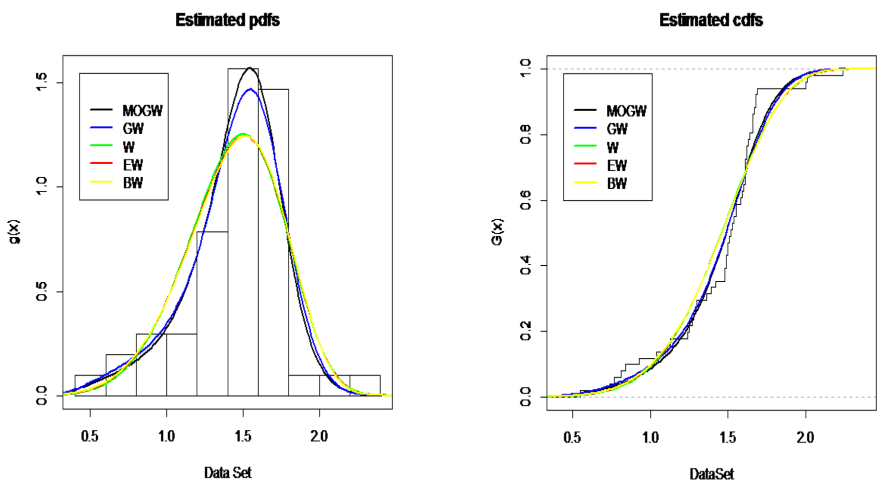

(a) (b)

Fig. 5: Estimated plots for the second data set of a) some competing density functions b) empirical *cdf* and some competing distribution functions

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