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Stein's method in two limit theorems involving the generalized inverse Gaussian distribution

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Abstract. On one hand, the generalized hyperbolic (*GH*) distribution converges in law to the generalized inverse Gaussian (*GIG*) distribution under certain conditions on the parameters. On the other hand, when the edges of an infinite rooted tree are equipped with independent resistances whose distributions are inverse Gaussian or reciprocal inverse Gaussian distributions, the total resistance converges almost surely to some random variable which follows the reciprocal inverse Gaussian (*RIG*) distribution. In this paper we provide explicit upper bounds for the distributional distance between the *GH* distribution (resp. the distribution of the total resistance of the tree) and their limiting GIG (resp. *RIG*) distribution applying Stein's method.

Key words: Generalized Inverse Gaussian distribution; generalized hyperbolic distribution; Reciprocal Inverse Gaussian distribution; Stein characterization; Stein equation, Taylor expansion.
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Résumé. (French Abstract) Sous certaines conditions sur ses paramètres, la loi hyperbolique généralisée (*GH*) converge vers la loi gaussienne inverse généralisée (*GIG*). Lorsque les arêtes d'un arbre infini sont munies de résistances aléatoires indépendantes, de loi gaussienne inverse ou de loi gaussienne inverse réciproque, la résistance équivalente converge presque sûrement vers une variable aléatoire de loi gaussienne inverse réciproque (*RIG*). Dans cet article, nous déterminons des majorants explicites de la distance probabiliste entre la loi *GH* (resp. la loi d'un circuit arborescent) et la loi limite *GIG* (resp. *RIG*) en appliquant la méthode de Stein.

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1. Introduction

The generalized inverse Gaussian (hereafter *GIG*) distribution with parameters $p \in \mathbb{R}, a > 0, b > 0$ has density

$$g_{p,a,b}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0, \tag{1}$$

where K_p is the modified Bessel function of the third kind. As readily seen from the expression of the normalizing constant in (1), if $W \sim GIG(p, a, b)$ then

$$\mathbb{E}(W^k) = \left(\frac{b}{a}\right)^{k/2} \frac{K_{p+k}(\sqrt{ab})}{K_p(\sqrt{ab})}. \tag{2}$$

For $p = -1/2$ (resp. $p = 1/2$), the *GIG* distribution is the inverse Gaussian ($IG(a, b)$) (resp. reciprocal inverse Gaussian $RIG(a, b)$) distribution, whose densities are respectively

$$g_{-1/2,a,b}(x) = \sqrt{\frac{b}{2\pi}} e^{\sqrt{ab}} x^{-3/2} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0,$$

and

$$g_{1/2,a,b}(x) = \sqrt{\frac{a}{2\pi}} e^{\sqrt{ab}} x^{-1/2} e^{-\frac{1}{2}(ax+b/x)}, \quad x > 0. \tag{3}$$

The *IG* and the *RIG* laws are respectively the distribution of the first and the last hitting time for a Brownian motion (see Bhattacharya and Waymire (1990)). We have the following well-known convolution property which can be easily proved from the Laplace transforms of the considered distributions:

$$\mathrm{RIG}(a_1, b) * \mathrm{IG}(a_2, b) = \mathrm{RIG}(a_1 + a_2, b). \tag{4}$$

Details on *GIG* distribution can be found in Jørgensen (1982), Barndorff *et al.*(1982), Koudou and Ley (2014), Koudou and Vallois (2012).

The generalized hyperbolic (*GH*) distribution was introduced by Barndorff-Nielsen Barndorff-Nielson (1977), who studied it in connection with the modelling of dune movements. For certain parameter values the *GH* distributions have semi-heavy tails which makes them appropriate for financial modelling (see for instance Bibby and Sørensen (2003), Eberlein and Keller (1995), Eberlein and Prause (1995)).

The *GH* distribution, with parameters $\lambda \in \mathbb{R}$, $\delta > 0$, $\mu \in \mathbb{R}$, $\alpha > |\beta| \geq 0$ has density

$$p_{\lambda,\alpha,\beta,\delta,\mu}(x) = \frac{\left(\alpha^2 - \beta^2\right)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \times \left(\delta^2 + (x-\mu)^2\right)^{(\lambda-\frac{1}{2})/2} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right) e^{\beta(x-\mu)}. \tag{5}$$

The class of *GH* distributions can be obtained by mean-variance mixtures of normal distributions where the mixing distribution is the *GIG* distribution. If Z follows the standard normal ($Z \sim \mathcal{N}(0,1)$) distribution, and $U \sim GIG(\lambda, \delta^2, \alpha^2 - \beta^2)$ are independent, then (see for example Bibby and Sørensen (2003), Paolella (2007))

$$\mu + \beta U + \sqrt{U} Z \sim GH(\lambda, \alpha, \beta, \delta, \mu). \tag{6}$$

The *GH* distributions are studied in some detail in Eberlein and Hammerstein (2004). The following special and limiting cases of the *GH* distribution can be found in Eberlein and Hammerstein (2004): Normal-inverse Gaussian distribution $\left(\lambda = -\frac{1}{2}\right)$, Hyperbolic distribution ($\lambda = 1$), GIG distribution with parameters $\lambda \in \mathbb{R}$, $a > 0$, $b > 0$ $\left(\beta = \alpha - \frac{a}{2},\ \alpha \to \infty,\ \delta \to 0,\ \alpha\delta^2 \to b,\ \mu = 0\right)$, Student's t-distribution with ν degrees of freedom ($\lambda = -\nu/2 < 0$, $\alpha, \beta \to 0$), Variance-gamma distribution with parameters ν, α, β, μ ($\nu = \lambda - 1/2 > -1/2$, $\delta \to 0$).

In this paper, the *GH*, the *GIG* and the *RIG* distributions are considered in the context of Stein's method. A first instance of the method was found in Stein (1972), where Stein showed that a random variable X has a standard normal distribution if and only if, for all real-valued absolutely continuous function f such that $\mathbb{E}\,|f'(Z)| < \infty$, with Z a random variable following the standard normal distribution,

$$\mathbb{E}\left[f'(X) - Xf(X)\right] = 0.$$

The corresponding Stein equation is

$$f'(x) - xf(x) = h(x) - \mathbb{E}h(Z) \tag{7}$$

where h is a bounded function. The corresponding Stein operator is $f \mapsto (T_f)(x) = f'(x) - xf(x)$. If f_h solves equation (7), then for any random variable X, we have

$$\left|\mathbb{E}\left[f_h'(X) - Xf_h(X)\right]\right| = \left|\mathbb{E}h(X) - \mathbb{E}h(Z)\right|.$$

Thus, we can bound $|\mathbb{E}h(X) - \mathbb{E}h(Z)|$ given h, by finding a solution f_h of the Stein equation (7) and bounding the left-hand side of the previous equation.

For instance, as shown in Ross (2011), if $X_1, \cdots, X_n$ are independent mean zero random variables such that, $\mathbb{E}X_i^2 = 1$ and $\mathbb{E}|X_i|^4 < \infty \;\; i = 1, \cdots, n$, then, by Stein's approach,

$$|\mathbb{E}h(W_n) - \mathbb{E}h(Z)| \leq \frac{1}{n^{\frac{3}{2}}} \sum_{i=1}^{n} \mathbb{E}|X_i|^3 + \frac{\sqrt{2}}{n\sqrt{\pi}} \sqrt{\sum_{i=1}^{n} \mathbb{E}|X_i|^4} \tag{8}$$

where h is 1-Lipschitz function,

$$W_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

and Z has the standard normal distribution. If the random variables X_i of (8) have common distribution, then the right-hand side is of order $n^{-1/2}$.
For more details on Stein's method, see Chen *et al.*(2011), Ross (2011), Konzou *et al.*(2020).

The aim of this paper is to provide a bound for the distance between a *GH* random variable (resp. a sequence of resistances) random variable and its limiting *GIG* (resp. *RIG*) variable, and therefore to give a contribution to the study of the rate of convergence, via Stein's method. For this purpose we make use of bounds established in Konzou and Koudou (2020), Konzou *et al.*(2019), Konzou *et al.*(2020) in the framework of Stein's method for the *GIG* distribution. These bounds are recalled in Section 2. Section 3 and 4 present our main results and Section 5 contains the proofs.

2. Stein characterization for the *GIG* distribution

Theorem 1. *A random variable X follows the GIG distribution with density g_{λ,b^2,a^2} if and only if, for all real-valued and differentiable function f such that*

$$\lim_{x\to\infty} g_{\lambda,b^2,a^2}(x)f(x) = \lim_{x\to 0} g_{\lambda,b^2,a^2}(x)f(x) = 0,$$

and such that the following expectation exists, we have:

$$\mathbb{E}\left[X^2 f'(X) + \left(\frac{b}{2} + (\lambda+1)X - \frac{a}{2}X^2\right) f(X)\right] = 0.$$

The corresponding Stein equation is

$$x^2 f'(x) + \left(\frac{a^2}{2} + (\lambda+1)x - \frac{b^2}{2}x^2\right) f(x) = h(x) - \mathbb{E}h(W) \tag{9}$$

where h is a bounded function and W a random variable following the *GIG* distribution with parameters λ, b^2, a^2.

Theorem 3.2 in Konzou and Koudou (2020) shows that the bounded solution of Stein's equation (9) is given by

$$\begin{aligned} f_h(x) =& \frac{1}{s(x)g_{\lambda,b^2,a^2}(x)} \int_0^x g_{\lambda,b^2,a^2}(t) \left[h(t) - \mathbb{E}h(W)\right] dt \\ =& \frac{-1}{s(x)g_{\lambda,b^2,a^2}(x)} \int_x^{+\infty} g_{\lambda,b^2,a^2}(t) \left[h(t) - \mathbb{E}h(W)\right] dt. \end{aligned} \tag{10}$$

Denote

$$f := f_h.$$

Konzou and Koudou (2020), Konzou *et al.*(2019), Konzou *et al.*(2020) give the following bounds of f, f', f'' and $f^{(3)}$.

(i) For any Lipschitz test function h,

$$\|f\| \le \frac{2}{ab}\frac{K_{\lambda+1}(ab)}{K_\lambda(ab)} \|h'\|.$$

(ii) For any bounded continuous function h,

$$\|f\| \leq M = \begin{cases} \max(V_1; V_2)\, \|h(\circ) - \mathbb{E}h(W)\| & \text{if} \quad \lambda \leq -1 \\ \max\left(\frac{2}{a^2}, \dfrac{b^4}{(\lambda+1)^2 g_{\lambda,b^2,a^2}(\frac{\lambda+1}{b^2})}\right) \|h(\circ) - \mathbb{E}h(W)\| & \text{if} \quad \lambda > -1, \end{cases} \tag{11}$$

where

$$V_1 = \frac{1}{\alpha^2 g_{\lambda,b^2,a^2}(\alpha)} \int_0^{\alpha} g_{\lambda,b^2,a^2}(t)dt, \quad V_2 = \frac{1}{\alpha^2 g_{\lambda,b^2,a^2}(\alpha)} \int_{\alpha}^{+\infty} g_{\lambda,b^2,a^2}(t)dt,$$

and

$$\alpha = \frac{\lambda + 1 + \sqrt{(\lambda+1)^2 + a^2 b^2}}{b^2}.$$

(iii) For any bounded function h, differentiable with bounded derivative h',

$$\|f'\| \leq M' = \begin{cases} \max\left(G_2^{(1)}, G_3^{(1)}\right) & \text{if} \quad \lambda \leq -3 \\ \max\left(G_1^{(1)}, G_3^{(1)}, G_4^{(1)}\right) & \text{if} \quad \lambda > -3 \end{cases} \tag{12}$$

where

$$G_1^{(1)} = \frac{2}{a^2}\left(\|h'\| + \max(2, |\lambda+1|) M \|h(.) - \mathbb{E}h(W)\|\right),$$

$$\begin{aligned} G_2^{(1)} &= \left(\|h'\| + (2 + \sqrt{(\lambda+3)^2 + a^2b^2}) M \|h(.) - \mathbb{E}h(W)\|\right) \\ &\times \frac{1}{\alpha_1^2 g_{\lambda+2,b^2,a^2}(\alpha_1)} \int_0^{\alpha_1} g_{\lambda+2,b^2,a^2}(t)dt, \end{aligned}$$

$$G_3^{(1)} = \frac{2}{\alpha_1^2} \|h(.) - \mathbb{E}h(W)\|,$$

$$G_4^{(1)} = \left(\|h'\| + (2 + \sqrt{(\lambda+3)^2 + a^2b^2}) M \|h(.) - \mathbb{E}h(W)\|\right) \frac{b^4}{(\lambda+3)^2 g_{\lambda+2,b^2,a^2}(\frac{\lambda+3}{b^2})},$$

$$\alpha_1 = \frac{\lambda + 3 + \sqrt{(\lambda+3)^2 + a^2b^2}}{b^2}.$$

M is given by (11).

(iv) For any bounded function h, twice differentiable such that h' and h'' are bounded,

$$\|f''\| \leq M'' = \begin{cases} \max\left(G_2^{(2)}, G_3^{(2)}\right) & \text{if} \quad \lambda \leq -5 \\ \max\left(G_1^{(2)}, G_3^{(2)}, G_4^{(2)}\right) & \text{if} \quad \lambda > -5 \end{cases} \tag{13}$$

where

$$G_1^{(2)} = \frac{2}{a^2}\left(\|h''\| + \max(6, |2\lambda+4|)M' + b^2 M \|h(.) - \mathbb{E}h(W)\|\right),$$

$$\begin{aligned} G_2^{(2)} &= \left(\|h''\| + (6 + 2\sqrt{(\lambda+5)^2 + a^2b^2})M' + b^2 M \|h(.) - \mathbb{E}h(W)\|\right) \\ &\times \frac{1}{\alpha_2^2 g_2(\alpha_2)} \int_0^{\alpha_2} g_2(t)dt, \end{aligned}$$

$$\alpha_2 = \frac{\lambda + 5 + \sqrt{(\lambda+5)^2 + a^2b^2}}{b^2},$$

$$G_3^{(2)} = \frac{2}{\alpha_2^2}\left(\|h'\| + \frac{4b^2 + b^2\sqrt{(\lambda+5)^2 + a^2b^2}}{4\lambda + 20 + 4\sqrt{(\lambda+5)^2 + a^2b^2}} \|h(.) - \mathbb{E}h(W)\|\right),$$

$$G_4^{(2)} = \frac{b^4\left(\|h''\| + (6 + 2\sqrt{(\lambda+5)^2 + a^2b^2})M' + b^2 M \|h(.) - \mathbb{E}h(W)\|\right)}{(\lambda+5)^2 g_2\left(\frac{\lambda+5}{b^2}\right)},$$

$g_2 := g_{\lambda+4,b^2,a^2}$ is the density of the $GIG(\lambda+4, b^2, a^2)$ distribution, M is given by (11) and M' by (12).

(v) For any bounded function h, three times differentiable such that h', h'' and $h^{(3)}$ are bounded,

$$\left\|f_h^{(3)}\right\| \leq M''' = \begin{cases} \max(G_1^{(3)}, G_2^{(3)}) & \text{if} \quad \lambda \leq -7 \\ \max(G_3^{(3)}, G_4^{(3)}, G_2^{(3)}) & \text{if} \quad \lambda > -7 \end{cases} \tag{14}$$

where

$$\begin{aligned} G_1^{(3)} &= \left(\left\|h^{(3)}\right\| + 3\max\left(|\lambda+3|, (4 + \sqrt{(\lambda+7)^2 + a^2b^2})\right) M'' + 3b^2 M'\right) \\ &\times \frac{1}{\alpha_3^2 g_3(\alpha_3)} \int_0^{\alpha_3} g_3(t)dt, \end{aligned}$$

$$\alpha_3 = \frac{\lambda + 7 + \sqrt{(\lambda+7)^2 + a^2b^2}}{b^2},$$

$$\begin{aligned} G_2^{(3)} &= \left(\|h'\| + \frac{6b^2 + b^2\sqrt{(\lambda+7)^2 + a^2b^2}}{6\lambda + 42 + 6\sqrt{(\lambda+7)^2 + a^2b^2}} \|h(.) - \mathbb{E}h(W)\| \right) \\ &\times \frac{5b^2 + b^2\sqrt{(\lambda+7)^2 + a^2b^2}}{\alpha_3^2\left(\lambda + 7 + \sqrt{(\lambda+7)^2 + a^2b^2}\right)} + \frac{2}{\alpha_3^2}\left(\|h''\| + b^2M\,\|h(.) - \mathbb{E}h(W)\|\right), \end{aligned}$$

$$G_3^{(3)} = \frac{2}{a^2}\left(\left\|h^{(3)}\right\| + 3\max\left(4, |\lambda+3|\right)M'' + 3b^2M'\right),$$

$$\begin{aligned} G_4^{(3)} &= \left(\left\|h^{(3)}\right\| + 3\max\left(|\lambda+3|, (4 + \sqrt{(\lambda+7)^2 + a^2b^2})\right)M'' + 3b^2M'\right) \\ &\times \frac{b^4}{(\lambda+7)^2\, g_3\left(\frac{\lambda+7}{b^2}\right)}, \end{aligned}$$

$g_3 := g_{\lambda+6,b^2,a^2}$ is the density of the $GIG(\lambda+6, b^2, a^2)$ distribution, M, M', M'' are given by (11), (12), (13) respectively.

Remark 1. The Stein's characterization of the *RIG*(δ,γ) distribution is the particular case $\lambda = \frac{1}{2}$, $a = \gamma^2$ and $b = \delta^2$ in the Stein's characterization of the GIG(λ, a, b) distribution.

3. Rate of convergence of the generalized hyperbolic distribution to the generalized inverse Gaussian distribution by Stein method

The purpose of this section is to estimate the rate of convergence of the generalized hyperbolic distribution to the generalized inverse Gaussian distribution by the Stein method. The following proposition establishes the convergence of the generalized hyperbolic distribution to the generalized inverse Gaussian distribution under certain conditions on the parameters.

Proposition 1. *Let $\lambda, a, b \in \mathbb{R}$. Consider (X_n) a sequence of random variables such that for any integer $n \geq 1$, X_n follows the generalized hyperbolic distribution with parameters λ, a^2n, $a^2n - \frac{b^2}{2}$, $\frac{1}{\sqrt{n}}$, 0. As $n \to \infty$, the sequence $(X_n)_{n\geq 1}$ converges in distribution to a random variable W following the generalized inverse Gaussian distribution with parameters λ, b^2, a^2.*

Proof of Proposition 1. We give the proof even if it is elementary. The density of $GH\left(\lambda, a^2n, a^2n - \frac{b^2}{2}, \frac{1}{\sqrt{n}}, 0\right)$ distribution is

$$\begin{aligned} f_n(x) &= \frac{\left(a^2b^2n - \frac{b^4}{4}\right)^{\lambda/2}}{\sqrt{2\pi}\,(a^2n)^{\lambda-\frac{1}{2}}\left(\frac{1}{\sqrt{n}}\right)^{\lambda} K_\lambda\left(\frac{1}{\sqrt{n}}\sqrt{a^2b^2n - \frac{b^4}{4}}\right)}\left(\frac{1}{n} + x^2\right)^{(\lambda-\frac{1}{2})/2} \\ &\quad \times K_{\lambda-\frac{1}{2}}\left(a^2n\sqrt{\frac{1}{n} + x^2}\right).e^{\left(a^2n - \frac{b^2}{2}\right)x} \end{aligned}$$

As $n \to +\infty$, we have the following equivalence (see Gaunt (2014)):

$$K_{\lambda-\frac{1}{2}}\left(a^2n\sqrt{\frac{1}{n} + x^2}\right) \sim \sqrt{\frac{\pi}{2a^2n\sqrt{\frac{1}{n} + x^2}}}e^{-a^2n\sqrt{\frac{1}{n}+x^2}}.$$

Hence for sufficiently large n, we have

$$\frac{\left(\frac{1}{n} + x^2\right)^{(\lambda-\frac{1}{2})/2}}{\sqrt{2\pi}\,(a^2n)^{\lambda-\frac{1}{2}}\left(\frac{1}{\sqrt{n}}\right)^{\lambda}}K_{\lambda-\frac{1}{2}}\left(a^2n\sqrt{\frac{1}{n} + x^2}\right) \sim \frac{\left(\frac{1}{n} + x^2\right)^{(\lambda-1)/2}}{2a^{2\lambda}n^{\lambda/2}}e^{-a^2n\sqrt{\frac{1}{n}+x^2}}$$

and

$$f_n(x) \sim \frac{\left(a^2b^2 - \frac{b^4}{4n}\right)^{\lambda/2}\left(\frac{1}{n} + x^2\right)^{(\lambda-1)/2}}{2a^{2\lambda}K_\lambda\left(\frac{1}{\sqrt{n}}\sqrt{a^2b^2n - \frac{b^4}{4}}\right)}e^{\left(a^2n - \frac{b^2}{2}\right)x - a^2n\sqrt{\frac{1}{n}+x^2}}.$$

For $x > 0$, as $n \to +\infty$, the Taylor series expansion gives

$$\sqrt{\frac{1}{n} + x^2} = x\sqrt{1 + \frac{1}{nx^2}} = x\left(1 + \frac{1}{2nx^2} + o\left(\frac{1}{nx^2}\right)\right).$$

Thus, as $n \to \infty$,

$$f_n(x) \sim \frac{\left(a^2b^2 - \frac{b^4}{4n}\right)^{\lambda/2}\left(\frac{1}{n} + x^2\right)^{(\lambda-1)/2}}{2a^{2\lambda}K_\lambda\left(\frac{1}{\sqrt{n}}\sqrt{a^2b^2n - \frac{b^4}{4}}\right)}e^{\left(a^2n - \frac{b^2}{2}\right)x - a^2nx\left(1 + \frac{1}{2nx^2}\right)}$$

and $f_n(x)$ converges to the function

$$g_{\lambda,b^2,a^2}(x) = \frac{\left(\frac{b}{a}\right)^{\lambda}}{2K_\lambda(ab)} x^{\lambda-1} e^{-\frac{1}{2}\left(a^2/x+b^2x\right)}, \quad x > 0$$

which is the density of the generalized inverse Gaussian distribution with parameters λ, b^2, a^2. ■

As the *GIG* and *GH* laws both have applications in finance for example, we think it is interesting to have a study on how close the *GH*$\left(\lambda,\ a^2n,\ a^2n - \frac{b^2}{2}, \frac{1}{\sqrt{n}}, 0\right)$ distribution of X_n is to the *GIG*(λ, b^2, a^2) of W for n big enough. Other sequences of independent random variables of *GH* law can be chosen, our choice is for convenience.

The following theorem provides a rate of convergence in the Proposition 1 by Stein method.

Theorem 2. *Let C_b^3 be the class of bounded functions h: $\mathbb{R}^+ \to \mathbb{R}$ for which h', h'', $h^{(3)}$ exist and are bounded. Consider (X_n) a sequence of random variables such that for each $n \geq 1$, $X_n \sim$ GH$\left(\lambda, a^2n, a^2n - \frac{b^2}{2}, \frac{1}{\sqrt{n}}, 0\right)$ and let $W \sim$GIG(λ, b^2, a^2). If $h \in C_b^3$, then*

$$\begin{aligned} |\mathbb{E}h(X_n) - \mathbb{E}h(W)| \leq & \frac{1}{nc_n} \times A_n + \frac{1}{n^2c_n^2} \times B_n \\ & + \frac{1}{n^2c_n^3}\sqrt{\frac{2}{\pi}}\left(\frac{2bK_{\lambda+1}(ab)}{aK_\lambda(ab)}\|h'\| + 2M'\right)\frac{\left|a^2 - \frac{b^2}{2n}\right| K_{\lambda+3/2}(c_n)}{K_\lambda(c_n)} \\ & + \frac{1}{n^3}\sqrt{\frac{2}{\pi}}M''\frac{\left(a^2 - \frac{b^2}{2n}\right)^2 K_{\lambda+5/2}(c_n)}{c_n^5 K_\lambda(c_n)} + \frac{1}{n^4}M'''\frac{\left(a^2 - \frac{b^2}{2n}\right)^2 K_{\lambda+3}(c_n)}{c_n^6 K_\lambda(c_n)} \\ & + \left|\mathbb{E}\left[(\beta_nY_n)^2 f'(\beta_nY_n) + \tau(\beta_nY_n)f(\beta_nY_n)\right]\right|, \end{aligned} \tag{15}$$

where

$$A_n = \frac{2\sqrt{2}}{ab\sqrt{\pi}}\|h'\|\,|\lambda+1|\frac{K_{\lambda+1}(ab)K_{\lambda+\frac{1}{2}}(c_n)}{K_\lambda(ab)K_\lambda(c_n)} + M'\left(\mathbb{E}\tau^2(\beta_nY_n)\right)^{1/2}\sqrt{\frac{K_{\lambda+1}(c_n)}{K_\lambda(c_n)}},$$

$$B_n = \left(M' + \frac{bK_{\lambda+1}(ab)}{aK_\lambda(ab)}\|h'\|\right)\frac{K_{\lambda+1}(c_n)}{K_\lambda(c_n)} + M''\left(\mathbb{E}\left(\tau^2(\beta_nY_n)\right)\right)^{1/2}\sqrt{\frac{K_{\lambda+2}(c_n)}{K_\lambda(c_n)}},$$

$$c_n = \sqrt{a^2b^2 - \frac{b^4}{4n}}, \qquad \beta_n = a^2n - \frac{b^2}{2},$$

$$\tau(\beta_n Y_n) = \frac{a^2}{2} + (\lambda+1)(\beta_n Y_n) - \frac{b^2}{2}(\beta_n Y_n)^2, \qquad Y_n \sim GIG\left(\lambda, \frac{1}{n}, a^2b^2n - \frac{b^4}{4}\right).$$

M', M'', M''' are such that, $\|f'\| \leq M'$, $\|f''\| \leq M''$, $\|f^{(3)}\| \leq M'''$, f is the solution of Stein equation for GIG(λ, b^2, a^2) distribution.

We have the following corollary :

Corollary 1. *The upper bound provided by Theorem 2 is of order n^{-1}.*

Proof of Corollary 1

We have :

$$\begin{aligned}\tau^2(\beta_n Y_n) &= \left(\frac{a^2}{2} + (\lambda+1)\beta_n Y_n - \frac{b^2}{2}(\beta_n Y_n)^2\right)^2 \\ &= \frac{a^4}{4} + (\lambda+1)^2\beta_n^2 Y_n^2 + \frac{b^4}{4}\beta_n^4 Y_n^4 + a^2(\lambda+1)\beta_n Y_n - \frac{a^2b^2}{2}\beta_n^2 Y_n^2 - b^2(\lambda+1)\beta_n^3 Y_n^3.\end{aligned}$$

$$\mathbb{E}\tau^2(\beta_n Y_n) = \frac{a^4}{4} + a^2(\lambda+1)\beta_n \mathbb{E}Y_n + \left((\lambda+1)^2 - \frac{a^2b^2}{2}\right)\beta_n^2 \mathbb{E}Y_n^2 - b^2(\lambda+1)\beta_n^3 \mathbb{E}Y_n^3 + \frac{b^4}{4}\beta_n^4 \mathbb{E}Y_n^4.$$

Since $Y_n \sim GIG\left(\lambda, \frac{1}{n}, a^2b^2n - \frac{b^4}{4}\right)$,

$$\mathbb{E}Y_n = \frac{1}{n^2\left(a^2b^2 - \frac{b^4}{4n}\right)} \frac{K_{\lambda+1}\left(\sqrt{a^2b^2 - \frac{b^4}{4n}}\right)}{K_\lambda\left(\sqrt{a^2b^2 - \frac{b^4}{4n}}\right)} = \frac{K_{\lambda+1}(c_n)}{n^2 c_n^2 K_\lambda(c_n)};$$

$$\mathbb{E}Y_n^2 = \frac{1}{n^4\left(a^2b^2 - \frac{b^4}{4n}\right)^2} \frac{K_{\lambda+2}\left(\sqrt{a^2b^2 - \frac{b^4}{4n}}\right)}{K_\lambda\left(\sqrt{a^2b^2 - \frac{b^4}{4n}}\right)} = \frac{K_{\lambda+2}(c_n)}{n^4 c_n^4 K_\lambda(c_n)};$$

$$\mathbb{E}Y_n^3 = \frac{1}{n^6\left(a^2b^2 - \frac{b^4}{4n}\right)^3} \frac{K_{\lambda+3}\left(\sqrt{a^2b^2 - \frac{b^4}{4n}}\right)}{K_\lambda\left(\sqrt{a^2b^2 - \frac{b^4}{4n}}\right)} = \frac{K_{\lambda+3}(c_n)}{n^6 c_n^6 K_\lambda(c_n)};$$

$$\mathbb{E}Y_n^4 = \frac{1}{n^8\left(a^2b^2 - \frac{b^4}{4n}\right)^4} \frac{K_{\lambda+4}\left(\sqrt{a^2b^2 - \frac{b^4}{4n}}\right)}{K_\lambda\left(\sqrt{a^2b^2 - \frac{b^4}{4n}}\right)} = \frac{K_{\lambda+4}(c_n)}{n^8 c_n^8 K_\lambda(c_n)}.$$

Hence

$$\mathbb{E}\tau^2(\beta_n Y_n) = \frac{a^4}{4} + a^2(\lambda+1)\frac{a^2 - \frac{b^2}{2n}}{nc_n^2}\frac{K_{\lambda+1}(c_n)}{K_\lambda(c_n)} + \left((\lambda+1)^2 - \frac{a^2b^2}{2}\right)\frac{\left(a^2 - \frac{b^2}{2n}\right)^2}{n^2c_n^4}\frac{K_{\lambda+2}(c_n)}{K_\lambda(c_n)}$$
$$- b^2(\lambda+1)\frac{\left(a^2 - \frac{b^2}{2n}\right)^3}{n^3c_n^6}\frac{K_{\lambda+3}(c_n)}{K_\lambda(c_n)} + \frac{b^4}{4n^4c_n^8}\frac{K_{\lambda+4}(c_n)}{K_\lambda(c_n)},$$

and

$$\lim_{n\to\infty} \mathbb{E}\tau^2(\beta_n Y_n) = \frac{a^4}{4} < \infty.$$

$Y_n \sim \mathrm{GIG}\left(\lambda, \frac{1}{n}, a^2b^2n - \frac{b^4}{4}\right)$ implies $\beta_n Y_n \sim \mathrm{GIG}\left(\lambda, \frac{1}{n\beta_n}, \beta_n\left(a^2b^2n - \frac{b^4}{4}\right)\right)$ and it is straightforward to prove that the sequence of random variables $(\beta_n Y_n)$ converges to some random variable following GIG(λ, b^2, a^2) distribution. Therefore, as $n \to \infty$, the Stein's characterization of the GIG(λ, b^2, a^2) distribution leads to

$$\mathbb{E}\left[(\beta_n Y_n)^2 f'(\beta_n Y_n) + \tau(\beta_n Y_n) f(\beta_n Y_n)\right] = 0.$$

□

We will prove Theorem 2 in Section 5.

4. Rate of convergence of a sequence of resistances to the reciprocal inverse Gaussian distribution

Proposition 2. *For any integer $n \geq 1$, let R_n and r_n be independent random variables such that,*
$R_n \sim RIG\left(\sum_{i=1}^{n}\delta_i, \gamma\right)$, $r_n \sim IG\left(\sum_{i=n+1}^{\infty}\delta_i, \gamma\right)$ *with* $\sum_{i=1}^{\infty}\delta_i = \delta < \infty$.

1. *As $n \to \infty$, the sequence $(R_n)_{n\geq1}$ converges almost surely (and therefore in distribution) to a random variable R following the $RIG\,(\delta, \gamma)$ distribution.*
2. *$R = R_n + r_n$ in distribution.*

Remark 2. The random variables considered in the previous Proposition appear in the context of a resistance model mentioned in the abstract. In fact, Barndorff-Nielsen (1994) have shown that, when the edges of a finite rooted tree are equipped with independent resistances that are inverse Gaussian for interior edges and the reciprocal inverse Gaussian for terminal edges, the total resistance is a reciprocal inverse Gaussian. This result has been extended to infinite trees by Barndorff-Nielson and Koudou (1998). In this case, if T_n is the subtree of the infinite tree formed of n first vertices, then the total resistance of T_n, converges almost surely to some random variable which follows the reciprocal inverse Gaussian distribution. However, the rate of convergence in this extended result remains unknown. We use Stein's approach to estimate the rate of convergence of the total resistance of an infinite tree to the reciprocal inverse Gaussian distribution.

We have the following Theorem.

Theorem 3. *With the notations of Proposition 2, if $h \in C_b^3$ then*

$$|\mathbb{E}h(R_n) - \mathbb{E}h(R)| \leq \left[\mathcal{Y}_{||h||} \times \mathcal{K} + \mathcal{Z}_{||h||} \times \mathcal{X}\right] \left(\sum_{i=n+1}^{\infty} \delta_i\right)^{1/2} \tag{16}$$

where

$$\mathcal{K} = \left(\frac{1}{\gamma^3} + \frac{1}{\gamma^2} \sum_{i=n+1}^{\infty} \delta_i\right)^{1/2},$$

$$\mathcal{X} = \left[\frac{15}{\gamma^7} + \frac{15}{\gamma^6} \sum_{i=n+1}^{\infty} \delta_i + \frac{6}{\gamma^5} \left(\sum_{i=n+1}^{\infty} \delta_i\right)^2 + \frac{1}{\gamma^4} \left(\sum_{i=n+1}^{\infty} \delta_i\right)^3\right]^{1/2},$$

$$\mathcal{Y}_{||h||} = M'' \sqrt{K_1} + \frac{1}{2} M' \sqrt{K_2} + M' \sqrt{K_3} + \frac{1}{2} M \left\| h(.) - \ Eh(R) \right\| \sqrt{K_4},$$

$$\mathcal{Z}_{||h||} = M''' \sqrt{K_1} + \frac{1}{2} M'' \sqrt{K_2},$$

$$K_1 = \frac{105}{\gamma^8} + \frac{105\delta}{\gamma^7} + \frac{45\delta^2}{\gamma^6} + \frac{10\delta^3}{\gamma^5} + \frac{\delta^4}{\gamma^4}, \quad K_2 = \frac{42}{\gamma^4} + \frac{42\delta}{\gamma^3} + \frac{24\delta^2}{\gamma^2} + \frac{10\delta^3}{\gamma},$$

$$K_3 = 4\left(\frac{3}{\gamma^4} + \frac{3\delta}{\gamma^3} + \frac{\delta^2}{\gamma^2}\right) + \left(\frac{1}{\gamma^3} + \frac{1}{\gamma^2} \sum_{i=n+1}^{\infty} \delta_i\right) \sum_{i=n+1}^{\infty} \delta_i$$
$$+ 4\left[\left(\frac{3}{\gamma^4} + \frac{3\delta}{\gamma^3} + \frac{\delta^2}{\gamma^2}\right) \left(\frac{1}{\gamma^3} + \frac{1}{\gamma^2} \sum_{i=n+1}^{\infty} \delta_i\right) \sum_{i=n+1}^{\infty} \delta_i\right]^{1/2},$$

$$K_4 = 9 + 4\delta^2\gamma^2 + 7\gamma \sum_{i=n+1}^{\infty} \delta_i + \gamma^2 \left(\sum_{i=n+1}^{\infty} \delta_i\right)^2$$
$$+ 4\gamma^4 \left[\left(\frac{3}{\gamma^4} + \frac{3\delta}{\gamma^3} + \frac{\delta^2}{\gamma^2}\right) \left(\frac{1}{\gamma^3} + \frac{1}{\gamma^2} \sum_{i=n+1}^{\infty} \delta_i\right) \sum_{i=n+1}^{\infty} \delta_i\right]^{1/2},$$

M, M', M'', M''' are such that $\|f\| \leq M\|h(.) - \mathbb{E}h(R)\|$, $\|f'\| \leq M'$, $\|f''\| \leq M''$, $\left\|f^{(3)}\right\| \leq M'''$, ($f$ is the solution of the Stein equation for the reciprocal inverse Gaussian distribution) and are given by

$$M = \max\left(\frac{2}{\delta^2}, \frac{4}{9}\sqrt{3\pi}\gamma^2 e^{\frac{(\delta\gamma)^2}{3} - \delta\gamma + \frac{3}{4}}\right),$$

$$M' = \max\left(C_1, C_2, C_3\right); \quad M'' = \max\left(C_4, C_5, C_6\right); \quad M''' = \max\left(C_7, C_8, C_9\right),$$

$$C_1 = \frac{2}{\delta^2}\left(\|h'\| + 2M\|h(.) - \mathbb{E}h(R)\|\right); \quad C_2 = \frac{8\gamma^4}{\left(7 + \sqrt{49 + (2\delta\gamma)^2}\right)^2}\|h(.) - \mathbb{E}h(R)\|,$$

$$\begin{aligned} C_3 &= \left(\|h'\| + \left(2 + \frac{1}{2}\sqrt{49 + (2\delta\gamma)^2}\right)M\|h(.) - \mathbb{E}h(R)\|\right) \\ &\times \frac{16\sqrt{2}}{343\sqrt{7}}\delta^2\gamma^4\sqrt{\delta\gamma}K_{5/2}(\delta\gamma)e^{(\delta\gamma)^2/7 + 7/4}, \end{aligned}$$

$$C_4 = \frac{2}{\delta^2}\left(\|h''\| + 6M' + \gamma^2 M\|h(.) - \mathbb{E}h(R)\|\right),$$

$$C_5 = \frac{8\gamma^4}{\left(11 + \sqrt{121 + (2\delta\gamma)^2}\right)^2}\left(||h'|| + \frac{\gamma^2}{4}\left(1 - \frac{3}{11 + \sqrt{121 + (2\delta\gamma)^2}}\right)\|h(.) - \mathbb{E}h(R)\|\right),$$

$$\begin{aligned} C_6 &= \left(\|h''\| + \left(6 + \sqrt{121 + (2\delta\gamma)^2}\right)M' + \gamma^2 M\|h(.) - \mathbb{E}h(R)\|\right) \\ &\times \frac{64\sqrt{2}}{11^5\sqrt{11}}\delta^4\gamma^6\sqrt{\delta\gamma}K_{9/2}(\delta\gamma)e^{(\delta\gamma)^2/11 + 11/4}, \end{aligned}$$

$$C_7 = \frac{2}{\delta^2}\left(\left\|h^{(3)}\right\| + 12M'' + 3\gamma^2 M'\right),$$

$$\begin{aligned} C_8 &= \frac{8\gamma^4}{\left(15 + \sqrt{225 + (2\delta\gamma)^2}\right)^2}\left(\|h''\| + \gamma^2 M\|h(.) - \mathbb{E}h(R)\|\right) + \frac{4\gamma^6}{15 + \sqrt{225 + (2\delta\gamma)^2}} \\ &\times \left(1 - \frac{5}{15 + \sqrt{225 + (2\delta\gamma)^2}}\right)\left[||h'|| + \left(\frac{\gamma^2}{6} - \frac{\gamma^2}{30 + 2\sqrt{225 + (2\delta\gamma)^2}}\right)\|h(.) - \mathbb{E}h(R)\|\right], \end{aligned}$$

$$\begin{aligned} C_9 &= \left(\left\|h^{(3)}\right\| + \left(12 + \frac{3}{2}\sqrt{225 + (2\delta\gamma)^2}\right)M'' + 3\gamma^2 M'\right) \\ &\times \frac{256\sqrt{2}}{15^7\sqrt{15}}\delta^6\gamma^8\sqrt{\delta\gamma}K_{13/2}(\delta\gamma)e^{(\delta\gamma)^2/15 + 15/4}. \end{aligned}$$

The proof of Theorem 3 will be given in the next Section.

Remark 3. Since the series $\sum \delta_i$ converges to δ, $\sum_{i=n+1}^{\infty} \delta_i$ tends to zero as n tends to infinity. As a consequence, the rate of convergence of the upper bound provided by Theorem 3 is of order $\left(\sum_{i=n+1}^{\infty} \delta_i\right)^{1/2}$.

In order to compare the rate of convergence in Theorem 3 with the rate obtained by using another approach, we have established the following result.

Theorem 4. *Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a Lipschitz function. Under the assumptions and notations of Proposition 2, we have*

$$|\mathbb{E}h(R_n) - \mathbb{E}h(R)| \leq \frac{1}{\gamma} \|h'\| \sum_{i=n+1}^{\infty} \delta_i. \tag{17}$$

Proof of Theorem 4. h' is bounded and

$$\begin{aligned} |\mathbb{E}h(R_n) - \mathbb{E}h(R)| &\leq \|h'\| \, \mathbb{E}|R_n - R| \\ &= \|h'\| \, \mathbb{E}|r_n| \\ &= \|h'\| \, \mathbb{E}r_n \\ &= \|h'\| \frac{1}{\gamma} \sum_{i=n+1}^{\infty} \delta_i. \end{aligned}$$

□

Remark 4. The upper bound provided by Theorem 4 is of order $\sum_{i=n+1}^{\infty} \delta_i$, which is more optimal than that of the Theorem 3.

5. Proofs

5.1. Proof of Theorem 2

We use the following result (deduced from (6)) to establish the proof of Theorem 2.

Proposition 3. *Let $Y_n \sim GIG\left(\lambda, \frac{1}{n}, a^2b^2n - \frac{b^4}{4}\right)$ independent of $Z \sim \mathcal{N}(0,1)$. Let $X_n \sim GH\left(\lambda, a^2n, a^2n - \frac{b^2}{2}, \frac{1}{\sqrt{n}}, 0\right)$. Then , $X_n = \beta_n Y_n + \sqrt{Y_n}Z$ in distribution where $\beta_n = a^2n - \frac{b^2}{2}$.*

The proof of Theorem 2 is based on the following lemmas.

Lemma 1. *X_n and Y_n defined in Proposition 3 verify*

$$\left|\mathbb{E}\left[(\beta_nY_n)^2f'(X_n)+\left(\frac{a^2}{2}+(\lambda+1)(\beta_nY_n)-\frac{b^2}{2}(\beta_nY_n)^2\right)f(X_n)\right]\right|$$
$$\le\left|\mathbb{E}\left[(\beta_nY_n)^2f'(\beta_nY_n)+\tau(\beta_nY_n)f(\beta_nY_n)\right]\right|$$
$$+\sqrt{\frac{2}{\pi}}\,\|f''\|\,\beta_n^2\mathbb{E}\left(Y_n^{5/2}\right)+\|f'\|\left(\mathbb{E}\tau^2(\beta_nY_n)\right)^{1/2}\left(\mathbb{E}Y_n\right)^{1/2}$$
$$+\|f''\|\left(\mathbb{E}\left(\tau^2(\beta_nY_n)\right)\right)^{1/2}\left(\mathbb{E}Y_n^2\right)^{1/2}+\left\|f^{(3)}\right\|\left|\beta_n^2\right|\mathbb{E}Y_n^3.$$

Proof of Lemma 1. By Taylor's expansion of $f'(X_n)$ and $f(X_n)$ in the neighborhood of β_nY_n, we have

$$(\beta_nY_n)^2f'(X_n)=(\beta_nY_n)^2f'(\beta_nY_n)+(\beta_nY_n)^2(X_n-\beta_nY_n)f''(\beta_nY_n)$$
$$+(\beta_nY_n)^2\int_{\beta_nY_n}^{X_n}(X_n-t)f^{(3)}(t)dt$$

and, noting

$$\tau(\beta_nY_n)=\frac{a^2}{2}+(\lambda+1)(\beta_nY_n)-\frac{b^2}{2}(\beta_nY_n)^2,$$

$$\tau(\beta_nY_n)f(X_n)=\tau(\beta_nY_n)f(\beta_nY_n)+\tau(\beta_nY_n)(X_n-\beta_nY_n)f'(\beta_nY_n)$$
$$+\tau(\beta_nY_n)\int_{\beta_nY_n}^{X_n}(X_n-t)f''(t)dt,$$

we have

$$\left|\mathbb{E}\left[(\beta_nY_n)^2f'(X_n)+\left(\frac{a^2}{2}+(\lambda+1)(\beta_nY_n)-\frac{b^2}{2}(\beta_nY_n)^2\right)f(X_n)\right]\right|$$
$$\le\left|\mathbb{E}\left[(\beta_nY_n)^2f'(\beta_nY_n)+\tau(\beta_nY_n)f(\beta_nY_n)\right]\right|+\rho_{n_1}+\rho_{n_2}$$

where

$$\rho_{n_1}=\left|\mathbb{E}\left[(\beta_nY_n)^2(X_n-\beta_nY_n)f''(\beta_nY_n)+\tau(\beta_nY_n)(X_n-\beta_nY_n)f'(\beta_nY_n)\right]\right|$$

and

$$\rho_{n_2} = \left| \mathbb{E}\left[(\beta_n Y_n)^2 \int_{\beta_n Y_n}^{X_n} (X_n - t) f^{(3)}(t) dt + \tau(\beta_n Y_n) \int_{\beta_n Y_n}^{X_n} (X_n - t) f''(t) dt \right] \right|.$$

By the triangular and Cauchy-Schwartz inequalities, we have

$$\begin{aligned}
\rho_{n_1} &\leq \|f''\| \, \mathbb{E}\left(\beta_n^2 Y_n^{5/2} |Z|\right) + \|f'\| \, \mathbb{E}\left|\tau(\beta_n Y_n) \sqrt{Y_n} Z\right| \\
&\leq \|f''\| \, \beta_n^2 \mathbb{E}\left(Y_n^{5/2}\right) \mathbb{E}|Z| + \|f'\| \left(\mathbb{E}\tau^2(\beta_n Y_n)\right)^{1/2} \left(\mathbb{E}\left(Y_n Z^2\right)\right)^{1/2} \\
&= \|f''\| \, \beta_n^2 \mathbb{E}\left(Y_n^{5/2}\right) \mathbb{E}|Z| + \|f'\| \left(\mathbb{E}\tau^2(\beta_n Y_n)\right)^{1/2} \left(\mathbb{E}Y_n\right)^{1/2}.
\end{aligned}$$

Since $Z \sim \mathcal{N}(0,1)$, $\mathbb{E}|Z| = \sqrt{\frac{2}{\pi}}$ and

$$\rho_{n_1} \leq \sqrt{\frac{2}{\pi}} \|f''\| \, \beta_n^2 \mathbb{E}\left(Y_n^{5/2}\right) + \|f'\| \left(\mathbb{E}\tau^2(\beta_n Y_n)\right)^{1/2} \left(\mathbb{E}Y_n\right)^{1/2}.$$

$$\begin{aligned}
\rho_{n_2} &\leq \left\|f^{(3)}\right\| \mathbb{E}\left[(\beta_n Y_n)^2 \int_{\beta_n Y_n}^{X_n} |X_n - t| dt \right] + \|f''\| \, \mathbb{E}\left[|\tau(\beta_n Y_n)| \int_{\beta_n Y_n}^{X_n} |X_n - t| dt \right] \\
&\leq \left\|f^{(3)}\right\| \mathbb{E}\left[(\beta_n Y_n)^2 \int_{\beta_n Y_n}^{X_n} |X_n - \beta_n Y_n| dt \right] \\
&+ \|f''\| \, \mathbb{E}\left[|\tau(\beta_n Y_n)| \int_{\beta_n Y_n}^{X_n} |X_n - \beta_n Y_n| dt \right] \\
&= \left\|f^{(3)}\right\| \mathbb{E}\left[(\beta_n Y_n)^2 Y_n Z^2 \right] + \|f''\| \, \mathbb{E}\left[|\tau(\beta_n Y_n)| Y_n Z^2 \right] \\
&= \left\|f^{(3)}\right\| \mathbb{E}\left[(\beta_n Y_n)^2 Y_n \right] \mathbb{E}Z^2 + \|f''\| \, \mathbb{E}\left[|\tau(\beta_n Y_n)| Y_n \right] \mathbb{E}Z^2 \\
&\leq \left\|f^{(3)}\right\| |\beta_n^2| \, \mathbb{E}Y_n^3 + \|f''\| \left(\mathbb{E}\left(\tau^2(\beta_n Y_n)\right)\right)^{1/2} \left(\mathbb{E}Y_n^2\right)^{1/2}.
\end{aligned}$$

□

Lemma 2. *Z and Y_n defined in Proposition 3 satisfy*

$$\begin{aligned}
|\mathbb{E}\left(U_1 + U_2\right)| \; \leq \; & \sqrt{\frac{2}{\pi}} \|f\| \, |\lambda + 1| \mathbb{E}\left(\sqrt{Y_n}\right) + \sqrt{\frac{2}{\pi}} \left(b^2 \|f\| + 2 \|f'\|\right) |\beta_n| \mathbb{E}\left(Y_n^{3/2}\right) \\
& + \left(\|f'\| + \frac{b^2 \|f\|}{2}\right) \mathbb{E}Y_n.
\end{aligned}$$

where

$$U_1 = (2\beta_n Y_n \sqrt{Y_n} Z + Y_n Z^2) f'(X_n)$$

and

$$U_2 = \left((\lambda+1)(\sqrt{Y_n} Z) - \frac{b^2}{2}(2\beta_n Y_n \sqrt{Y_n} Z + Y_n Z^2) \right) f(X_n).$$

Proof of Lemma 2.By the triangular inequality and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|\mathbb{E}\,(U_1+U_2)| &\leq \|f\| \left(|\lambda+1|\mathbb{E}\left(\sqrt{Y_n}|Z|\right) + b^2|\beta_n|\mathbb{E}\left(Y_n^{3/2}|Z|\right) + \frac{b^2}{2}\mathbb{E}Y_n Z^2 \right) \\
&+ \|f'\| \left(2|\beta_n|\mathbb{E}\left(Y_n^{3/2}\,|Z|\right) + \mathbb{E}Y_n Z^2 \right) \\
&= \mathbb{E}|Z| \left[\|f\|\,|\lambda+1|\mathbb{E}\left(\sqrt{Y_n}\right) + \left(b^2\,\|f\| + 2\,\|f'\|\right)|\beta_n|\mathbb{E}\left(Y_n^{3/2}\right) \right] \\
&+ \left(\frac{b^2}{2}\,\|f\| + \|f'\| \right) \mathbb{E}Y_n \mathbb{E}Z^2 \\
&= \sqrt{\frac{2}{\pi}} \left[\|f\|\,|\lambda+1|\mathbb{E}\left(\sqrt{Y_n}\right) + \left(b^2\,\|f\| + 2\,\|f'\|\right)|\beta_n|\mathbb{E}\left(Y_n^{3/2}\right) \right] \\
&+ \left(\frac{b^2}{2}\,\|f\| + \|f'\| \right) \mathbb{E}Y_n. \blacksquare
\end{aligned}$$

Proof of Theorem 2. By Stein characterization for *GIG* distribution, we have

$$\begin{aligned}
\mathbb{E}h(X_n) - \mathbb{E}h(W) &= \mathbb{E}\left[X_n^2 f'(X_n) + \left(\frac{a^2}{2} + (\lambda+1)X_n - \frac{b^2}{2}X_n^2 \right) f(X_n) \right] \\
&= \mathbb{E}\left[\left(\frac{a^2}{2} + (\lambda+1)(\beta_n Y_n + \sqrt{Y_n}Z) - \frac{b^2}{2}(\beta_n Y_n + \sqrt{Y_n}Z)^2 \right) f(X_n) \right] \\
&+ \mathbb{E}\left[(\beta_n Y_n + \sqrt{Y_n}Z)^2 f'(X_n) \right]
\end{aligned}$$

and

$$\begin{aligned}
|\mathbb{E}h(X_n) - \mathbb{E}h(W)| \leq &\left| \mathbb{E}\left[(\beta_n Y_n)^2 f'(X_n) + \left(\frac{a^2}{2} + (\lambda+1)(\beta_n Y_n) - \frac{b^2}{2}(\beta_n Y_n)^2 \right) f(X_n) \right] \right| \\
&+ |\mathbb{E}\,[U_1+U_2]|\,.
\end{aligned}$$

By Lemma 1 and Lemma 2, we have

$$\begin{aligned}|\mathbb{E}h(X_n)-\mathbb{E}h(W)| \le{}& \sqrt{\frac{2}{\pi}}\|f\|\,|\lambda+1|\mathbb{E}\left(\sqrt{Y_n}\right)+\|f'\|\left(\mathbb{E}\tau^2(\beta_nY_n)\right)^{1/2}\left(\mathbb{E}Y_n\right)^{1/2}\\ &+\left(\|f'\|+\frac{b^2\|f\|}{2}\right)\mathbb{E}Y_n+\|f''\|\left(\mathbb{E}\left(\tau^2(\beta_nY_n)\right)\right)^{1/2}\left(\mathbb{E}Y_n^2\right)^{1/2}\\ &+\left\|f^{(3)}\right\||\beta_n^2|\,\mathbb{E}Y_n^3+\sqrt{\frac{2}{\pi}}\left(b^2\|f\|+2\|f'\|\right)|\beta_n|\mathbb{E}\left(Y_n^{3/2}\right)\\ &+\sqrt{\frac{2}{\pi}}\|f''\|\,\beta_n^2\mathbb{E}\left(Y_n^{5/2}\right)+\left|\mathbb{E}\left[(\beta_nY_n)^2f'(\beta_nY_n)+\tau(\beta_nY_n)f(\beta_nY_n)\right]\right|.\end{aligned} \tag{18}$$

Since $Y_n \sim GIG\left(\lambda, \frac{1}{n}, a^2b^2n-\frac{b^4}{4}\right)$,

$$\mathbb{E}\left(\sqrt{Y_n}\right)=\frac{K_{\lambda+1/2}(c_n)}{nc_nK_\lambda(c_n)};\qquad \mathbb{E}Y_n=\frac{K_{\lambda+1}(c_n)}{n^2c_n^2K_\lambda(c_n)};\qquad \mathbb{E}\left(Y_n^{3/2}\right)=\frac{K_{\lambda+3/2}(c_n)}{n^3c_n^3K_\lambda(c_n)};$$

$$\mathbb{E}Y_n^2=\frac{K_{\lambda+2}(c_n)}{n^4c_n^4K_\lambda(c_n)};\qquad \mathbb{E}\left(Y_n^{5/2}\right)=\frac{K_{\lambda+5/2}(c_n)}{n^5c_n^5K_\lambda(c_n)};\qquad \mathbb{E}Y_n^3=\frac{K_{\lambda+3}(c_n)}{n^6c_n^6K_\lambda(c_n)};$$

$$\mathbb{E}Y_n^4=\frac{K_{\lambda+4}(c_n)}{n^8c_n^8K_\lambda(c_n)};\qquad c_n=\sqrt{a^2b^2-\frac{b^4}{4n}}.$$

We have :

$$\begin{aligned}|\mathbb{E}h(X_n)-\mathbb{E}h(W)| \le{}& \sqrt{\frac{2}{\pi}}\|f\|\,|\lambda+1|\frac{K_{\lambda+1/2}(c_n)}{nc_nK_\lambda(c_n)}+\left(\|f'\|+\frac{b^2\|f\|}{2}\right)\frac{K_{\lambda+1}(c_n)}{n^2c_n^2K_\lambda(c_n)}\\ &+\|f'\|\left(\mathbb{E}\tau^2(\beta_nY_n)\right)^{1/2}\left(\frac{K_{\lambda+1}(c_n)}{n^2c_n^2K_\lambda(c_n)}\right)^{1/2}\\ &+\|f''\|\left(\mathbb{E}\left(\tau^2(\beta_nY_n)\right)\right)^{1/2}\left(\frac{K_{\lambda+2}(c_n)}{n^4c_n^4K_\lambda(c_n)}\right)^{1/2}\\ &+\left\|f^{(3)}\right\||\beta_n^2|\frac{K_{\lambda+3}(c_n)}{n^6c_n^6K_\lambda(c_n)}+\sqrt{\frac{2}{\pi}}\|f''\|\,\beta_n^2\left(\frac{K_{\lambda+5/2}(c_n)}{n^5c_n^5K_\lambda(c_n)}\right)\\ &+\sqrt{\frac{2}{\pi}}\left(b^2\|f\|+2\|f'\|\right)|\beta_n|\left(\frac{K_{\lambda+3/2}(c_n)}{n^3c_n^3K_\lambda(c_n)}\right)\\ &+\left|\mathbb{E}\left[(\beta_nY_n)^2f'(\beta_nY_n)+\tau(\beta_nY_n)f(\beta_nY_n)\right]\right|.\end{aligned}$$

Therefore

$$\begin{aligned}
&|\mathbb{E}h(X_n) - \mathbb{E}h(W)| \\
&\quad \leq \frac{1}{nc_n}\left[\sqrt{\frac{2}{\pi}}\,\|f\|\,|\lambda+1|\frac{K_{\lambda+\frac{1}{2}}(c_n)}{K_\lambda(c_n)} + \|f'\|\left(\mathbb{E}\tau^2(\beta_n Y_n)\right)^{1/2}\sqrt{\frac{K_{\lambda+1}(c_n)}{K_\lambda(c_n)}}\right] \\
&\quad + \frac{1}{n^2c_n^2}\left[\left(\|f'\| + \frac{b^2\|f\|}{2}\right)\frac{K_{\lambda+1}(c_n)}{K_\lambda(c_n)} + \|f''\|\left(\mathbb{E}\left(\tau^2(\beta_n Y_n)\right)\right)^{1/2}\sqrt{\frac{K_{\lambda+2}(c_n)}{K_\lambda(c_n)}}\right] \\
&\quad + \frac{1}{n^2c_n^3}\sqrt{\frac{2}{\pi}}\left(b^2\|f\| + 2\|f'\|\right)\frac{\left|a^2 - \frac{b^2}{2n}\right| K_{\lambda+3/2}(c_n)}{K_\lambda(c_n)} \\
&\quad + \frac{1}{n^3}\sqrt{\frac{2}{\pi}}\,\|f''\|\,\frac{\left(a^2 - \frac{b^2}{2n}\right)^2 K_{\lambda+5/2}(c_n)}{c_n^5 K_\lambda(c_n)} + \frac{1}{n^4}\left\|f^{(3)}\right\|\frac{\left(a^2 - \frac{b^2}{2n}\right)^2 K_{\lambda+3}(c_n)}{c_n^6 K_\lambda(c_n)} \\
&\quad + \left|\mathbb{E}\left[(\beta_n Y_n)^2 f'(\beta_n Y_n) + \tau(\beta_n Y_n) f(\beta_n Y_n)\right]\right|.
\end{aligned}$$

To end the proof of the theorem, it suffices to substitute $\|f\|$, $\|f'\|$, $\|f''\|$ and $\|f^{(3)}\|$ by their upper bound. ■

5.2. Proof of Theorem 3

The proof of Theorem 3 is based on the following lemmas. The notations are those of Proposition 2.

Lemma 3. *Let* $E_1 = \mathbb{E}\left[R^2 f'(R_n) + \frac{1}{2}\left(\delta^2 + 3R - \gamma^2 R^2\right) f(R_n)\right]$. *We have*

$$\begin{aligned}
|E_1| \leq &\left(\|f''\|\left[\mathbb{E}\left(R^4\right)\right]^{1/2} + \frac{1}{2}\|f'\|\left[\mathbb{E}\left(\delta^2 + 3R - \gamma^2R^2\right)^2\right]^{1/2}\right)\left[\mathbb{E}(r_n^2)\right]^{1/2} \\
&+ \left(\left\|f^{(3)}\right\|\left[\mathbb{E}(R^4)\right]^{1/2} + \frac{1}{2}\|f''\|\left[\mathbb{E}\left(\delta^2 + 3R - \gamma^2R^2\right)^2\right]^{1/2}\right)\left[\mathbb{E}(r_n^4)\right]^{1/2}.
\end{aligned} \tag{19}$$

Proof of Lemma 3. Taylor's expansion of $f'(R)$ and $f(R)$ in the neighbourhood of R_n gives

$$f'(R) = f'(R_n) + (R - R_n)f''(R_n) + \int_{R_n}^{R}(R-t)f^{(3)}(t)dt$$

and

$$f(R) = f(R_n) + (R - R_n)f'(R_n) + \int_{R_n}^{R}(R-t)f''(t)dt.$$

We have

$$R^2 f'(R_n) = R^2 f'(R) - R^2(R-R_n)f''(R_n) - R^2\int_{R_n}^{R}(R-t)f^{(3)}(t)dt;$$

$$\frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)f(R_n) = -\frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)(R-R_n)f'(R_n) + \frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)f(R) - \frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)\int_{R_n}^{R}(R-t)f''(t)dt.$$

Hence

$$\begin{aligned}
E_1 &= \mathbb{E}\left[R^2 f'(R_n) + \frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)f(R_n)\right]\\
&= \mathbb{E}\left[R^2 f'(R) + \frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)f(R)\right]\\
&- \mathbb{E}\left[R^2(R-R_n)f''(R_n) + \frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)(R-R_n)f'(R_n)\right]\\
&- \mathbb{E}\left[R^2\int_{R_n}^{R}(R-t)f^{(3)}(t)dt + \frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)\int_{R_n}^{R}(R-t)f''(t)dt\right].
\end{aligned}$$

Since R $\sim$RIG(δ,γ), by Stein characterization for *RIG* distribution deduced from Proposition 1 (just take $p=\frac{1}{2}, a=\gamma^2$ and $b=\delta^2$),

$$\mathbb{E}\left[R^2 f'(R) + \frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)f(R)\right] = 0.$$

Let

$$e_1 = \mathbb{E}\left[R^2(R-R_n)f''(R_n) + \frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)(R-R_n)f'(R_n)\right]$$

and

$$e_2 = \mathbb{E}\left[R^2\int_{R_n}^{R}(R-t)f^{(3)}(t)dt + \frac{1}{2}\left(\delta^2+3R-\gamma^2R^2\right)\int_{R_n}^{R}(R-t)f''(t)dt\right].$$

Using the triangular inequality, Cauchy-Schwartz inequality and the fact that $R = R_n + r_n$ in distribution, we have

$$\begin{aligned}
|e_1| &= \left|\mathbb{E}\left[R^2 r_n f''(R_n) + \frac{1}{2}\left(\delta^2 + 3R - \gamma^2 R^2\right) r_n f'(R_n)\right]\right| \\
&\leq \|f''\| \, \mathbb{E}|R^2 r_n| + \frac{1}{2}\|f'\| \, \mathbb{E}\left|\left(\delta^2 + 3R - \gamma^2 R^2\right) r_n\right| \\
&\leq \|f''\| \left[\mathbb{E}\left(R^4\right)\mathbb{E}(r_n^2)\right]^{1/2} + \frac{1}{2}\|f'\| \left[\mathbb{E}\left(\delta^2 + 3R - \gamma^2 R^2\right)^2 \mathbb{E}(r_n^2)\right]^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
|e_2| &= \left|\mathbb{E}\left[R^2 \int_{R_n}^{R} (R-t) f^{(3)}(t)dt + \frac{1}{2}\left(\delta^2 + 3R - \gamma^2 R^2\right) \int_{R_n}^{R} (R-t) f''(t)dt\right]\right| \\
&\leq \left\|f^{(3)}\right\| \mathbb{E}\left[R^2 \int_{R_n}^{R} |R-t|dt\right] + \frac{1}{2}\|f''\| \, \mathbb{E}\left[\left|\delta^2 + 3R - \gamma^2 R^2\right| \int_{R_n}^{R} |R-t|dt\right] \\
&\leq \left\|f^{(3)}\right\| \mathbb{E}\left[R^2 \int_{R_n}^{R} |R-R_n|dt\right] + \frac{1}{2}\|f''\| \, \mathbb{E}\left[\left|\delta^2 + 3R - \gamma^2 R^2\right| \int_{R_n}^{R} |R-R_n|dt\right] \\
&= \left\|f^{(3)}\right\| \mathbb{E}\left[R^2 (R-R_n)^2\right] + \frac{1}{2}\|f''\| \, \mathbb{E}\left[\left|\gamma^2 + 3R - \gamma R^2\right| (R-R_n)^2\right] \\
&= \left\|f^{(3)}\right\| \mathbb{E}\left[R^2 r_n^2\right] + \frac{1}{2}\|f''\| \, \mathbb{E}\left[\left|\delta^2 + 3R - \gamma^2 R^2\right| r_n^2\right] \\
&\leq \left\|f^{(3)}\right\| \left[\mathbb{E}\left(R^4\right)\mathbb{E}\left(r_n^4\right)\right]^{1/2} + \frac{1}{2}\|f''\| \left[\mathbb{E}\left(\left(\delta^2 + 3R - \gamma^2 R^2\right)^2\right)\mathbb{E}\left(r_n^4\right)\right]^{1/2}.
\end{aligned}$$

Hence

$$\begin{aligned}
|E_1| &\leq |e_1| + |e_2| \\
&\leq \left(\|f''\| \left[\mathbb{E}\left(R^4\right)\right]^{1/2} + \frac{1}{2}\|f'\| \left[\mathbb{E}\left(\left(\delta^2 + 3R - \gamma^2 R^2\right)^2\right)\right]^{1/2}\right) \left[\mathbb{E}(r_n^2)\right]^{1/2} \\
&+ \left(\left\|f^{(3)}\right\| \left[\mathbb{E}\left(R^4\right)\right]^{1/2} + \frac{1}{2}\|f''\| \left[\mathbb{E}\left(\left(\delta^2 + 3R - \gamma^2 R^2\right)^2\right)\right]^{1/2}\right) \left[\mathbb{E}(r_n^4)\right]^{1/2}. \blacksquare
\end{aligned}$$

Using the triangular inequality and Cauchy-Schwartz inequality as in the previous lemma, we have the following lemma:

Lemma 4. *Let* $E_2 = \mathbb{E}\left[\left((r_n - 2R)f'(R_n) + \frac{1}{2}\left(2\gamma^2 R - \gamma^2 r_n - 3\right) f(R_n)\right) r_n\right]$. *Then*

$$|E_2| \leq \left(\|f'\| \left[\mathbb{E}(r_n - 2R)^2\right]^{1/2} + \frac{1}{2}\|f\| \left[\mathbb{E}\left(2\gamma^2 R - \gamma^2 r_n - 3\right)^2\right]^{1/2}\right) \left[\mathbb{E}(r_n^2)\right]^{1/2}. \tag{20}$$

We are now able to prove Theorem 3.

Proof of Theorem 3. Since the Stein equation for **RIG** distribution is

$$x^2 f'(x) + \frac{1}{2}\left(\delta^2 + 3x - \gamma^2 x^2\right) f(x) = h(x) - \mathbb{E}h(R) \tag{21}$$

where $R \sim$RIG(δ, γ) distribution, we have

$$\mathbb{E}\left[R_n^2 f'(R_n) + \frac{1}{2}\left(\delta^2 + 3R_n - \gamma^2 R_n^2\right) f(R_n)\right] = \left|\mathbb{E}h(R_n) - \mathbb{E}h(R)\right|.$$

Thus, we can bound $|\mathbb{E}h(R_n) - \mathbb{E}h(R)|$ by bounding the left-hand side of the previous equation. We have

$$\begin{aligned}
|\mathbb{E}h(R_n) - \mathbb{E}h(R)| &= \mathbb{E}\left[R_n^2 f'(R_n) + \frac{1}{2}\left(\delta^2 + 3R_n - \gamma^2 R_n^2\right) f(R_n)\right] \\
&= \left|\mathbb{E}\left[(R - r_n)^2 f'(R_n) + \frac{1}{2}\left(\delta^2 + 3R - 3r_n - \gamma^2 (R - r_n)^2\right) f(R_n)\right]\right| \\
&\leq \left|\mathbb{E}\left[R^2 f'(R_n) + \frac{1}{2}\left(\delta^2 + 3R - \gamma^2 R^2\right) f(R_n)\right]\right| \\
&+ \left|\mathbb{E}\left[\left((r_n - 2R) f'(R_n) + \frac{1}{2}\left(2\gamma^2 R - \gamma^2 r_n - 3\right) f(R_n)\right) r_n\right]\right| \\
&= |E_1| + |E_2|.
\end{aligned}$$

The second equality is obtained by using the fact that $R = R_n + r_n$ in distribution.

By the Lemmas 3 and 4, we have

$$\begin{aligned}
|\mathbb{E}h(R_n) - \mathbb{E}h(R)| \leq &\left(\|f''\| \left[\mathbb{E}\left(R^4\right)\right]^{1/2} + \frac{1}{2}\|f'\| \left[\mathbb{E}\left(\left(\delta^2 + 3R - \gamma^2 R^2\right)^2\right)\right]^{1/2}\right) \left[\mathbb{E}(r_n^2)\right]^{1/2} \\
&+ \left(\left\|f^{(3)}\right\| \left[\mathbb{E}\left(R^4\right)\right]^{1/2} + \frac{1}{2}\|f''\| \left[\mathbb{E}\left(\left(\delta^2 + 3R - \gamma^2 R^2\right)^2\right)\right]^{1/2}\right) \left[\mathbb{E}(r_n^4)\right]^{1/2} \\
&+ \left(\|f'\| \left[\mathbb{E}(r_n - 2R)^2\right]^{1/2} + \frac{1}{2}\|f\| \left[\mathbb{E}\left(2\gamma^2 R - \gamma^2 r_n - 3\right)^2\right]^{1/2}\right) \left[\mathbb{E}(r_n^2)\right]^{1/2}.
\end{aligned} \tag{22}$$

Since $r_n \sim IG\left(\sum_{i=n+1}^{\infty} \delta_i, \gamma\right)$ and $R \sim RIG\left(\delta, \gamma\right)$, we have

$$\mathbb{E}(r_n) = \frac{1}{\gamma} \sum_{i=n+1}^{\infty} \delta_i; \quad \mathbb{E}(r_n^2) = \left(\frac{1}{\gamma^3} + \frac{1}{\gamma^2} \sum_{i=n+1}^{\infty} \delta_i\right) \sum_{i=n+1}^{\infty} \delta_i;$$

$$\mathbb{E}(r_n^3) = \left[\frac{3}{\gamma^5} + \frac{3}{\gamma^4}\sum_{i=n+1}^{\infty}\delta_i + \frac{1}{\gamma^3}\left(\sum_{i=n+1}^{\infty}\delta_i\right)^2\right]\sum_{i=n+1}^{\infty}\delta_i;$$

$$\mathbb{E}(r_n^4) = \left[\frac{15}{\gamma^7} + \frac{15}{\gamma^6}\sum_{i=n+1}^{\infty}\delta_i + \frac{6}{\gamma^5}\left(\sum_{i=n+1}^{\infty}\delta_i\right)^2 + \frac{1}{\gamma^4}\left(\sum_{i=n+1}^{\infty}\delta_i\right)^3\right]\sum_{i=n+1}^{\infty}\delta_i;$$

$$\mathbb{E}(R) = \frac{1}{\gamma^2} + \frac{\delta}{\gamma}; \qquad \mathbb{E}(R^2) = \frac{3}{\gamma^4} + \frac{3\delta}{\gamma^3} + \frac{\delta^2}{\gamma^2};$$

$$\mathbb{E}(R^3) = \frac{15}{\gamma^6} + \frac{15\delta}{\gamma^5} + \frac{6\delta^2}{\gamma^4} + \frac{\delta^3}{\gamma^3}; \qquad \mathbb{E}(R^4) = \frac{105}{\gamma^8} + \frac{105\delta}{\gamma^7} + \frac{45\delta^2}{\gamma^6} + \frac{10\delta^3}{\gamma^5} + \frac{\delta^4}{\gamma^4}.$$

As a consequence,

$$\begin{aligned}\mathbb{E}\left(\delta^2 + 3R - \gamma^2R^2\right)^2 &= \mathbb{E}\left[\delta^4 + 6\delta^2R + \left(9 - 2\delta^2\gamma^2R^2\right) - 6\gamma^2R^3 + \gamma^4R^4\right]\\ &= \frac{42}{\gamma^4} + \frac{42\delta}{\gamma^3} + \frac{24\delta^2}{\gamma^2} + \frac{10\delta^3}{\gamma},\end{aligned}$$

$$\begin{aligned}\mathbb{E}(r_n - 2R)^2 &= \mathbb{E}\left(r_n^2 + 4R^2 - 4Rr_n\right)\\ &\le 4\mathbb{E}\left(R^2\right) + \mathbb{E}\left(r_n^2\right) + 4\left[\mathbb{E}\left(R^2\right)\mathbb{E}\left(r_n^2\right)\right]^{1/2}\\ &= 4\left(\frac{3}{\gamma^4} + \frac{3\delta}{\gamma^3} + \frac{\delta^2}{\gamma^2}\right) + \left(\frac{1}{\gamma^3} + \frac{1}{\gamma^2}\sum_{i=n+1}^{\infty}\delta_i\right)\sum_{i=n+1}^{\infty}\delta_i\\ &+ 4\left[\left(\frac{3}{\gamma^4} + \frac{3\delta}{\gamma^3} + \frac{\delta^2}{\gamma^2}\right)\left(\frac{1}{\gamma^3} + \frac{1}{\gamma^2}\sum_{i=n+1}^{\infty}\delta_i\right)\sum_{i=n+1}^{\infty}\delta_i\right]^{1/2},\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(2\gamma^2R - \gamma^2r_n - 3\right)^2 &= 4\gamma^4\mathbb{E}(R^2) - 12\gamma^2\mathbb{E}(R) + 9 + \gamma^4\mathbb{E}(r_n^2) + 6\gamma^2\mathbb{E}(r_n) - 4\gamma^4\mathbb{E}(Rr_n)\\ &= 9 + 4\delta^2\gamma^2 + \left(\gamma + \gamma^2\sum_{i=n+1}^{\infty}\delta_i\right)\sum_{i=n+1}^{\infty}\delta_i + 6\gamma\sum_{i=n+1}^{\infty}\delta_i - 4\gamma^4\mathbb{E}(Rr_n)\\ &\le 9 + 4\delta^2\gamma^2 + 7\gamma\sum_{i=n+1}^{\infty}\delta_i + \gamma^2\left(\sum_{i=n+1}^{\infty}\delta_i\right)^2\\ &+ 4\gamma^4\left[\left(\frac{3}{\gamma^4} + \frac{3\delta}{\gamma^3} + \frac{\delta^2}{\gamma^2}\right)\left(\frac{1}{\gamma^3} + \frac{1}{\gamma^2}\sum_{i=n+1}^{\infty}\delta_i\right)\sum_{i=n+1}^{\infty}\delta_i\right]^{1/2}.\end{aligned}$$

To end the proof of the theorem, it suffices to substitute each term of the right-hand side of the inequality (22) by its upper bound or its equal. ■

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