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# A dynamic markov regime-switching asymmetric GARCH model and its cumulative impulse response function

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**Abstract.** In this paper, we consider the Markov regime-switching GJR-GARCH(1,1) model to capture both the cumulative impulse response and the asymmetry of the dynamic behavior of financial market volatility in stationary and explosive states. The model can capture regime shifts in volatility between two regimes as well as the asymmetric response to negative and positive shocks. A Monte Carlo simulation is conducted to validate the main theory and find that the regime-switching GJR-GARCH model performs better than the standard GJR-GARCH model. Applications to Brazilian stock market data show that the proposed model performs well in terms of cumulative impulse response.

**Résumé.** Dans cet article, nous examinons le modèle GJR-GARCH(1,1) à changement de régime de Markov pour capturer à la fois la réponse impulsionnelle cumulative et l'asymétrie du comportement dynamique de la volatilité des marchés financiers dans les états stationnaires et explosifs. Le modèle peut capturer les changements de régime de la volatilité entre deux régimes ainsi que la réponse asymétrique aux chocs négatifs et positifs. Une simulation de Monte Carlo est menée pour valider la théorie principale et trouver que le modèle GJR-GARCH à changement de régime est plus performant que le modèle GJR-GARCH standard. Les applications aux données du marché boursier brésilien montrent que le modèle proposé est performant en termes de réponse impulsionnelle cumulative.

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## 1. Introduction

Volatility in financial markets has been the subject of many developments and applications over the past four decades. Volatility in financial time series data has characteristics such as volatility concentration, time variation, heavy-tailed distribution, and leverage. Financial returns are well known to have a non-normal distribution that tends to have a fat-tailed. Leptokurtic and volatility clustering (Brooks (2002)) are commonly observed in financial time series (Mandelbrot (1963)). The normal distribution for asset return data was strongly rejected by Mandelbrot (1963), conjecturing that financial return processes behave as stable non-Gaussian processes. Another phenomenon most often encountered is the leverage effect. Changes in stock returns generally tend to be negatively correlated with changes in the volatility of returns. Black (1976) was the first to note that volatility tends to increase in response to "bad news" and decrease in response to "good news". This phenomenon is called, "leverage" and can only be partially interpreted by fixed costs such as financial and operational leverage (Black (1976) and Christie (1982)). The asymmetry in the volatility of stock returns is not fully explained by leverage, but also by the property of "long memory" (Harris and Sollis (2003)) and time-varying volatility or "heteroscedasticity" of the data. Long memory implies that current information has a persistent impact on future accounts. Heteroscedasticity refers to variable volatility.

Leverage is not easily detected in stock market index and a company's leverage ratio increases when its stock price decreases. If the company's cash flow is constant, it will increase the volatility of the stock return. In this case, we can expect negative returns today to lead to greater volatility tomorrow, and vice versa for positive returns. This behavior cannot be captured by standard GARCH(1,1) models. Symmetric GARCH class VaR models have difficulties in correctly modeling the tails of the return distribution (Giot and Laurent (2003)) due to leverage effects. The use of asymmetric conditional models that contain a skewness parameter in the conditional variance equation and the use of asymmetric density functions for the error terms allows for leverage in the volatility forecast. While these approaches offer an improvement in fit over symmetric models, empirical evidence suggests that the persistence of the conditional variance is likely to be significantly biased upward.

A Markov regime-switching (MS) approach solves this problem by endogenizing changes in the data generation process. Many authors have attempted to combine the Markov chain with the GARCH models to simulate the dynamic properties of financial market data and to obtain better performance in predicting volatility. Hamilton and Samuel (1994) and Cai (1994) propose an ARCH model with speed switching parameters. Gray (1996) proposes a class of GARCH models (RS-GARCH) with regime change with a variable probability over time, but estimates an approximation to the model. For many details, see also Bollen *et al.* (2000), Klaassen (2002) and Haas *et al.* (2004). Gray (1996) presents a tractable Markov-switched GARCH model and Klaassen (2002) modifies his model to improve the GRS-GARCH model by allowing greater flexibility in capturing the persistence of volatility shocks and by providing a recursive form of multi-stage volatility forecasts. Haas *et al.* (2004) proposed a new traceable approach to Markov-switched GARCH models to overcome serious estimation difficulties and understand the dynamic properties of non-linearity, and Marcucci (2005) compared a set of different standard GARCH models with a group of Markov-switched GARCH models.

To measure the persistence of volatility, Baillie *et al.* (1996) introduced the cumulative impulse response into volatility and the fractional integration GARCH process and discussed its limitations in stationary or explosive GARCH processes. Conrad and Kranasos (2006) also obtained practical representations of the impulse response function of long memory GARCH processes. Park *et al.* (2010) also examined a general form of the cumulative impulse response function of asymmetric GARCH processes at the threshold.

Since volatility in GARCH does not differentiate between positive and negative past values, as long as they are of the same magnitude, the GARCH class fails to capture asymmetric volatility. Consequently, there is a growing interest in asymmetric GARCH modeling in response to empirical evidence of asymmetric volatility resulting mainly from financial time series. See Rabemananjara and Zakoian (1993), Hwang and Basawa (2004), Pan *et al.* (2008) and Park *et al.* (2009) with their references.

Based on the previous literature, we then introduce an asymmetric regime switching model. This paper studies Glosten (1993) GJR-GARCH with two-state Markov change regimes and examines its cumulative impulse response function, which measures the long-term effect of current shocks on future volatility. We apply the Markov regime change over two consecutive discrete times $t-1$ and $t$, which represent the time intervals $[t-1, t]$ and $[t, t+1]$ respectively, and we use continuous functions of the parameters in these two consecutive intervals. It is assumed that the GJR-GARCH model has two states: the stationary state and the explosive state, so that the persistence parameter of the GJR-GARCH model has a value less than 1 and a value above 1 respectively. It is also assumed that the current state remains constant in a unit of time and that at the next instant the state can move with the probability of transition from the Markov chain to parameters that are always changing over time. A Monte Carlo study is conducted to see the time series plots of the GRS-GJR-GARCH along with the volatility, the conditional probability of Markov-chain state given the past information, and the behavior of the cumulative impulse response functions. This work extends the result of Kim and Hwang (2018) to the Markov regime-switching GJR-GARCH model with time-varying switching probabilities, in order to take into account asymmetry and persistence in the dynamic of the volatility. A real data application is given to compare the GJR-GARCH and the GRS-GJR-GARCH's cumulative impulse response functions for the Brazilian stock market IBOVESPA due to the world health crisis.

The remainder of the paper is organized as follows. Section 2 describes the Markov regime-switching GJR-GARCH model, presents assumptions, and the main result. In Section 3, a Monte Carlo Study is conducted to verify the dynamics of volatility and the empirical study in section 4. Proof is given after the conclusion in section 5.

## 2. Model and main results

### *2.1. Generalized regime-switching GJR-GARCH(1,1) process*

In order to describe two-state Markov regime-switching GJR-GARCH model, we start with a GJR-GARCH model given in (1) below:

$$\begin{cases} y_t = \xi_t \sqrt{h_t} \quad ; \qquad t = 1, 2, \ldots \\ h_t = \omega + (\alpha + \gamma \mathbb{I}_{(y_{t-1}<0)}) y_{t-1}^2 + \beta h_{t-1}; \qquad \omega > 0, \quad \alpha, \gamma, \beta \geq 0 \end{cases} \tag{1}$$

with the indicator function

$$\mathbb{I}_{(y_{t-1}<0)} = \begin{cases} 1 & if \quad y_{t-1} < 0, \\ 0 & if \quad y_{t-1} \geq 0, \end{cases}$$

where the conditional variance $h_t = h(\theta_h, \Psi_{t-1})$, with $\theta_h = (\omega, \alpha, \gamma, \beta)$ being vector of parameters and $\Psi_{t-1}$ being the entire past history of the data up to time $t-1$, and $\xi_t$ is a stationary sequence of random variables with mean zero and variance one.

If $\gamma > 0$ then a leverage effect exists, that is negative news has a bigger impact on volatility than positive news. If $\gamma \neq 0$, the news impact is asymmetric. The leverage effect is often described as a falling equity price which leads to an increase in a firm's debt to equity ratio which increases the volatility of returns to equity holders.

Now we introduce a GRS (Generalized Regime Switching) of Gray (1996) to the GJR–GARCH(1,1) model, proposed by Glosten (1993) and consider the GRS-GJR-GARCH(1,1) model in this work. To propose the GRS-GJR-GARCH(1,1) process, we adopt two-state Markov switching regimes with time-varying switching probabilities for the GJR-GARCH(1,1) model, which is expressed as $y_t = \xi_t\sqrt{h_t}$, $h_t = h[\theta_h(S_t), \Psi_{t-1}]$ where $S_t$ is unobserved regime at time $t$, $\Psi_{t-1}$ is the entire past history of the data up to time $t-1$, and $\theta_h(S_t) = (\omega(S_t), \alpha(S_t), \gamma(S_t), \beta(S_t))$, parameter vector depending on $S_t$. As seen in Gray (1996), $\Psi_{t-1}$ does not contain $S_t$ or lagged values of $S_t$. We consider two-state Markov process for regimes, i.e., $S_t \in \{1, 2\}$, with the following conditions: for $i = 1, 2$,

$$S_t = i \Longleftrightarrow \theta_h(S_t) = (\omega_i, \alpha_i, \gamma_i, \beta_i),$$

with $\alpha_i + \frac{\gamma_i}{2} + \beta_i < 1$, or $\alpha_i + \frac{\gamma_i}{2} + \beta_i > 1$ if $i = 1, 2$, respectively. These conditions imply stationary and explosive state, respectively. For $i = 1, 2$, and $t \in \mathbb{Z}$, let $h_{it} = h[\theta_h(S_t), \Psi_{t-1}]$ i.e., $h_{it} = \omega_i + \alpha_i y_{t-1}^2 + \gamma_i y_{t-1}^2 \mathbb{I}_{(y_{t-1}<0)} + \beta_i h_{t-1}$.

Let $p_{1t} = Pr(S_t = 1|\Psi_{t-1})$ and $p_{2t} = 1 - p_{1t} = Pr(S_t = 2|\Psi_{t-1})$. These probabilities are determined by transition probabilities of first-order Markov process, following Hamilton (1989), Hamilton (1990), which are assumed to be time-dependent as in Gray (1996).

$$\begin{array}{ll} Pr(S_t = 1|S_{t-1} = 1) = P_t, & Pr(S_t = 2|S_{t-1} = 1) = 1 - P_t, \\ Pr(S_t = 2|S_{t-1} = 2) = Q_t, & Pr(S_t = 1|S_{t-1} = 2) = 1 - Q_t. \end{array} \quad (2)$$

In this work we assume that $0 < p_{1t} < 1$ for taking account of two distinct regimes.

Then we have the conditional variance:

$$\begin{aligned} h_t = h_{it} &= Var[y_t/\Psi_{t-1}] = \mathbb{E}[y_t^2/\Psi_{t-1}] \\ &= \sum_{i=1}^{2} Pr(S_t = i/\Psi_{t-1})\mathbb{E}[y_t^2/\Psi_{t-1}, S_t = i] = p_{1t}h_{1t} + p_{2t}h_{2t} =: \mathbf{P}'\mathbf{h_t}, \end{aligned}$$

where $\mathbf{P}_t = (p_{1t}, p_{2t})'$ and $\mathbf{h}_t = (h_{1t}, h_{2t})'$.

If conditional normality is assumed for each regime, then, (w.p. below stands for 'with probability')

$$y_t|\Psi_{t-1} \sim \begin{cases} N(0, h_{1t}) & \text{w.p} \quad p_{1t}, \\ N(0, h_{2t}) & \text{w.p} \quad p_{2t}. \end{cases}$$

We see

$$\mathbf{h}_t = (h_{1t}, h_{2t})' = \begin{bmatrix} \omega_1 + (\alpha_1 + \gamma_1 \mathbb{I}_{(y_{t-1}<0)}) y_{t-1}^2 + \beta_1 h_{t-1} \\ \omega_2 + (\alpha_2 + \gamma_2 \mathbb{I}_{(y_{t-1}<0)}) y_{t-1}^2 + \beta_2 h_{t-1} \end{bmatrix}$$
$$= \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 + \gamma_1 \mathbb{I}_{(y_{t-1}<0)} \\ \alpha_2 + \gamma_2 \mathbb{I}_{(y_{t-1}<0)} \end{bmatrix} y_{t-1}^2 + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} h_{t-1}$$
$$=: \mathbf{w} + (\mathbf{a_1} + \mathbf{a_2} \mathbb{I}_{(\mathbf{y_{t-1}}<\mathbf{0})}) y_{t-1}^2 + \mathbf{b} h_{t-1},$$

where

$$\mathbf{w} = (\omega_1, \omega_2)',\ \mathbf{a_1} = (\alpha_1, \alpha_2)',\ \mathbf{a_2} = (\gamma_1, \gamma_2)',\ \mathbf{b} = (\beta_1, \beta_2),$$

and

$$h_{t-1} = p_{1,t-1} h_{1,t-1} + p_{2,t-1} h_{2,t-1} = \mathbf{P'_{t-1} h_{t-1}}.$$

We multiply by $\mathbf{p'_t}$ to obtain

$$h_t = \mathbf{P'_t h_t} = \mathbf{P'_t w} + \mathbf{P'_t} (\mathbf{a_1} + \mathbf{a_2} \mathbb{I}_{(y_{t-1}<0)}) y_{t-1}^2 + h_{t-1} \mathbf{P'_t b} \tag{3}$$
$$= W_t + A_t y_{t-1}^2 + B_t h_{t-1}, \tag{4}$$

where

$$\begin{aligned} W_t = \mathbf{P'_t w} = p_{1t}\omega_1 + (1-p_{1t})\omega_2 \quad , \quad B_t = \mathbf{P'_t b} = p_{1t}\beta_1 + (1-p_{1t})\beta_2, \\ A_t = \mathbf{P'_t}(\mathbf{a_1} + \mathbf{a_2} \mathbb{I}_{(\mathbf{y_{t-1}}<\mathbf{0})}) = p_{1t}\alpha_1 + (1-p_{1t})\alpha_2 + [p_{1t}\gamma_1 + (1-p_{1t})\gamma_2] \mathbb{I}_{(y_{t-1}<0)}. \end{aligned} \tag{5}$$

We adopt $\tau \in [0, 1+\tau_0)$ for some $\tau_0 > 0$, and assume that the value $\tau$ moves continuously in two time intervals $[t-1, t)$ and $[t, t+1)$ which represents two consecutive times $t-1$ and $t$, respectively, and also assume that the present state remains without change within one unit time. For the value $\tau$ of the "persistence parameter" $\triangle_j := \alpha_j + \frac{\gamma_j}{2} + \beta_j$, $j = 1, 2$. Specifically we express and assume as follows: Let $\tau(t)$ be a function of time $t \in \mathbb{R}$ given by $\tau(.) : \mathbb{R} \to [0, 1+\tau_0)$ for some $\tau_0 > 0$ such that if $t \in \mathbb{Z}$, then $\tau(t) = \triangle(S_t) = \alpha_j + \frac{\gamma_j}{2} + \beta_j$ if $S_t = j$ for $j = 1, 2$ and if $s \in \mathbb{R}/\mathbb{Z}$ and if $t < s < t+1$ for some $t \in \mathbb{Z}$, then $|\tau(t) - \tau(s)| < \epsilon$ for some small $\epsilon > 0$. It implies the following condition for the Markov chain.

(A1): Letting one-step transition time-dependent probabilities $Q_{ij,t} := Pr(S_t = j | S_{t-1} = i)$, the one-step transition probability matrix $\mathbf{Q}_t$ is given as follows:

$$\mathbf{Q}_t = \begin{pmatrix} Q_{11,t} & Q_{12,t} \\ Q_{21,t} & Q_{22,t} \end{pmatrix}.$$

The probabilities $p_{jt} = Pr(S_t = j | \Psi_{t-1})$, $j = 1, 2$, are determined by the one-step transition matrix $\mathbf{Q}_t$. We assume the conditional normality for each regime as follows to apply the two-state Markov chain to the GJR-GARCH model:

(A2): Conditional normality is assumed for each regime with each probability $p_{jt}$, $j = 1, 2$.

$$y_t|\Psi_{t-1} \sim \begin{cases} N(0, h_{1t}) & \text{with} \quad \text{pdf} \quad f(y_t|S_t = 1, \Psi_{t-1}) \quad \text{w.p} \quad p_{1t}, \\ N(0, h_{2t}) & \text{with} \quad \text{pdf} \quad f(y_t|S_t = 2, \Psi_{t-1}) \quad \text{w.p} \quad p_{2t}, \end{cases} \tag{6}$$

where

$$f(y_t|S_t = i, \Psi_{t-1}) = \frac{1}{\sqrt{2\pi h_{it}}} \exp\left\{\frac{-y_t^2}{2h_{it}}\right\}, \quad i = 1, 2.$$

**Lemma 1.** *We assume (A1) and (A2). Under the conditional normality in (6) of (A2), the probability $p_{jt}$ is obtained recursively in terms of $Q_{ij,t}$ and $p_{i,t-1}$ with some initial probabilities $p_{j0}$:*

$$p_{jt} = \sum_{i=1}^{2} Q_{ij,t} \frac{g_{i,t-1} P_{i,t-1}}{\sum_{l=1}^{2} g_{l,t-1} P_{l,t-1}}, \quad j = 1, 2, \tag{7}$$

*where*

$$g_{i,t-1} = f(y_{t-1}|S_{t-1} = i, \Psi_{t-2}). \tag{8}$$

**Lemma 2.** *We assume (A1) and (A2). For $W_t$, $A_t$, $B_t$ in (5), the conditional variance $h_t$ in (4) satisfies the following two expressions:*

*(a)* $h_t = W_t + \sum_{k=1}^{t-1} W_k \prod_{j=1}^{t-k} (B_{t-j+1} + A_{t-j+1}\xi_{t-j}^2) + \prod_{i=0}^{t-1} (B_{i+1} + A_{i+1}\xi_i^2) h_0.$

*(b)* $h_t = \left[\prod_{i=0}^{t-1} (B_{i+1} + A_{i+1}\xi_i^2)\right] \left[h_0 + \sum_{k=1}^{t} \prod_{j=0}^{k-1} \frac{W_k}{(B_{j+1} + A_{j+1}\xi_j^2)}\right].$

### *2.2. Forecasting and cumulative impulse response function*

In the following theorem we present the $\ell$-step ahead of forecasting of the normal parameter $h_{t+\ell}$ of the GRS-GJR-GARCH(1,1) process, given the information $\Psi_t$ at the present time $t$.

**Theorem 1.** *We suppose that:*

*(i) data $\{y_1, ..., y_t\}$ are given and,*

*(ii) the GJR-GARCH parameters are known.*

*We assume (A1)–(A2).*

*Let $h_t(\ell)$ be the $\ell$-step ahead volatility for $\ell = 1, 2, \ldots$, given by $h_t(\ell) = \mathbb{E}[h_{t+\ell}|\Psi_t]$. For $\ell = 1$, the forecasting is given as*

$$h_t(1) = W_{t+1} + A_{t+1}y_t^2 + B_{t+1}h_t,$$

*where* $W_{t+1} = p_{t+1}w$, $A_{t+1} = p_{t+1}(\mathbf{a_1} + \mathbf{a_2}\mathbb{I}_{(\mathbf{y_{t-1}}<\mathbf{0})})$, $B_{t+1} = p_{t+1}b$, *with* $p_{t+1} = (p_{1,t+1}, p_{2,t+1})'$, *and for* $\ell = 2, 3, \ldots$,

$$h_t(\ell) = \hat{W}_{t+\ell} + \sum_{k=1}^{\ell-1} \hat{W}_{k+\ell} \prod_{j=k+1}^{\ell} (\hat{A}_{j+1} + \hat{B}_{j+1}) + \prod_{j=1}^{\ell-1}(\hat{A}_{t+j+1} + \hat{B}_{t+j+1})h_t(1),$$

*where*

$$\hat{W}_{t+j} = \hat{\mathbf{P}}_{\mathbf{t+j}}\mathbf{w}, \quad \hat{A}_{t+j} = \hat{\mathbf{P}}_{\mathbf{t+j}}(\mathbf{a_1} + \mathbf{a_2}\mathbb{I}_{(\mathbf{y_{t-1}}<\mathbf{0})}), \quad \hat{B}_{t+j} = \hat{\mathbf{P}}_{\mathbf{t+j}}\mathbf{b}, \tag{9}$$

*with* $\hat{P}_{t+j} = (\hat{P}_{1,t+j}, \hat{P}_{2,t+j})'$,

$$\hat{P}_{s,t+j} = \sum_{i=1}^{2} Q_{is,t+j} \frac{\hat{g}_{i,t+j-1}\hat{P}_{i,t+j-1}}{\sum_{l=1}^{2} \hat{g}_{l,t+j-1}\hat{P}_{l,t+j-1}}, \quad s = 1, 2, \tag{10}$$

*where*

$$\hat{g}_{i,t+j} = f(\tilde{y}_{t+j}|S_{t+j} = i, \Psi_{t+j-1}) = \frac{1}{\sqrt{2\pi\tilde{h}_{i,t+j}}}\exp\left\{\frac{-\tilde{y}_{t+j}^2}{2\tilde{h}_{i,t+j}}\right\}, \tag{11}$$

*with* $\tilde{h}_{i,t+j} = \omega_i + (\alpha_i + \gamma_i\mathbb{I}_{(y_{t-1}<0)})\tilde{y}_{t+j-1}^2 + \beta_i\tilde{h}_{t+j-1}$ *and* $\tilde{y}_{t+j} = \xi_{t+j}\sqrt{\tilde{h}_{t+j}}$ *for* $j = 2, 3, \ldots, \ell$, *and for* $i = 1, 2$.

**Remark**

In this work, $p_{jt}$ are computed by Lemma 1 with some initial values $p_{jt}$, for which, say, $p_{jt}$ randomly from uniform distribution $U(0, 1)$ can be chosen when the data $\{y_1, \cdots, y_t\}$ were given at the present time $t$. To evaluate the $\ell$-step ahead volatility $h_t(\ell)$ in Theorem 1, the initial probabilities $\hat{P}_{jt}$ are necessary in (10), for which the $p_{jt}$ obtained by Lemma 1 are used. Note that for the non-stationary GRS-GJR-GARCH(1,1) model with a regime having explosive values in volatility, the probabilities $p_{jt}$ converge to a constant related with the limit of the transition probabilities. However, for the stationary cases, the probabilities might be dynamics depending on the randomness of the process. In the recursive formulas in (7) and (10), the probabilities should be evaluated along and are computed in the Section of a Monte Carlo simulation as well as their dynamics in the two cases: constant transition probabilities and time-dependent transition probabilities of the Markov-chain regime switching.

Now we derive the representation for the cumulative impulse response function of the GRS-GJR-GARCH(1,1) process. Impulse response function of a dynamic system is its output when presented with a brief input signal, called an impulse. An impulse response refers to the reaction of the dynamic system in response to some external change. The cumulative impulse response in volatility denoted by $\lambda_\ell$ measures a certain contribution of innovation $\eta_t$ at time $t$ to the $\ell$-step ahead volatility. Baillie *et al.* (1996) introduced Fractionally Integrated GARCH processes and noted that cumulative impulse response in volatility denoted by $\lambda_\ell$ goes to zero as $\ell \to \infty$ for a class of stable GARCH processes, while $\lambda_\ell \to \infty$ as $\ell \to \infty$ for a class of explosive GARCH processes where the cumulative impulse response is given by the partial derivative of $h_t(\ell)$:

$$\lambda_\ell = \frac{\partial h_t(\ell)}{\partial \eta_t}$$

with $\eta = y_t^2 - h_t = y_t^2 - \mathbb{E}[y_t^2|\Psi_{t-1}]$, the prediction error for the squared observations $y_t^2$.

Considering the following sequence $\eta_t = y_t^2 - h_t$ which constitutes a sequence of zero mean martingale differences, we can express

$$y_t^2 = W_t + (A_t + B_t)y_{t-1}^2 + \eta_t - B_t\eta_{t-1}.$$

**Proof.**

$\eta_t = y_t^2 - h_t$ in (4), we have

$$\begin{aligned} y_t^2 &= W_t + A_t y_{t-1}^2 + B_t h_{t-1} + \eta_t \\ &= W_t + A_t y_{t-1}^2 + B_t h_{t-1} + B_t y_{t-1}^2 - B_t y_{t-1}^2 + \eta_t \\ &= W_t + (A_t + B_t)y_{t-1}^2 - B_t(y_{t-1}^2 - h_{t-1}) + \eta_t, \end{aligned}$$

then $y_t^2 = W_t + (A_t + B_t)y_{t-1}^2 + \eta_t - B_t\eta_{t-1}$.

Note that $y_t^2$ is an ARMA(1,1) process with time-varying coefficients, and for $\ell \geq 2$,

$$\begin{aligned} h_t(\ell) = \mathbb{E}[y_{t+\ell}^2|\Psi_t] &= \mathbb{E}[W_{t+\ell} + (A_{t+\ell} + B_{t+\ell})y_{t+\ell-1}^2 + \eta_{t+\ell} - B_{t+\ell}\eta_{t+\ell-1}|\Psi_t] \\ &= W_{t+\ell} + (A_{t+\ell} + B_{t+\ell})\mathbb{E}[y_{t+\ell-1}^2|\Psi_t] + \mathbb{E}[\eta_{t+\ell}|\Psi_t] - B_{t+\ell}\mathbb{E}[\eta_{t+\ell-1}|\Psi_t]. \end{aligned}$$

We know that $\eta_t = y_t^2 - \mathbb{E}[y_t^2|\Psi_{t-1}]$ is a martingale difference sequence.

We denote that $\Psi_{t+\ell-1} \subset \Psi_t$ with $\ell \geq 2$,

$$\begin{aligned} \mathbb{E}[\eta_{t+\ell}|\Psi_t] &= \mathbb{E}[y_{t+\ell}^2 - \mathbb{E}[y_{t+\ell}^2|\Psi_{t+\ell-1}]|\Psi_t] \\ &= \mathbb{E}[y_{t+\ell}^2|\Psi_t] - \mathbb{E}[\mathbb{E}[y_{t+\ell}^2|\Psi_{t+\ell-1}]|\Psi_t] \\ &= E[y_{t+\ell}^2|\Psi_t] - \mathbb{E}[y_{t+\ell}^2|\Psi_t] \\ &= 0, \end{aligned}$$

and $\Psi_{t+\ell-2} \subset \Psi_t$ with $\ell \geq 2$,

$$\begin{aligned}\mathbb{E}[\eta_{t+\ell-1}|\Psi_t] &= \mathbb{E}[y_{t+\ell-1}^2 - \mathbb{E}[y_{t+\ell-1}^2|\Psi_{t+\ell-2}]|\Psi_t] \\ &= \mathbb{E}[y_{t+\ell-1}^2|\Psi_t] - \mathbb{E}[\mathbb{E}[y_{t+\ell-1}^2|\Psi_{t+\ell-2}]|\Psi_t] \\ &= \mathbb{E}[y_{t+\ell-1}^2|\Psi_t] - \mathbb{E}[y_{t+\ell-1}^2|\Psi_t] \\ &= 0.\end{aligned}$$

We have then $h_t(\ell) = W_{t+\ell} + (A_{t+\ell} + B_{t+\ell})h_t(\ell-1)$ and thus $\lambda_\ell = (A_{t+\ell} + B_{t+\ell})\lambda_{\ell-1}$. The following result presents the general solution of the linear difference equation for $\lambda_\ell$ for $\ell = 1, 2, \ldots$, straightforwardly from Theorem 1, where the estimated parameters are used.

**Theorem 2.** *We assume (A1) and (A2). Then*

*(a)* $\lambda_1 = A_{t+1}$ *and for* $\ell = 2, 3, \ldots$, $\lambda_\ell = \prod_{j=1}^{\ell-1}(\hat{A}_{t+j+1} + \hat{B}_{t+j+1})A_{t+1}$.

*(b) If* $\hat{A}_{t+j} + \hat{B}_{t+j} \equiv \hat{P}_{1,t+j} \triangle_1 + \hat{P}_{2,t+j}\triangle_2 < 1$ *for all* $j$ *but finitely many times, then* $\lambda_\ell \to 0$ *as* $\ell \to \infty$ *and the GRS-GJR-GARCH process is stable in volatility.*

*(c) If* $\hat{A}_{t+j} + \hat{B}_{t+j} \equiv \hat{P}_{1,t+j} \triangle_1 + \hat{P}_{2,t+j}\triangle_2 > 1$ *infinitely often in* $j$*, then* $\lambda_\ell \to \infty$ *as* $\ell \to \infty$ *and the GRS-GJR-GARCH process is explosive in volatility.*

There exist in the literature for volatilities of non-stationary GARCH models such as Nelson (1990), Linton *et al.* (2010) and Li *et al.* (2014), we includes also in this work much wider classes of time series models than the existing. Hong and Hwang (2016) and Kim and Hwang (2018) established recently the asymptotic normality of the logarithm of the volatility under the non-stationary condition of the GRS-–GARCH models and four-state Markov regime switching GARCH model respectively.

According to them, under condition that $A_t + B_t \equiv p_{1t} \triangle_1 + p_{2t}\triangle_2 > 1$ infinitely often in $t$, which implies the non-stationary of our two-state Markov regime switching GARCH model, we have the following normality theory of the logarithm of the volatility:
Let $X_t = \log(B_{t+1} + A_{t+1}\xi_t^2)$, $m_t = \mathbb{E}[X_t]$, and

$$\sigma^2 = \lim_{T\to\infty} Var\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(X_t - m_t)\right].$$

**Theorem 3.** *(Asymptotic Normality Under Non-stationary Condition).*

*Under some appropriate assumptions on* $\xi_t$*, the asymptotic normality holds as* $T \to \infty$,

$$\frac{1}{\sigma\sqrt{T}}\left[\log h_T - \sum_{t=0}^{T-1} m_t\right] \xrightarrow{d} N(0,1).$$

## 3. Monte Carlo Simulation

In this section, we consider time-dependent transition probabilities of the Markov-chain regime-switching, which determine crucially the behavior of the cumulative impulse response functions $\lambda_\ell$. To simulate Markov Switching's GJR-GARCH models, we use Chuffart (2017) MSGtool, which is a MATLAB toolbox. This box provides a set of functions for simulating and estimating a wide variety of Markov Switching (MSG) GARCH models. The toolbox is very flexible and user-friendly with a large number of possible options. We choice the parameter values to enforce conditions for the parameters so that the model becomes stationary and nonstationary.

We first start with the time series plot of the GJR-GARCH(1,1) model. Fig. 1 (a), (b) depicts the GJR-GARCH(1,1) process and its volatility with $\omega = 0.4$, $\alpha = 0.2$, $\gamma = 0.3$, and $\beta = 0.4$, which is a stationary case while Fig. 1(c), (d) with $\omega = 0.4$, $\alpha = 0.5$, $\gamma = 0.2002$, and $\beta = 0.5$, which is a non-stationary case.

Figure 2 depicts the conditional probabilities $p_{1t}$ and $p_{2t} = 1 - p_{1t}$ of regime 1 and regime 2, respectively, given the past information, which is given in Lemma 1, where the parameters of GRS-GJR-GARCH(1,1) process are $\omega_1 = 0.5$, $\alpha_1 = 0.1$, $\gamma_1 = 0.2$, $\beta_1 = 0.7$; $\omega_2 = 0.4$, $\alpha_2 = 0.2$, $\gamma_2 = 0.3$, $\beta_2 = 0.4$, that is a stationary case. Figure 2(a), (b) use constant transition probabilities $P_t = 0.6$, $Q_t = 0.8$ in (a) and $P_t = 0.3$, $Q_t = 0.4$ in (b), while Figure 2(c), (d) use time-dependent transition probabilities $P_t = 0.8-0.5/t$, $Q_t = 0.4+0.1/t$ in (c) and $P_t = 0.6+0.2/t$, $Q_t = 0.9-0.6/t$ in (d). We choose the initial values of $p_{1t}$ randomly from uniform distribution $U(0,1)$ and generate values of $p_{1t}$ recursively by using (7). Under the assumption of the conditional normality for each regime, conditional probabilities $p_{jt}$, $j = 1,2$, of each state, which are formulated in Lemma 1, are given in Figure 3, where the red line is the probability $p_{1t}$ of state 1 and blue one $p_{2t}$ of state 2. As seen in Figure 2(c),(d), the probabilities vary dynamically as time goes, even though the transition probabilities are constant.
In Figures 3 and 4, we see the time series plots of the stationary GRS-GJR-GARCH(1,1) process and its volatility. Figure 3(a)-(d) use the constant transition probabilities $P_t$ and $Q_t$ as in Figure 2(a),(b), while Figure 4(a)-(d) use the time-dependent transition probabilities as in Figure 2(c),(d). Figures 5 and 6 repeat the same way as Figures 3 and 4, but with one of the two regimes being a non-stationary case. Figures 3 and 4 are related to switching two stationary GJR-GARCH(1,1) models, while Figures 5 and 6 are related to switching a stationary GJR-GARCH(1,1) process and a non-stationary GJR-GARCH(1,1) process with the same transition probabilities as those in Figures 3 and 4.
Finally, we observe the behavior of the cumulative impulse response functions discussed in Theorem 2.4. Figure 7(a) shows the plot of the cumulative impulse re-

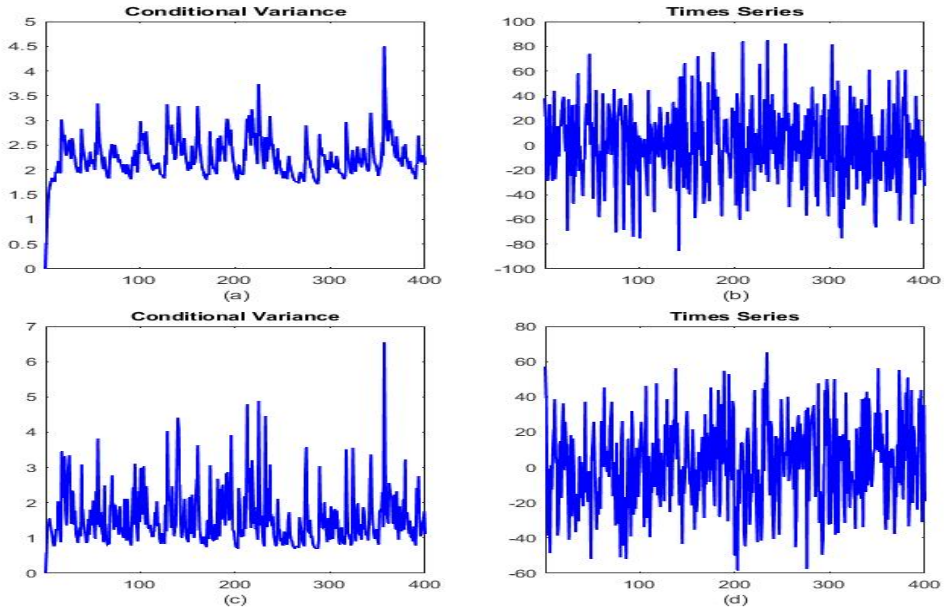


**Fig. 1.** GJR-GARCH(1,1) process and its volatility in stationary and non-stationary, (a),(b) $\omega = 0.5$, $\alpha = 0.1$, $\gamma = 0.2$, $\beta = 0.7$; and (c), (d) $\omega = 0.4$, $\alpha = 0.502$, $\gamma = 0.2$, $\beta = 0.4$.

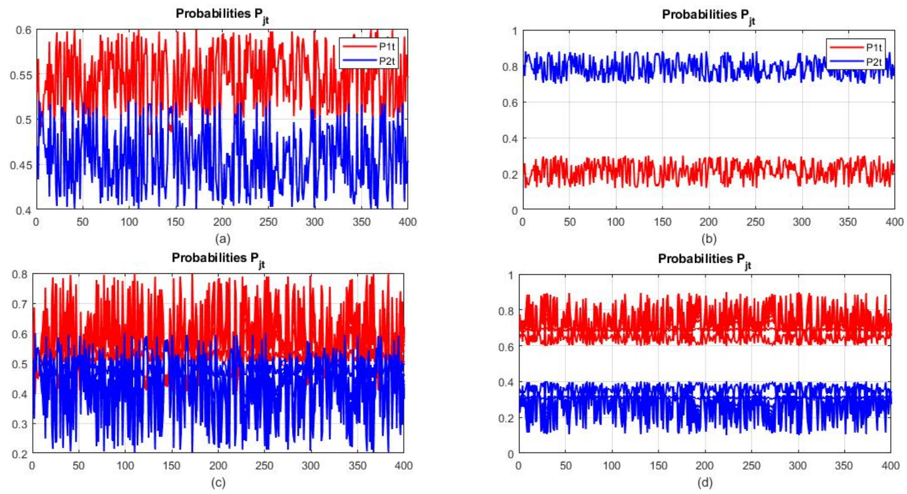


**Fig. 2.** Conditional probability in the GRS-GJR-GARCH(1,1) process with $\omega_1 = 0.5$, $\alpha_1 = 0.1$, $\gamma_1 = 0.2$, $\beta_1 = 0.7$; $\omega_2 = 0.4$, $\alpha_2 = 0.2$, $\gamma_2 = 0.3$, $\beta_2 = 0.4$: (a) $P_t = 0.6$, $Q_t = 0.8$; (b) $P_t = 0.3$, $Q_t = 0.4$ ; (c) $P_t = 0.8 - 0.5/t$, $Q_t = 0.4 + 0.1/t$; (d) $P_t = 0.6 + 0.2/t$, $Q_t = 0.9 - 0.6/t$.

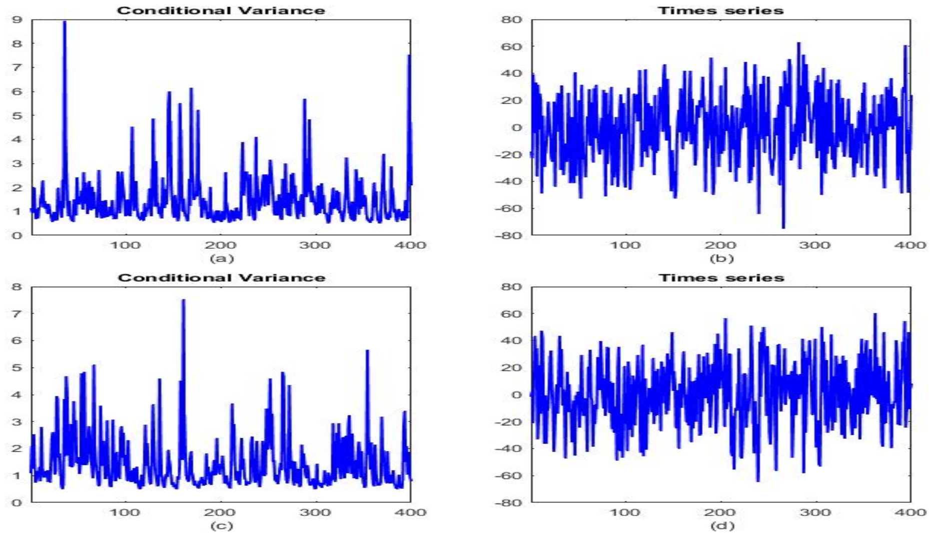


**Fig. 3.** GRS-GJR-GARCH(1,1) process and its volatility with $\omega_1 = 0.5$, $\alpha_1 = 0.1$, $\gamma_1 = 0.2$, $\beta_1 = 0.7$; $\omega_2 = 0.4$, $\alpha_2 = 0.2$, $\gamma_2 = 0.3$, $\beta_2 = 0.4$ and with transition probabilities in (a), (b) $P_t = 0.6$, $Q_t = 0.8$ and in (c), (d) $P_t = 0.3$, $Q_t = 0.4$.

sponse functions for two cases : stationary ($\omega = 0.5$, $\alpha = 0.1$, $\gamma = 0.2$, $\beta = 0.7$) and explosive ($\omega = 0.3$, $\alpha = 0.45$, $\gamma = 0.2$, $\beta = 0.55$) cases of the GJR-GARCH(1,1) process, while Figure 7(b) for the stationary GRS-GJR-GARCH(1,1) with the constant transition probabilities $P_t = 0.6$, $Q_t = 0.8$; and $P_t = 0.3$, $Q_t = 0.4$ with the GJR-GARCH(1,1) parameters as in Figure 3 and 4. Figure 8(a) illustrate the non-stationary GJR-GARCH(1,1), whereas Figure 8(b) the non-stationary GRS-GJR-GARCH(1,1) model with transition probability $P_t = 0.8 - 0.5/t$, $Q_t = 0.4 + 0.1/t$.

## 4. An empirical study of IBOVESPA stock index

We, therefore, illustrate our regime-switching Markov model with time-varying transition probabilities in an empirical study of the Brazilian stock market index IBOVESPA. The time series for IBOVESPA is obtained from Yahoo finance; we have daily observations adjusted for changes from January 2019 to December 2020, T = 493. Unsurprisingly, the COVID-19 pandemic has also affected the Brazilian economy and growth forecasts. The South American powerhouse was showing the first signs of recovery after a severe economic crisis that hit the country in 2014. When the World Health Organization declared the new coronavirus a pandemic on 11 March 2020, Brazil was still a week away from declaring its first death due to COVID-19. Nevertheless, the largest Latin American nation quickly attracted the world's attention, as the number of cases and deaths due to COVID-19 in the country increased exponentially, reaching the third-highest figure in the world, behind only the United States and India. The online site Le Figaro writes in mid-December that Brazil has passed the threshold of seven million cases

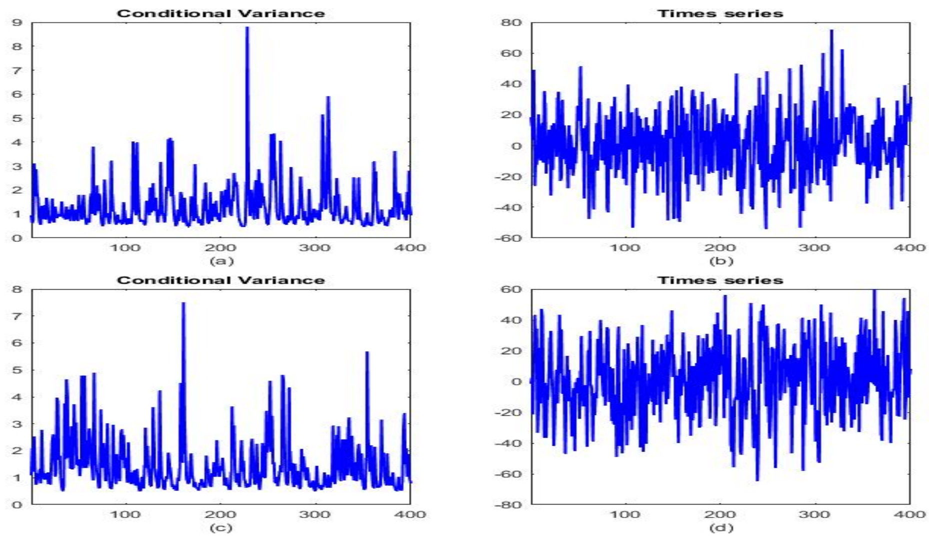


**Fig. 4.** GRS-GJR-GARCH(1,1) process and its volatility with $\omega_1 = 0.5$, $\alpha_1 = 0.1$, $\gamma_1 = 0.2$, $\beta_1 = 0.7$; $\omega_2 = 0.4$, $\alpha_2 = 0.2$, $\gamma_2 = 0.3$, $\beta_2 = 0.4$ and with transition probabilities in (a), (b) $P_t = 0.8 - 0.5/t$, $Q_t = 0.4 + 0.1/t$ and in (c), (d) $P_t = 0.6 + 0.2/t$, $Q_t = 0.9 - 0.6/t$.

and 180,000 deaths. This is confirmed by WHO statistics (see www.who.int). Brazil's main stock market is located in the city of São Paulo, which is also the region of Brazil where the majority of coronavirus infections are registered. Just after the COVID-19 pandemic reached Brazil with the first confirmed case on February 25, 2020, the IBOVESPA stock market index fell to 102,984 Brazilian reals on February 27. On March 23, 2020, the index has reached its lowest value since the beginning of the year, at 63,570 Brazilian reals. In August 2020, the stock market began to stabilize, with an average value of 95,000 Brazilian reals.

A study is being carried out on the IBOVESPA stock market index to see how cumulative impulse response functions behave in the face of the shocks of the health crisis. We examine impulse response functions that identify for each market the impact of a shock on volatility and reaction time. In order to know what the amplitude of the shock response would be and how long it would take the IBOVESPA stock market to absorb the effect of a random shock, the study of impulse response functions will allow us to provide some answers. The impulse response function traces the effect of a unitary residual shock on the present and future values of endogenous variables. The effect of the shock is transmitted in the system by the dynamics of the model we have defined for the financial series. In this paper, we, therefore, compare the impulse response function of the GRS-GJR-GARCH model with the standard GJR-GARCH model using the constant and time-dependent probabilities in Figure 9 and Figure 10 in order to show the efficiency of our model in taking

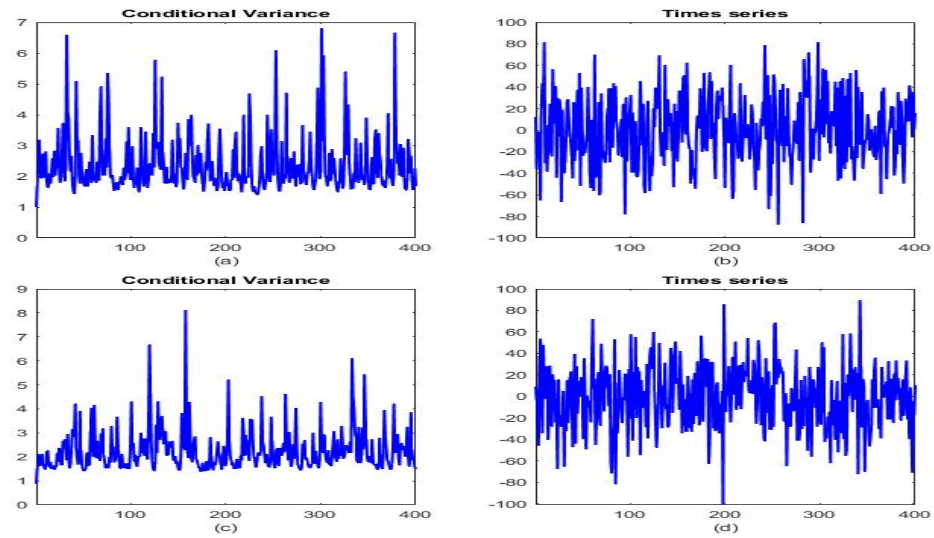


**Fig. 5.** GRS-GJR-GARCH(1,1) process and its volatility with $\omega_1 = 2$, $\alpha_1 = 0.2$, $\gamma_1 = 0.1$, $\beta_1 = 0.6$; $\omega_2 = 0.4$, $\alpha_2 = 0.502$, $\gamma_2 = 0.1$, $\beta_2 = 0.5$ and with time-dependent transition probabilities (a), (b) $P_t = 0.6$, $Q_t = 0.8$ and in (c), (d) $P_t = 0.3$, $Q_t = 0.4$.

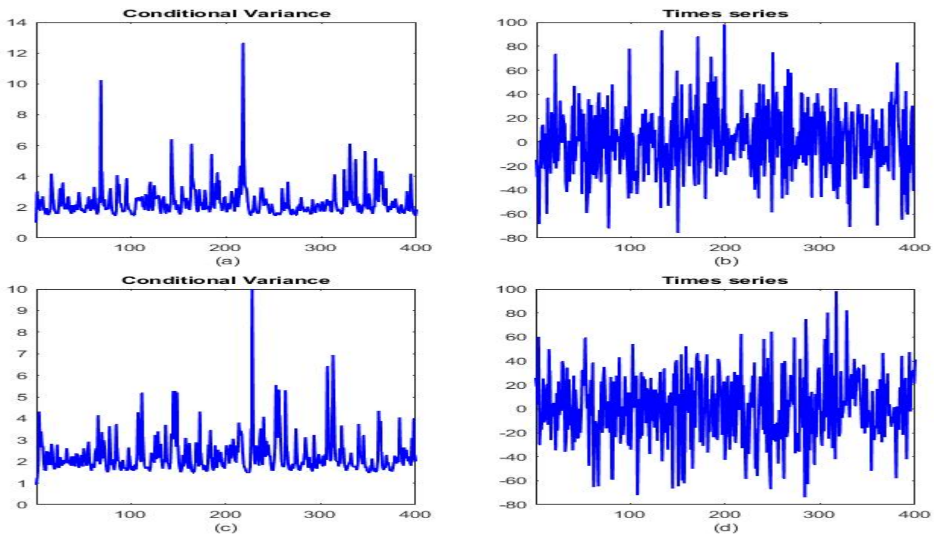


**Fig. 6.** GRS-GJR-GARCH(1,1) process and its volatility with $\omega_1 = 2$, $\alpha_1 = 0.2$, $\gamma_1 = 0.1$, $\beta_1 = 0.6$; $\omega_2 = 0.4$, $\alpha_2 = 0.502$, $\gamma_2 = 0.1$, $\beta_2 = 0.5$ and with time-dependent transition probabilities (a), (b) $P_t = 0.8 - 0.5/t$, $Q_t = 0.4 + 0.1/t$ and in (c), (d) $P_t = 0.6 + 0.2/t$, $Q_t = 0.9 - 0.6/t$.

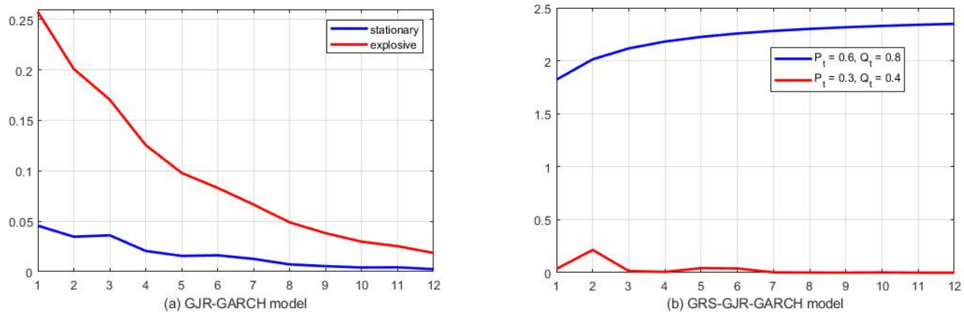


**Fig. 7.** Cumulative impulse response functions of (a) stationary, explosive GJR-GARCH(1,1) and (b) non-stationary GRS-GJR-GARCH(1,1) with transition probabilities $P_t = 0.6$, $Q_t = 0.8$ and $P_t = 0.3$, $Q_t = 0.4$.

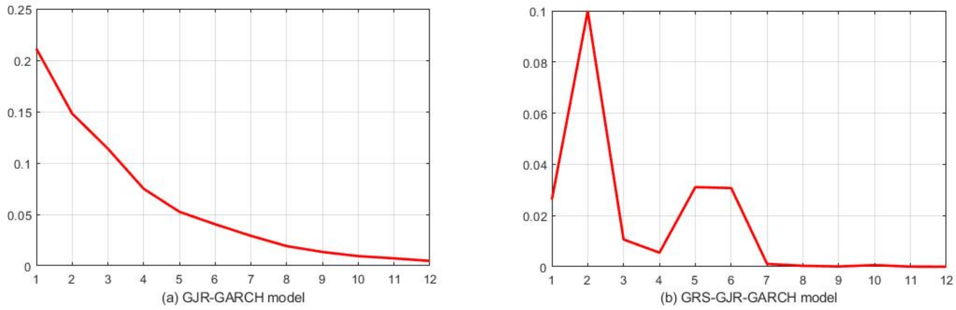


**Fig. 8.** Cumulative impulse response functions of (a) non-stationary GJR-GARCH(1,1) and (b) non-stationary GRS-GJR-GARCH(1,1) with transition probabilities $P_t = 0.8 - 0.5/t$, $Q_t = 0.4 + 0.1/t$.

into account the asymmetry of the shocks received by the IBOVESPA stock market index.

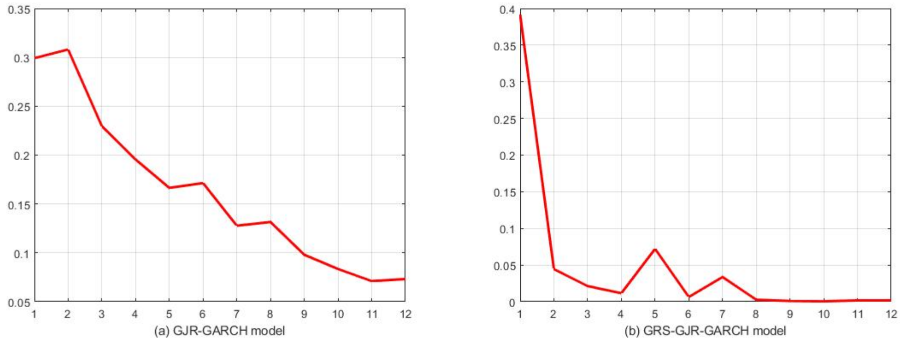


**Fig. 9.** Cumulative impulse response functions of (a) non-stationary GJR-GARCH(1,1) and (b) non-stationary GRS-GJR-GARCH(1,1) with transition probabilities $P_t = 0.6$, $Q_t = 0.8$.

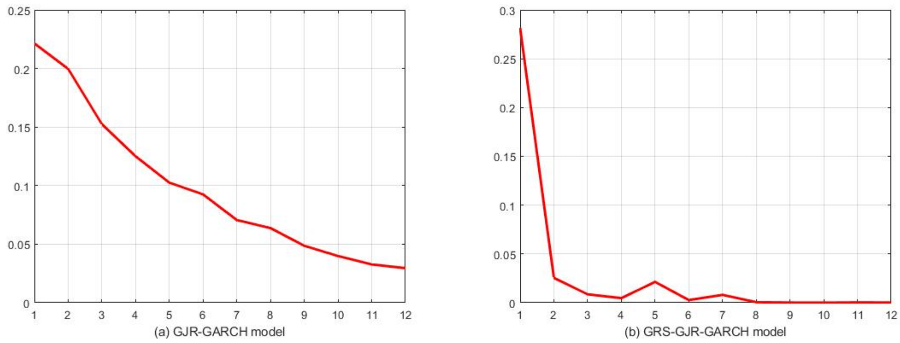


**Fig. 10.** Cumulative impulse response functions of (a) explosive GJR-GARCH(1,1) and (b) explosive GRS-GJR-GARCH(1,1) with transition probabilities $P_t = 0.8-0.5/t$, $Q_t = 0.4 + 0.1/t$.

**Discussion**

The stock market of Brazil IBOVESPA, has reacted positively to the shock of the global health crisis of Coronavirus. This can be seen in Figure 9(b) and Figure 10(b) with our GRS-GJR-GARCH model. The amplitude of the GRS-GJR-GARCH response to the shock is however larger than that of the GJR-GARCH and after the shock, the effects are dissipated just from horizon 8. Concerning the impulse response of the standard GJR-GARCH model, which has a rather similar reaction behavior, the impact of the shock is less important, but the damping time is relatively long for the GJR-GARCH which goes beyond horizon 12 to dampen the shock (See Figure 9(a) and Figure 10(a)). Thus, we observe that the cumulative impulse response function of the GRS-GJR-GARCH model with time-varying transition probability is less persistent than that of the GRS-GJR-GARCH model with constant probability. The GRS-GJR-GARCH model with time-varying transition probability, therefore, better addresses the asymmetry problem generally observed in the volatility of financial series.

## 5. Conclusion

This work proposed an asymmetric GJR-GARCH with a two-state Markov regime-switching model with two components for the dynamics: one is stationary and the other is explosive, using time-varying transition probabilities to capture the volatility dynamics with different characteristics of practical financial markets such as financial crisis and international politics. The novelty of this model is to capture the asymmetry in the volatility dynamics and the impact on the persistence parameter observed through the impulse response function. An empirical study carried out on real data from the IBOVESPA stock market, shows that the GRS-GJR-GARCH takes better into account the asymmetry and presents a slight persistence in the volatility than the standard GJR-GARCH. We show here the interest of using the GRS-GJR-GARCH model with time-dependent probabilities through the cumulative impulse response function which is less persistent and drops quickly to zero than its constant probability counterpart and the GJR-GARCH model.

In the future, it would be interesting to see the implication of asymmetric innovations such as asymmetric innovations as in Park *et al.* (2010) and Hwang *et al.* (2010) for the Markov-chain states, and we will complete the novelty of our model with a work on an estimation problem.

## Appendix

This appendix contains the proofs of the theorems and the lemma.

## A Proofs

*A.1 Proof of Lemma 1*

Derivation of the recursive formula of $p_{it}$: we observe

$$\begin{aligned} p_{1t} &= Pr(S_t = 1|\Psi_{t-1}) \\ &= Pr(S_t = 1|S_{t-1} = 1)Pr(S_{t-1} = 1|\Psi_{t-1}) + Pr(S_t = 1|S_{t-1} = 2)Pr(S_{t-1} = 2|\Psi_{t-1}) \\ &= P_t Pr(S_{t-1} = 1|\Psi_{t-1}) + (1 - Q_t)Pr(S_{t-1} = 2|\Psi_{t-1}) \\ &= Q_{11,t}Pr(S_{t-1} = 0|\Psi_{t-1}) + Q_{21,t}Pr(S_{t-1} = 1|\Psi_{t-1}). \end{aligned}$$

For $j = 1, 2$, $Pr(S_{t-1} = j|\Psi_{t-1}) = Pr(S_{t-1} = j|y_{t-1}, \Psi_{t-2})$

$$= \frac{f(y_{t-1}|S_{t-1} = j, \Psi_{t-2})Pr(S_{t-1} = j|\Psi_{t-2})}{\sum_{i=1}^{2} f(y_{t-1}|S_{t-1} = i, \Psi_{t-2})Pr(S_{t-1} = i|\Psi_{t-2})} = \frac{f(y_{t-1}|S_{t-1} = j)p_{j,t-1}}{\sum_{i=0}^{2} f(y_{t-1}|S_{t-1} = i)p_{i,t-1}}$$

by Bayes Rule, where $g_{i,t-1} = f(y_{t-1}|S_{t-1} = i)$ is the likelihood function of Markov regime switching GJR-GARCH model at time $t-1$ given $S_{t-1} = i$.

Therefore we have

$$p_{1t} = Q_{11,t}\left[\frac{g_{1,t-1}p_{1,t-1}}{g_{1,t-1}p_{1,t-1} + g_{2,t-1}(1-p_{1,t-1})}\right] + Q_{21,t}\left[\frac{g_{2,t-1}(1-p_{1,t-1})}{g_{1,t-1}p_{1,t-1} + g_{2,t-1}(1-p_{1,t-1})}\right] \quad (12)$$

*A.2 Proof of Lemma 2*

(a)

$$\begin{aligned} h_t &= W_t + A_t(y_{t-1} - m_{t-1})^2 + B_t h_{t-1} = W_t + A_t(\xi_{t-1}^2 h_{t-1}) + B_t h_{t-1} \\ &= W_t + (B_t + A_t\xi_{t-1}^2)h_{t-1} \\ &= W_t + (B_t + A_t\xi_{t-1}^2)[W_{t-1} + (B_{t-1} + A_{t-1}\xi_{t-2}^2)h_{t-2}] \\ &= W_t + W_{t-1}(B_t + A_t\xi_{t-1}^2) + (B_t + A_t\xi_{t-1}^2)(B_{t-1} + A_{t-1}\xi_{t-2}^2)h_{t-2} \\ &\vdots \\ &= W_t + W_{t-1}(B_t + A_t\xi_{t-1}^2) + \ldots + W_1(B_t + A_t\xi_{t-1}^2)(B_{t-1} + A_{t-1}\xi_{t-2}^2)\ldots(B_2 + A_2\xi_1^2) \\ &+ (B_t + A_t\xi_{t-1}^2)(B_{t-1} + A_{t-1}\xi_{t-2}^2)\ldots(B_1 + A_1\xi_0^2)h_0. \end{aligned}$$

Thus

$$h_t = W_t + \sum_{k=1}^{t-1} W_k \prod_{j=1}^{t-k}(B_{t-j+1} + A_{t-j+1}\xi_{t-j}^2) + \prod_{i=0}^{t-1}(B_{i+1} + A_{i+1}\xi_i^2)h_0.$$

(b)

$$\begin{aligned} h_t &= \left[\prod_{i=0}^{t-1}(B_{i+1} + A_{i+1}\xi_i^2)\right]\left[h_0 + \sum_{k=1}^{t}\prod_{j=0}^{k-1}\frac{W_k}{(B_{j+1} + A_{j+1}\xi_j^2)}\right] \\ &= \prod_{i=0}^{t-1}(B_{i+1} + A_{i+1}\xi_i^2)h_0 + \sum_{k=1}^{t} W_k \prod_{i=0}^{t-1}(B_{i+1} + A_{i+1}\xi_i^2)\prod_{j=0}^{k-1}\frac{1}{(B_{j+1} + A_{j+1}\xi_j^2)}, \end{aligned}$$

by equating equations (a) and (b), we see that $\prod_{i=0}^{t-1}(B_{i+1} + A_{i+1}\xi_i^2)h_0$ is common. We show that

$$W_t + \sum_{k=1}^{t-1} W_k \prod_{j=1}^{t-k}(B_{t-j+1} + A_{t-j+1}\xi_{t-j}^2) = \sum_{k=1}^{t} W_k \prod_{i=0}^{t-1}(B_{i+1} + A_{i+1}\xi_i^2)\prod_{j=0}^{k-1}\frac{1}{(B_{j+1} + A_{j+1}\xi_j^2)} \quad (13)$$

We will show that the right-hand term of (9) can be written like the left-hand term

$$\sum_{k=1}^{t} W_k \prod_{i=0}^{t-1}(B_{i+1} + A_{i+1}\xi_i^2) \prod_{j=0}^{k-1} \frac{1}{(B_{j+1} + A_{j+1}\xi_j^2)}$$

$$= \sum_{k=1}^{t-1} W_k \prod_{i=0}^{t-1}(B_{i+1}+A_{i+1}\xi_i^2) \prod_{j=0}^{k-1} \frac{1}{(B_{j+1} + A_{j+1}\xi_j^2)} + W_t \prod_{i=0}^{t-1}(B_{i+1}+A_{i+1}\xi_i^2) \prod_{j=0}^{t-1} \frac{1}{(B_{j+1} + A_{j+1}\xi_j^2)}$$

$$\begin{aligned}
&= \sum_{k=1}^{t-1} W_k \frac{(B_1 + A_1\xi_0^2)\dots(B_k + A_k\xi_{k-1}^2)(B_{k+1} + A_{k+1}\xi_k^2)\dots(B_t + A_t\xi_{t-1}^2)}{(B_1 + A_1\xi_0^2)\dots(B_k + A_k\xi_{k-1}^2)} \\
&= W_t + \sum_{k=1}^{t-1} W_k(B_{k+1} + A_{k+1}\xi_k^2)\dots(B_t + A_t\xi_{t-1}^2) \\
&= W_t + \sum_{k=1}^{t-1} W_k \prod_{l=k+1}^{t} (B_l + A_l\xi_{l-1}^2) \\
&= W_t + \sum_{k=1}^{t-1} W_k \prod_{j=1}^{t-1}(B_{t-j+1} + A_{t-j+1}\xi_{t-j}^2).
\end{aligned}$$

Hence we have the desired equality in (9) and complete proof of Lemma 2.

### *A.3 Proof of Theorem 1*

Note that given the data $\{y_1, \dots, y_t\}$ at the present time $t$, probabilities $p_{j,t+1}$ are obtained by Lemma 1, and so are $W_{t+1}$, $A_{t+1}$, $B_{t+1}$. Let $\tilde{h}_{t+\ell} \equiv h_t(\ell)$ be the $\ell$-step ahead forecast of the volatility. First, we observe

$$\begin{aligned}
\tilde{h}_{t+1} \equiv h_t(1) = \mathbb{E}[h_{t+1}|\mathcal{F}_t] &= \mathbb{E}[W_{t+1} + A_{t+1}y_t^2 + B_{t+1}h_t|\mathcal{F}_t] \\
&= W_{t+1} + A_{t+1}\mathbb{E}[y_t^2|\mathcal{F}_t] + B_{t+1}\mathbb{E}[h_t|\mathcal{F}_t] \\
&= W_{t+1} + A_{t+1}y_t^2 + B_{t+1}h_t.
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
\tilde{h}_{t+2} \equiv h_t(2) = \mathbb{E}[h_{t+2}|\mathcal{F}_t] &= \mathbb{E}[W_{t+2} + A_{t+2}y_{t+1}^2 + B_{t+2}h_{t+1}|\mathcal{F}_t] \\
&= W_{t+2} + A_{t+2}\mathbb{E}[y_{t+1}^2|\mathcal{F}_t] + B_{t+2}\mathbb{E}[h_{t+1}|\mathcal{F}_t] \\
&= W_{t+2} + (A_{t+2} + B_{t+2})\tilde{h}_{t+1}.
\end{aligned}$$

However, we estimate $W_{t+2}$, $A_{t+2}$, $B_{t+2}$ by means of the estimates of $p_{s,t+2}$, $s = 1, 2$. We use the formula given in (7) of Lemma 1 recursively to obtain (9) and (10) with j = 2. Thus we use (8) with $j = 2$ to get $\tilde{h}_{t+2} \equiv h_t(2) = \hat{W}_{t+2} + (\hat{A}_{t+2} + \hat{B}_{t+2})\tilde{h}_{t+1}$. For general $\ell = 2, 3, \cdots$, we have

$$\tilde{h}_{t+\ell} \equiv h_t(\ell) = \mathbb{E}[h_{t+\ell}|\mathcal{F}_t] = W_{t+\ell} + (A_{t+\ell} + B_{t+\ell})\tilde{h}_{t+\ell-1} \tag{14}$$

$$= W_{t+\ell} + (A_{t+\ell} + B_{t+\ell})[W_{t+\ell-1} + (A_{t+\ell-1} + B_{t+\ell-1})\tilde{h}_{t+\ell-2}] = \cdots =$$
$$= W_{t+\ell} + \sum_{k=1}^{\ell-1} W_{t+k} \prod_{j=k+1}^{\ell} (A_{j+1} + B_{j+1}) + \prod_{j=1}^{\ell-1}(A_{t+j+1} + B_{t+j+1})\tilde{h}_{t+1},$$

provided all $p_{s,t+j}$, $s = 1, 2$, are given. These probabilities are estimated recursively by using Lemma 1 and the results are obtained along with (9) and (10) in Theorem 1.

*A.4 Proof of Theorem 2*

(a) For $\eta_t = y_t^2 - h_t$, since $h_t(1) = W_{t+1} + A_{t+1}y_t^2 + B_{t+1}h_t = W_{t+1} + (A_{t+1} + B_{t+1})y_t^2 - B_{t+1}\eta_t$, we have $\lambda_1 = (A_{t+1}+B_{t+1}) - B_{t+1} = A_{t+1}$ in theorem 1 where $\dfrac{\partial y_t^2}{\partial \eta_t} = 1$ is used.

In effect $\lambda_1 = \dfrac{\partial h_t(1)}{\partial \eta_t} = \dfrac{\partial h_t(1)}{\partial y_t^2} \cdot \dfrac{\partial y_t^2}{\partial \eta_t}$ with $y_t^2 = \eta_t + h_t$,

where $\dfrac{\partial h_t(1)}{\partial y_t^2} = A_{t+1} + B_{t+1}$ and $\dfrac{\partial y_t^2}{\partial \eta_t} = 1$,

hence $\lambda_1 = (A_{t+1} + B_{t+1}) - B_{t+1} = A_{t+1}$.

For $\ell \geq 2$, by $\tilde{h}_{t+\ell} \equiv h_t(\ell) = \mathbb{E}[h_{t+\ell}|\mathcal{F}_t] = W_{t+1} + (A_{t+\ell} + B_{t+\ell})\tilde{h}_{t+\ell-1}$,

we have $\dfrac{\partial h_t(\ell)}{\partial \eta_t} = (A_{t+\ell} + B_{t+\ell})\dfrac{\partial h_t(\ell-1)}{\partial \eta_t}$.

By the same reason as the argument in the proof of Theorem 2, we use the estimates $\hat{A}_{t+j}$, $\hat{B}_{t+j}$ and obtain the result in Theorem 2(a).

(b) If $\hat{A}_{t+j} + \hat{B}_{t+j} \equiv \hat{P}_{1,t+j} \triangle_1 + \hat{P}_{2,t+j}\triangle_2 < 1$, we have
$\lambda_\ell = \prod_{j=1}^{\ell-1}(\hat{A}_{t+j+1} + \hat{B}_{t+j+1})A_{t+1} \longrightarrow 0$ when $\ell \to \infty$.

(c) If $\hat{A}_{t+j} + \hat{B}_{t+j} \equiv \tilde{P}_{1,t+j} \triangle_1 + \hat{P}_{2,t+j}\triangle_2 > 1$, we have
$\lambda_\ell = \prod_{j=1}^{\ell-1}(\hat{A}_{t+j+1} + \hat{B}_{t+j+1})A_{t+1} \longrightarrow \infty$ when $\ell \to \infty$.

Theorem 2(b)(c) are clear by Theorem 2(a).

*A.5 Proof of Theorem 3*

The proof is technically the same as one in the Proof of Theorem 2.2 of Hong and Hwang (2016) and so the proof is omitted.

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