Afrika Statistika

Afrika Statistika
Vol. 16 (1), 2021, pages 2587 - 2602.
DOI: http://dx.doi.org/10.6929/as/2021.2587.175

ISSN 2316-090X

Log-Exponential Power Distribution for Accelerated Failure Time Model in Survival Analysis and Its Application

Akinlolu Adeseye Olosunde [1*] **and Chidimma Florence Ejiofor** [2]

[1,2]Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria.

Received on March 29, 2021; Accepted on June 28, 2021

Abstract. We proposed the log-exponential power density function as baseline distribution for accelerated failure time model (*AFT*) used in analysis of survival data with covariates. This model generalizes the log-normal and some exponential family due to flexibility at the tail region. It has log-concavity property, accommodates the four basic shapes of hazard function which is an attractive property compared with other distributions that cannot accommodate same. The model's goodness of fit relative to some existing models was tested using data from chronic liver disease patients monitored at Obafemi Awolowo University Teaching Hospital, Ile-Ife.

Key words: Log-exponential power distribution; log-concavity; accelerated failure time model; survival analysis.
AMS 2010 **Mathematics Subject Classification Objects :** 62Nxx; 60E05; 62P10.

---

*Corresponding author Olosunde, A. A: aolosunde@oauife.edu.ng
Ejiofor, C. F : cflorence007@gmail.com

Résumé. (Abstract in French) Dans ce papier, nous proposons la fonction de densité de puissance log-exponentielle comme distribution comme fonction de base pour le modèle du temps défaillance accéléré *AFT* en Analyse des données de survie avec des covariables. Ce modèle généralise la famille log-normale et une famille exponentielle en raison de la flexibilité de la queue de la distribution. Ce modèle a une propriété de concavité logarithmique, s'adapte aux quatre formes de base de la fonction de danger, ce qui est une propriété attrayante par rapport aux autres distributions qui n'ont pas. La qualité de l'ajustement du modèle par rapport à certains modèles existants a été testée à l'aide de données provenant de patients atteints d'une maladie hépatique chronique suivis à l'hôpital universitaire Obafemi Awolowo, Ile-Ife.

The authors.
Akinlolu Adeseye Olosunde, Ph.D., is a Reader at Obafemi Awolowo University, Ile-Ife, Nigeria.
Chidimma Florence Ejiofor, M.Sc. in Statistics, is a Researcher at Obafemi Awolowo University, Ile-Ife, Nigeria.

1. Introduction

The Accelerated Failure Time model (*AFT*), which is a parametric survival regression model in which the effects of covariates on the response variable (the logarithm of survival times) is to accelerate or decelerate survival time has received substantial attention in recent times. This is attributable to the straight-forward method of estimation based on the maximum likelihood estimate as against the partial likelihood as in semi-parametric ones Lawless (2005). Khan and Khosa (2016) affirmed that parametric survival models give rise to more efficient parameter estimates than the semi-parametric ones. The *AFT* model also serves as an alternative approach in modeling survival times when assumptions of the commonly used Proportional Hazard model flops Wei (1992).

Several *AFT* models exist such as the generalized gamma *AFT* model as in Cox *et al.* (2007) and the inverse gaussian *AFT* model Lemeshko *et al.* (2010). For further discussion on parametric survival models, see Kalbfleisch and Prentice (2002), Collett (2003), Hashimoto *et al.* (2016), Rezaei *et al.* (2014), Pescim *et al.* (2013), Reed (2011), Mahmoud *et al.* (2015) and Ortega *et al.* (2012). Dey *et al.* (2019) proposed alpha-power transformed Lomax (*APTL*) distribution which generalized the existing Lomax distribution to provide better fits in modeling survival and lifetime time data.

In a review on *AFT* models, Saikia and Barman (2017) suggested that since gamma, weibull, log-logistic, lognormal and exponential distributions have been widely used as *AFT* models, other distributions such as skew normal, generalized exponential, and the likes should be explored. Hence, the choice of baseline distribution in *AFT* model is vital in that each distribution has its hazard function which could take on a variety of shapes. That is why Maxim (2008) and Tableman and Kim (2004) remarked that although different distributions do have same basic shape of survival function (which is monotone decreasing), their hazard functions could change dramatically and therefore proves to be much more informative and a better way to depict mechanism of failure than the survival function.

It is on this basis that we developed and studied log-exponential power distribution which was derived from the transformation of exponential power distribution. The new model has flexible tails and log-concavity properties and equally accommodates the four basic shapes of hazard function: monotone increasing, monotone decreasing, increasing and then decreasing, decreasing and then increasing all these in a single model we proposed. These properties are attractive when compared with other distributions that cannot provide all these four different shapes for hazard function.

Some univariate and multivariate extensions of the exponential power distribution has been studied by Subbotin (1923), Olosunde (2013), Nadarajah (2005), Wenhao (2013), Kanichukattu and Paul (2018) and Hutson (2019). Finally, we applied the log-exponential power distribution as baseline distribution in accelerated failure time model in the analysis of chronic liver disease data.

2. Log-Exponential Power Distribution

Definition 2.1: Suppose a random variable X follow the univariate exponential power distribution defined as

$$f_X(x) = \frac{1}{\Gamma(1+\frac{1}{2\beta})2^{1+\frac{1}{2\beta}}} \exp\left\{ -\frac{1}{2}\left|\frac{x-\mu}{\sigma}\right|^{2\beta}\right\}, \qquad x, \mu \in R; \quad \sigma, \beta > 0, \tag{1}$$

where β is the shape parameter, μ and σ represent the location and scale parameters respectively, without loss of generality let $\mu = 0$, $\sigma = 1$ then we have the standardized exponential power density function given as

$$f_X(x) = \frac{1}{\Gamma(1+\frac{1}{2\beta})2^{1+\frac{1}{2\beta}}} \exp\{-\frac{1}{2}|x|^{2\beta}\}, \quad x \in R, \beta > 0$$

where β is the shape parameter. See Hutson (2019) for further reviews on exponential power distribution.

We derived the log-exponential power distribution through transformation of the univariate exponential power distribution. Hence, a random variable T is said to have log-exponential power distribution which satisfies all necessary and sufficient conditions if

$$f_T(t) = \frac{1}{t\sigma\Gamma(1+\frac{1}{2\beta})2^{1+\frac{1}{2\beta}}} \exp\left\{-\frac{1}{2}\left(\frac{\ln t}{\sigma}\right)^{2\beta}\right\}, \quad t>0,\ \sigma>0,\ \beta>0,$$

where σ and β represents the scale and shape parameters respectively.

Equation (1) will henceforth be referred to as the Log-Exponential Power distribution (*LEP*). We denote this distribution as $T \sim$ *LEP*(σ, β). We note that if β=1, Equation (1) reduces to the log-normal distribution. For higher values of β, such as β=2, β=3, ... we observe the high kurtosis which shows the ability of the distribution to capture inherent dynamic data. Figure 1 (top left) shows the plot of the density function for various values of β.

The plots of the probability density function, cumulative distribution function, survival and hazard functions were given in Figure 1. The probability density plot reveals flexibility in its tails for each increase in value of the shape parameter as well as increase at the peaks. This shows the distribution's ability to capture inherent dynamic data. More so, the plots of its hazard function shows that for various values of the shape parameter, the four basic shapes of hazard function were exhibited namely monotone increasing, decreasing, unimodal and bathtub shapes.

Proposition 1. *Let T be a random variable having log-exponential power distribution given in equation (3). Then the cumulative distribution function is*

$$F(t) = \frac{\gamma\left(\frac{1}{2\beta}, \frac{1}{2}(\frac{\ln t}{\sigma})^{2\beta}\right)}{\Gamma(\frac{1}{2\beta})}. \tag{2}$$

Proof: See Appendix.

Figure 1 (top right) shows the various shape of the *cdf* with different β. In what follows, we present a corollary to Proposition 1 to derive the survival and hazard functions.

Corollary 1. *Let T be a random variable having (1) as probability density function and cumulative distribution function in equation 2, then the survival and hazard functions are respectively*

$$S(t) = \frac{\Gamma(\frac{1}{2\beta}) - \gamma\left(\frac{1}{2\beta}, \frac{1}{2}(\frac{\ln t}{\sigma})^{2\beta}\right)}{\Gamma(\frac{1}{2\beta})} \tag{3}$$

and

$$h(t) = \frac{2^{1-\frac{1}{2\beta}} \exp(-\frac{1}{2}(\frac{\ln t}{\sigma})^{2\beta})}{\frac{\sigma t}{\beta}\left[\Gamma(\frac{1}{2\beta}) - \gamma\left(\frac{1}{2\beta}, \frac{1}{2}(\frac{\ln t}{\sigma})^{2\beta}\right)\right]}. \tag{4}$$

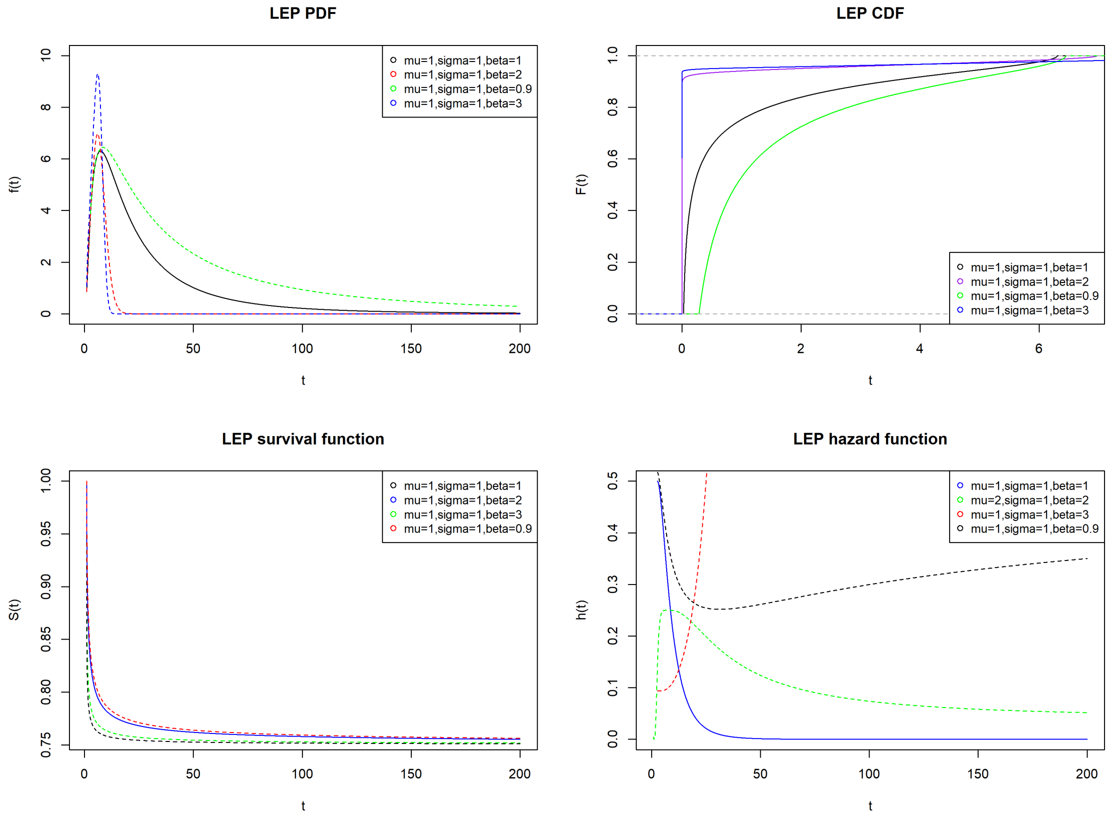

Fig. 1. Log-exponential power plot of functions

The plot of survival(down left) and hazard functions (down right) are presented in above Figure 1. The next proposition is to establish the log-concavity of the *pdf* (1). Bagnoli and Bergstrom (2005) studied the properties and importance of log-concavity of some probability density function. Olosunde (2020) presented properties and applications of log concave exponential power distribution which is the baseline *pdf* of (1) in the present study. Generally, a function is said to be log-concave if twice-differentiable real-valued function, k whose domain is an interval on the extended real line is a function that satisfies the condition $(\ln k(.))'' < 0$. Now, for the case of log-exponential power distribution (1) we have the following:

Definition 2.2: The random variable X is said to have increasing (decreasing) hazard rate (failure rate) if $h(x)$ is a decreasing (increasing) function of x.

The following theorem connect the log-concavity of the reliability function and the monotonicity of the hazard components.

Theorem 1. *Ma (2000) Let X be an absolutely continuous random vector with joint reliability function $\bar{\mathbf{F}}(\mathbf{x}) = \mathbb{P}(X_1 > x_1, ..., X_n > x_n)$. If $\bar{\mathbf{F}}(\mathbf{x})$ is log-concave, then the random vector X has increasing failure (hazard rate).*

Proposition 2. *Suppose T is random variable with pdf (1), then (1), (2), (3) and (4) are log-concave or log-convex functions depending on the value of β and for $t = 1$ it is log-convex.*

Proof. It is sufficient to prove that (3) is log-concave, consequently the result can be extended to other derivations of (3) namely (4) to (6). The Log-exponential power distribution with probability density function, $f(t)$ is said to be log-concave, if $(\ln f(t))'' < 0$ Bagnoli and Bergstrom (2005). From density 3 we could deduce that

$$\begin{aligned}(\ln f(t))'' &= \frac{d}{dt}\left[-\frac{1}{t} - \frac{\beta}{t}(\ln t)^{2\beta-1}\right] \\ &= \frac{1}{t^2} - \frac{\beta}{t^2}(2\beta-1)(\ln t)^{2\beta-2} + \frac{\beta}{t^2}(\ln t)^{2\beta-1} \\ &= \frac{1}{t^2} - \frac{\beta}{t^2}(\ln t)^{2\beta-2}(2\beta - 1 - \ln t), , \ t > 0 \, . \end{aligned} \quad (5)$$

Remark 1. From 5 the log-concavity or log-convexity depends on the values of the shape parameter β except at the point $t = 1$, where it is log-convex. For $\beta = \frac{1}{2}$, $(\ln f(t))'' > 0$ which is log-convexity and for $\beta = 1$, it is mixed, i.e. mixture of log-concavity and log-convexity. This results are useful in establishing the (increasing or decreasing) monotonicity of hazard functions of *LEP*.

Corollary 2. *The density function (1), survival function (3) and hazard function (4) have monotonically:*

i. *decreasing failure rate for $t \neq 1$ and $\beta = \frac{1}{2}$;*
ii. *increasing failure rate for $t \neq 1$ and $\beta > 1$; and*
iii. *decreasing failure rate for $t = 1$ for all values of β.*

Proof: The results are immediate from (5) and bringing the definitions and the theorem into consideration. These results generalized the cases of exponential, log normal and Weibull distributions etc. commonly used in reliability studies.

3. Some Inferential Aspects

3.1. Moments of Log-exponential power distribution

The r^{th} moment of a continuous random variable, T, which follows the log-exponential power distribution is given by,

Table 1. Moments of simulated sample from Log-exponential power distribution

β	expectation	variance	skewness	kurtosis
2	1.2609	0.7718	1.0147	4.7926
3	1.2145	0.5774	0.9659	3.2674
4	1.1986	0.5145	0.1147	2.8428
5	1.1911	0.4873	0.7903	2.6572
6	1.1869	0.4723	0.6358	2.5544
7	1.1843	0.4630	0.7421	2.4929
8	1.1825	0.4570	0.7259	2.4525
9	1.1813	0.4530	0.7154	2.4170
10	1.1803	0.4495	0.7088	2.4020

$$\begin{aligned} E(T^r) &= \int_0^\infty t^r \frac{1}{t\ \sigma\ \Gamma(1+\frac{1}{2\beta})\ 2^{1+\frac{1}{2\beta}}} \exp\left\{-\frac{1}{2}\left(\frac{\ln t}{\sigma}\right)^{2\beta}\right\} dt \\ &= \frac{1}{\sigma\ \Gamma(1+\frac{1}{2\beta})\ 2^{1+\frac{1}{2\beta}}} \int_0^\infty t^{r-1} \exp\left\{-\frac{1}{2}\left(\frac{\ln t}{\sigma}\right)^{2\beta}\right\} dt \end{aligned}$$

When β is 1, the log-exponential power distribution reduces to the lognormal distribution, same applies to the moments. The moments for higher values of β are given below.

Table 1 below gives the numerical results of the expectation, variance, skewness and kurtosis of simulated sample from log-exponential power distribution for $\beta = 2, \cdots, 10$.

3.2. Maximum Likelihood Estimate of Parameters of Log-exponential power Distribution

Given a random sample of n observations $t_1, t_2, \cdots, t_n$ from log-exponential power distribution, the likelihood function is given as:

$$L(t_i; \sigma, \beta) = \prod_{i=1}^{n} f(t_i; \sigma, \beta).$$

The log-likelihood function is given as

$$\begin{aligned} \ell(t_i; \sigma, \beta) &= n\ \ln \sum_{i=1}^{n} \left(\frac{1}{t_i \sigma \Gamma(1+\frac{1}{2\beta}) 2^{1+\frac{1}{2\beta}}}\right) - \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\ln t_i}{\sigma}\right)^{2\beta} \\ &= -\sum_{i=1}^{n} \ln t_i - n \ln \sigma - n \ln \Gamma(1+\frac{1}{2\beta}) - n\left(1+\frac{1}{2\beta}\right) \ln 2 - \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\ln t_i}{\sigma}\right)^{2\beta}. \end{aligned} \quad (6)$$

Differentiating (6) with respect to each of the parameters, we have

$$\frac{\partial \ell(t_i;\sigma,\beta)}{\partial \sigma} = \frac{-n}{\sigma} + \frac{\beta \sum_{i=1}^{n} (\ln t_i)^{2\beta}}{\sigma^{2\beta+1}};$$
$$\hat{\sigma} = \left(\frac{\beta \sum_{i=1}^{n} (\ln t_i)^{2\beta}}{n} \right)^{\frac{1}{2\beta}};$$
$$\frac{\partial \ell(t_i;\sigma,\beta)}{\partial \beta} = \frac{n\psi\left(1+\frac{1}{2\beta}\right)}{2\beta^2} + \frac{n \ln 2}{2\beta^2} - \sum_{i=1}^{n} \left(\frac{\ln t_i}{\sigma}\right)^{2\beta} \ln\left(\frac{\ln t_i}{\sigma}\right), \qquad (7)$$

where

$$\psi\left(1+\frac{1}{2\beta}\right) = \frac{d \ln\left(\Gamma(1+\frac{1}{2\beta})\right)}{d\beta}.$$

An explicit solution is not easily obtained for (7). We therefore adopt numerical approach and substitute for σ in the resulting equation to obtain estimated value for $\hat{\beta}$, codes were written in R environment to achieve this.

3.3. Fisher information

The fisher information matrix for a random variable T, that follows *LEP* and its parameters space, $P = (\sigma, \beta)$, the observed fisher information matrix is given by

$$I(P) = \begin{bmatrix} L_{\sigma\sigma} & L_{\beta\sigma} \\ L_{\sigma\beta} & L_{\beta\beta} \end{bmatrix}$$

where, the elements of $I(P)$ are

$$L_{\sigma\sigma} = \frac{\partial^2 \ell}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{\beta \sum_{i=1}^{n} (\ln t_i)^{2\beta}(2\beta+1)}{\sigma^{2\beta+2}},$$

$$L_{\beta\sigma} = \frac{\partial^2 \ell}{\partial \beta \partial \sigma} = \frac{\sum_{i=1}^{n} (\frac{\ln t_i}{\sigma})^{2\beta}}{\sigma} + \frac{2\beta \sum_{i=1}^{n} (\frac{\ln t_i}{\sigma})^{2\beta} \ln(\frac{\ln t_i}{\sigma})}{\sigma},$$

and

$$L_{\beta\beta} = \frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n\psi'(1+\frac{1}{2\beta})}{\beta^3} - \frac{n \ln 2}{\beta^3} - 2\sum_{i=1}^{n} \left(\frac{\ln t_i}{\sigma}\right)^{2\beta} \left(\ln\left(\frac{\ln t_i}{\sigma}\right)\right)^2..$$

4. Log-Exponential Power for Accelerated Failure Time Model (*AFT*)

To derive the log-exponential power *AFT* model, the log-exponential power density function (1) as well as its survival function (3) will be of great use. The two functions will be used as baseline distribution by substituting them in the likelihood function meant for estimating parameters of Accelerated Failure Time model. This likelihood function is peculiar to the type of data censoring involved. In this present work a case of right censored data will be considered. The likelihood function for any *AFT* model whose data is right censored is given as

$$L = \prod_{i=1}^{n} [f(t_i \mid c_i)]^{\delta_i} \quad [S(t_i \mid c_i)]^{1-\delta_i}, \tag{8}$$

where $f(t_i \mid c_i)$ and $S(t_i \mid c_i)$ are the density function and survival function of the *AFT* model respectively; t_i represents the survival time; c_i is $(c_1, \cdots, c_p)$, a vector of covariates under study; and ω_i is $(\omega_1, ..., \omega_p)$, a vector of regression coefficient.

The *AFT* model in terms of survival function is given as

$$S(t \mid c) = S_0(\exp\{c'\omega\}t), \tag{9}$$

where $S_0(.)$ is the baseline survival function.

Likewise, in terms of hazard function, *AFT* model is given as

$$h(t \mid c) = \exp\{c'\omega\} \; h_0(\exp\{c'\omega\}t),$$

where $h_0(.)$ is the baseline hazard function.

Then, the density function of *AFT* model is given by

$$\begin{aligned} f(t \mid c) = h(t \mid c) \; S(t \mid c) \\ &= \exp\{c'\omega\} \; h_0(\exp\{c'\omega\}t) \; S_0(\exp\{c'\omega\}t) \\ &= \exp\{c'\omega\} \; f_0(\exp\{c'\omega\}t), \end{aligned} \tag{10}$$

where $f_0(.)$ is the baseline density function. δ_i is the censoring indicator defined as

$$\delta_{\mathrm{i}} = \begin{cases} 1 & \text{if} \quad \text{the ith individual has the event of interest} \\ 0 & \text{if} \quad \text{the ith individual is censored (alive or discharged)} \end{cases}$$

4.1. Estimate of parameters of log-exponential power AFT model

Substituting Equations (9) and (10) in Equation (8), the likelihood function of Log-exponential power *AFT* model is given as

$$L = \prod_{i=1}^{n} \left[\exp\{c_i'\omega\} \frac{1}{\exp\{c_i'\omega\}t_i \; \sigma \; \Gamma(1+\frac{1}{2\beta}) \; 2^{1+\frac{1}{2\beta}}} \; \exp\left\{ -\frac{1}{2} \left(\frac{\ln(\exp\{c_i'\omega\}t_i)}{\sigma} \right)^{2\beta} \right\} \right]^{\delta_i}$$

$$\times \left[1 - \frac{1}{\Gamma(\frac{1}{2\beta})}\gamma\left(\frac{1}{2\beta}, \frac{1}{2}\left(\frac{\ln(\exp\{c_i'\omega\}t_i)}{\sigma}\right)^{2\beta}\right)\right]^{1-\delta_i}.$$

The log-likelihood function of *LEP AFT* model is given as

$$\ell(t_i, \delta_i, c_i, \sigma, \beta, \omega) = \delta \ln \tau - Z(.) - \sum_{i=1}^{n} \delta_i \ln t_i + (1-\delta_i)\ \ln(1 - K(.)),$$

where $\delta = \sum_{i=1}^{n} \delta_i$ and $\tau = (\sigma\Gamma(1+\frac{1}{2\beta})2^{1+\frac{1}{2\beta}})^{-1}$.

$$Z(t_i, \delta_i, c_i, \sigma, \beta, \omega) = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{c'\omega\ +\ \ln t_i}{\sigma}\right)^{2\beta},$$

and

$$K(t_i, \delta_i, c_i, \sigma, \beta, \omega) = \left(1 - \frac{1}{\Gamma(\frac{1}{2\beta})}\gamma\left(\frac{1}{2\beta}, \frac{1}{2}\frac{\ln(\exp\{c_i'\omega\}t_i)}{\sigma}^{2\beta}\right)\right).$$

Differentiating equation (24) with respect to each of the parameters and solving simultaneously, we obtain estimators of σ, β, and ω using numerical approach, Broyden-Fletcher-Goldfarb-Shanno (*BFGS*) algorithm, with codes written in *R* environment.

5. Application to data on chronic liver disease

The aforementioned methods were applied to data on chronic liver disease. Chronic liver disease is a disease in which the liver damages little by little from one stage to another starting with the liver tissue inflammation (known as Hepatitis) to liver fibrosis (which is the very first stage of liver scarring), then cirrhosis (a condition in which the usual healthy liver tissue is replaced by scar tissue) and finally to Hepatocellular Carcinoma (most commonly occurring primary liver cancer). Liver disease could be hereditary or caused by factors such as alcohol abuse, virus causing hepatitis A,B,C and so on.

Although the liver has the ability to heal itself (referred to as regeneration), continuous scarring which leads to cirrhosis impedes the liver's ability to completely heal. However, positive adjustments in lifestyle as well as medications can decelerate fibrosis build up.

Kim *et al.* (2015) stated that for most chronic diseases, age is a major risk factor. They also stated that longer interventions are needed in treatment of older patients who have liver disease. Furthermore, Guy and Peters (2013) asserted that men are twice more probable to die from chronic liver disease than women.

On this basis, two covariates namely age and sex were considered in this study. The data on chronic liver disease patients was obtained from Obafemi Awolowo University Teaching Hospital Complex, Ile-Ife, Osun State, Nigeria. It contains information on 45 patients which comprises 37 males and 8 females monitored at the facility and it is right censored. The survival times (measured in days) starts from date of admission to death. Those who were discharged are censored. Status denotes the censoring indicator such that those who died were coded as '1' and those who were discharged were coded '0'. Also, two covariates namely age (in years) and sex (males coded as '1' and females '0'were considered for each patient.

Codes are written in *R* environment with supplementary packages *normalp*, *fitdistrplus* and *survival* were used in estimating parameters of the distribution. They are all available in *R* software.

Table 2 shows summary of the dataset. It was observed that the minimum survival time is 1 day, the mean survival time is 15 days, median survival time is 12 days and the maximum survival time is 93 days.

Also, patients who experienced the event of interest are 11 males and just 1 female while the remaining 33 patients were censored. The minimum age is 28 years, maximum age is 80 years and mean age is 48 years.

Table 3 shows the comparison of log-exponential power distribution with some existing models when only the survival times of chronic liver disease patients were modeled. The comparison was made using the values for *AIC* and *BIC* respectively. Using this data, the maximum Likelihood estimate of parameters of log-exponential power distribution are $\sigma = 2.186$ and $\beta = 1.119$. The model with the least *AIC* and *BIC* values is considered as the model that best fits the data. From the result, the Log-exponential power distribution has the least *AIC* and *BIC* values compared to the log-logistic, lognormal, weibull and inverse weibull distribution. Table (4) shows the comparison of log-exponential power distribution with some existing models when covariates were incorporated into the survival model in analysis of chronic liver disease data. We observed that the *LEP AFT* model has the largest *AIC* and *BIC* values compared to the log-logistic, lognormal and weibull distributions.

The coefficients estimates of the covariates for all of the considered models were presented in Table 4. $exp(-0.03) \;\; = 0.9704$ implies that age decelerates survival time by 97%.

For sex effect $(male = 1, female = 0)$, $exp(0.59) \;\; = 1.7968$ implies that sex accelerates survival time by approximately 80%. That is, males have longer survival time than females.

Table 2. Summary of data on chronic liver disease patients who reported at the facility

Total number of patients	45
Number of males	37
Number of females	8
Number of deaths recorded	12
Number censored	33
minimum survival time	1 day
maximum survival time	93 days
mean survival time	15 days
median survival time	12 days
minimum age	28 years
maximum age	80 years
mean age	48 years

Table 3. Comparison of log-exponential power distribution with some existing models when only the survival times of chronic liver disease patients were modeled

Distribution	log-likelihood	*AIC*	*BIC*
log-exponential power	-68.913	141.827	144.439
log-logistic	-167.167	338.333	341.946
lognormal	-167.290	338.580	342.194
weibull	-165.587	335.175	338.788

Table 4. Comparison of log-exponential power distribution with some existing models when covariates were incorporated into the survival model in analysis of chronic liver disease data

Distribution	covariates		log-likelihood	*AIC*	*BIC*
	age	sex			
log-exponential power	-0.03(0.011)	0.586(0.426)	-64.838	137.675	144.902
lognormal	-0.047(0.025)	-0.528(1.263)	-57.244	122.489	129.716
log-logistic	-0.041(0.026)	-0.386(1.262)	-57.795	123.589	130.876
weibull	-0.036(0.026)	-0.335(1.399)	-58.269	124.539	131.767

6. Conclusion

This study introduced a new parametric model for the analysis of survival data called the Log-exponential power distribution which is flexible in tails, generalizes the existing Log-normal distribution and some other distributions and most importantly, it accommodated four basic properties of hazard function namely; monotone increasing, monotone decreasing, increasing and then decreasing, decreasing and then increasing. These properties are attractive comparing with other distribution that cannot provide all these four different shapes for hazard function. It fits the observed chronic liver disease patients data better when only survival times were modeled compared with the log-normal, log-logistic, weibull and inverse weibull distributions. We therefore strongly advocate its use as alternative flexible parametric model in survival analysis.

Conflict of interest

The authors declare no potential conflict of interests.

Funding

No funding was received for conducting this study.

Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Appendix

Proof of Proposition 1. We have

$$F(t) = \int_0^t \frac{1}{x\sigma\Gamma(1+\frac{1}{2\beta})2^{1+\frac{1}{2\beta}}} \exp\left\{-\frac{1}{2}\left(\frac{\ln x}{\sigma}\right)^{2\beta}\right\}dx$$
$$= \frac{1}{\sigma\Gamma(1+\frac{1}{2\beta})2^{1+\frac{1}{2\beta}}} \int_0^t \frac{1}{x} \exp\left\{-\frac{1}{2}\left(\frac{\ln x}{\sigma}\right)^{2\beta}\right\}dx. \tag{11}$$

Let

$$w = \frac{1}{2}\left(\frac{\ln\ x}{\sigma}\right)^{2\beta}, \quad i.e., \quad (2w)^{\frac{1}{2\beta}} = \frac{\ln x}{\sigma}.$$

Then

$$x = exp\ \{\sigma(2w)^{\frac{1}{2\beta}}\}. \tag{12}$$

Differentiating Equation (12) with respect to w leads to

$$\frac{dx}{dw} = exp\ \{\sigma(2w)^{\frac{1}{2\beta}}\} \times \frac{\sigma}{\beta}(2w)^{\frac{1}{2\beta}-1}. \tag{13}$$

Bu substituting Equations (12) and (13) into Equation (11), we have

$$\frac{1}{\Gamma(\frac{1}{2\beta})}\int_0^t \exp\{-w\}\ w^{\frac{1}{2\beta}-1}dw.$$

Therefore,

$$F(t) = \frac{\gamma\left(\frac{1}{2\beta}, \frac{1}{2}(\frac{\ln t}{\sigma})^{2\beta}\right)}{\Gamma(\frac{1}{2\beta})},$$

where γ (.) is the lower incomplete gamma function defined as

$$\gamma(a,t) = \int_0^t w^{a-1}exp(-w)dw.$$

References

Bagnoli, M. and Bergstrom, T. (2005). Log-concave probability and its applications. *Economic Theory*, 26(2), 445-469 .

Cox, C., Chu, H., Schneider, M.F. and Muñoz, A. (2007). Parametric survival analysis and taxonomy of hazard functions for the generalized gamma distribution. *Statistics in Medicine*, 26(23), 4352-4374.

Collett, D. (2003). Modeling survival data in medical research. Chapman and Hall/CRC, Florida.ISBN 1584883251

Dey, S., Nassar, M., Kumar, D., Alzaatreh, A and Tahir, M. (2019). A new lifetime distribution with decreasing and upside-down bathtub-shaped hazard rate function. *Statistica*, 79(4), 399-426.

Guy, J. and Peters, M. (2013). Liver disease in women: the influence of gender on epidemiology, natural history and patient outcomes. *Gastroenterol Hepatol (NY)*, 9(10), 633-639.

Hashimoto, M.E., Cordeiro, G.M., Ortega, E.M.M. and Hamedani, G.G. (2016). New flexible regression models generated by gamma random variables with censored data. *International Journal of Statistics and Probability*, 5(3), 9-31.

Hutson, A.D. (2019). An alternative skew-exponential power distribution formulation. *Communications in Statistics-Theory and Methods*,48(12), 3005-3024.

Kalbfleisch, J. D. and Prentice, R. L.(2002). The statistical analysis of failure time data. Wiley, New Jersey.ISBN 0-471-36357-X

Kanichukattu, J. and Paul, A. (2018). Marshall Olkin exponential power distribution and its generalization: theory and applications. *IAPQR Transactions*, 23(1), 1-29.

Khan, S.A. and Khosa, S.K. (2016).Generalized log-logistic proportional hazard model with applications in survival analysis. *Journal of Statistical Distributions and Applications*, 3(1), 1-18.

Kim, H., Kisseleva, T. and Brenner, D. (2015). Aging and liver disease. *Current Opinion in Gastroenterology*, 31(3), 184-191.

Lawless, J.F. (2005). Parametric models in survival analysis. Encyclopedia of Biostatistics. John Wiley and Sons, New York.

Lemeshko, B.Yu., Lemeshko, S.B., Akushkina, K.A., Nikulin, M.S., Saaidia, S. (2010). Inverse gaussian model and its applications in reliability and survival analysis. Mathematical and statistical models and methods in reliability: applications to medicine, Finance, and quality control, Statistics for industry and technology. Springer Science+Business Media,LLC, 433-453.

Ma, C. (2000): A note on on the multivariate normal hazard. *Journal of Multivariate Analysis*, 73(2), 282-283.

Mahmoud, M.R., El-Sheikh, A.A., Morad, N.A., Moshera, A. and Ahmad, M. (2015). Log-beta log-logistic regression model. *International Journal of Sciences: Basic and Applied Research*, 22(2), 389-405.

Maxim, F. (2008). Failure rate modeling for reliability and risk. Springer Series in Reliability Engineering, London. ISBN 9781848009851.

Nadarajah, S. (2005). A generalized normal distribution. *Journal of Applied Statistics*, 32 (7), 685-694.

Olosunde, A.A. (2013). On exponential power distribution and poultry feeds data. *J.Iran.Stat.Soc.* (JIRSS), 12(2), 253-269.

Olosunde, A. A. (2020). On log-concavity density function: a case of exponential power distribution and its application. *Missouri Journal of Mathematical Sciences*,32 (1), 39-48

Ortega, E.M., Cordeiro, G.M. and Lemonte, A.J. (2012). A Log-linear regression model for the Birnbaum-Saunders distribution with censored data. *Computational Statistics and Data Analysis*, 56(3), 698-718.

Pescim, R.R., Ortega, E.M.M., Cordeiro, G.M. and Demétrio, C.G.B. (2013). The Log-beta generalized half-normal regression model. *Journal of Statistical Theory and Applications*, 12(4), 330-343.

Reed, W.J. (2011). A flexible parametric survival model which allows a bathtub shaped hazard rate function. *Journal of Applied Statistics*, 38(8), 1665-1680.

Rezaei, S., Hashami, S. and Najjar, L. (2014). Extended exponential geometric proportional hazard model. *Ann. Data. Sci.*, 1(2), 173-189.

Saikia, R. and Barman, M.P. (2017). A review on accelerated failure time models. *International Journal of Statistics and Systems*, 12(2), 311-322.

Subbotin, M.T. (1923). On the law of frequency of error. *Mathematicheskii Sbornik*, 296-300.

Tableman, M. and Kim, J.S. (2004). Survival analysis using S: Analysis of time-to-event data, Chapman and Hall/CRC, New York. ISBN 9781584884088

Wei, L.J. (1992). The accelerated failure time model: a useful alternative to the Cox regression in survival analysis. *Statistics in Medicine*, 11, 1871-1879.

Wenhao, G. (2013). Statistical inferences and applications of the half exponential power distribution. *Journal of Quality and Reliability Engineering*, Vol. 2013, Article ID 219473. Hindawi Publishing Corporation.