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On some Inferences of Lévy Distribution

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Abstract. The Lévy distribution is one of the three stable distributions that has probability density function in simple closed form. This distribution is used in modeling stock prices. In this paper, we present some properties of this distribution. Based on the basic properties some characterizations of this distribution are given.

Résumé (Abstract in French) La loi de probabilité Lévy figurent parmi les lois stables ayant un expression explicite de la densité de probabilité. Elle est souvent utilisée pour modéliser le prix des actions en Finance. Dans ce papier, nous présentos quelques de ses proprietes à partir desquelles des charactérisations sont données.

Key words: moments; characterizations; stable distribution.
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1. Introduction

Paul Lévy(1921) introduced the notion of the stable distribution.

A non degenerate distribution is stable distribution if it satisfies the following property:

Let X_1 and X_2 be independent copies of a random variable X. Then X is said to be stable if for any constants $a > 0$ and $b > 0$ the random variable $aX_1 + bX_2$ has the same distribution as $cX + d$ for some constants $c > 0$ and $-\infty < d < \infty$. The distribution is said to be strictly stable if this definition holds with $d = 0$.

The Lévy distribution is strictly stable with $a = b = 1/4$, $c = 1$ and $d = 0$. The Lévy distribution is widely used as models in many types of physical and economic problems.

The probability distribution density function (*pdf*) $f_{(0,1)}(x)$ of the standard Lévy $L(0,1)$-distribution is given as

$$f_{(0,1)}(x) = \frac{1}{\sqrt{2\pi}x^{3/2}} e^{-\frac{1}{2x}}, x > 0. \quad (1)$$

More generally, the Lévy distribution associated with the location parameter $\mu \in \mathbb{R}$ and the scale parameter $0 < \sigma < +\infty$ is the distribution with *pdf* as

$$f_{(\mu,\sigma)}(x) = \sqrt{\frac{\sigma}{2\pi}} \frac{1}{(x-\mu)^{3/2}} \exp\left(-\frac{\sigma}{2(x-\mu)}\right), \quad -\infty < \mu < x < +\infty. \quad (2)$$

The corresponding cumulative distribution function (**cdf**) $F_{(\mu,\sigma)}(x)$ is given by the equality

$$F_{(\mu,\sigma)}(x) = 2\left(1 - \Phi\left(\sqrt{\frac{\sigma}{(x-\mu)}}\right)\right), \quad -\infty < \mu < x < +\infty, \ \sigma > 0.$$

where $\Phi(x)$ is distribution function of the standard normal distribution. When $\mu = 0$ and $\sigma = 1$, $L(0,1)$ is called the standard Lévy distribution. The standard Lévy distribution satisfies the condition : for any n independent copies of $X \sim L(0,1)$ ($n > 1$), defined on the same probability space, we have

$$(X_1 + X_2 + ... + X_n)/n^2 =^d X,$$

where $=^d$ denotes the equality in distribution. The Lévy distribution belongs to the class of infinitely divisible distributions. If the random variable X has Lévy distribution then $X^{-1/2}$ is distributed as the standard half normal distribution.

O'Reilly and Rueda (1998) considered the problem of testing the fit of the Lévy distribution with an unknown scale parameter. Ahsanullah and Nevzorov (2014, 2019) presented some characterizations based on the linear functions of two independent copies of the random variable X.

Hamedani et *al.* (2015) gave some characterizations based on the sub-independence of the random variables, on a simple relationship between two truncated moments and on the conditional expectation of a certain function of the random variables.

In this paper, we present some basic properties and several characterizations of Lévy distribution by the truncated negative moments and we give a generalization of the results of Ahsanullah and Nevzorov (2014, 2019)

2. Main results

The *pdf*'s $f_{(\mu,\sigma)}(x)$ of the Lévy distribution are given for some selected values of the parameters μ and σ as given in Fig. 1.

The survival function $SR(\mu,\sigma,x)$ is given by

$$SR(\mu,\sigma,x) = \textit{erf}\left(\sqrt{\frac{\sigma}{2(x-\mu)}}\right)$$

where the error function

$$\textit{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \; x > 0.$$

The survival functions for $L(\mu,\sigma)$ are given in Fig. 2 for selected values of μ and σ.

The m-th moment exists if $m < 0.5$. It appears for any $m = 1, 2, ...$ that

$$\mathbb{E}\left(X^{-m}\right) = \int_0^\infty \frac{1}{\sqrt{2\pi}} \times \frac{1}{x^{m+3/2}} \exp\left(-1/2x\right)dx \quad (3)$$

$$= (2m-1)\int \frac{1}{\sqrt{2\pi}}\frac{1}{x^{m+1/2}} \exp\left(-1/2x\right)dx \quad (4)$$

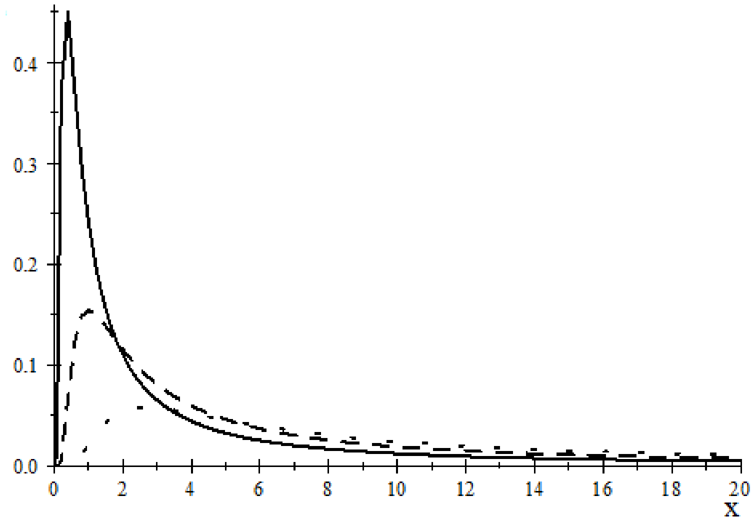

Fig. 1. *Pdf*'s of Lévy distributions: $f_{(0,1)}$ is solid, $f_{(9,3)}$ is dash

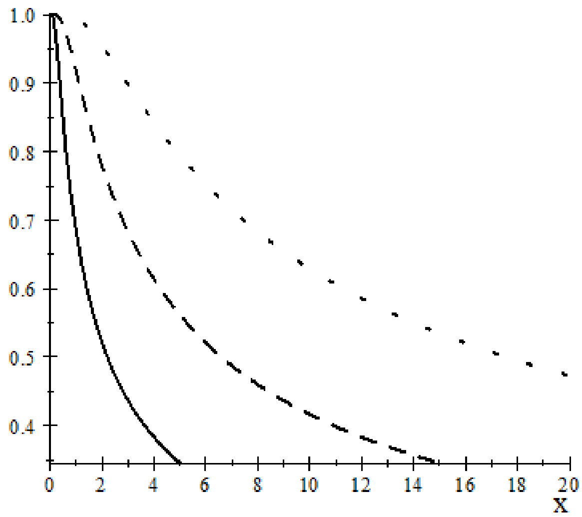

Fig. 2. Survival functions for $f_{(0,\sigma)}$: $f_{(0,1)}$ is solid, $f_{(0,3)}$ is dash, $f_{(0,8)}$ is dots

$$
\begin{aligned}
&= (2m-1)\,E\left(X^{-(m-1)}\right) \\
&= \cdots \\
&= \Pi_{k=1}^{m}\,(2k-1) \\
&= (2m-1)!!,
\end{aligned}
$$

where we used the notation, for any integer $m \geq 1$,

$$(2m+1)!! = \prod_{0=1}^{m}(2j+1) \quad and \quad (2m)!! = \prod_{j=1}^{m}(2j).$$

For $m = 0, 1, 2, \cdots$, one can get that

$$\begin{aligned} \mathbb{E}\left(X^{-(m+1/2)}\right) &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \times \frac{1}{x^{m+2}} \exp\left(-1/2x\right) dx \\ &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \times 2^{m+1} \exp\left(-x\right) x^m dx \\ &= \frac{2^{m+1/2}}{\sqrt{\pi}} \Gamma\left(m+1\right) \\ &= \sqrt{\frac{2}{\pi}}\,(2m)!! \end{aligned} \tag{5}$$

If X and Y are independently distributed as $L(\mu, \sigma_1)$ and $L(\mu, \sigma_2)$ then

$$\mathbb{P}(X < Y) = \frac{2}{\pi} \arcsin\left(\frac{\sigma_1}{\sigma_1 + \sigma_2}\right).$$

If $\sigma_1 = \sigma_2$ then $\mathbb{P}(X < Y) = \frac{1}{2}$

Let $X(n,n) = \max(X_1, X_2, ..., X_n)$ and $X(1,n) = \min(X_1, X_2, ..., X_n)$.

It is shown in Ahsanullah and Nevzorov (2014) that the distribution of $X_{1,n}$ belongs to the domain of attraction of type 1 extreme distributions of minimum while $X_{n,n}$ belongs to the domain of attraction a type 2 distribution of maximum (Frechet domain).

Ahsanullah and Nevzorov (2014) proved the following theorem.

Theorem 1. *Let X_1, X_2, X_3 be independent identically distributed absolutely continuous random variables with cdf $F(x)$and pdf $f(x)$. We assume $F(0) = 0$ and $F(x) > 0$ for all $x > 0$.*

Then X_1 and $(X_2 + X_3)/4$ are identically distributed if and only if $F(x)$ has the $L(0, \sigma)$ distribution.

The following theorem gives a generalization of Theorem 1.

Theorem 2. *Suppose that $X,\ X_1,\ X_2,\ \cdots,\ X_n$ are independent and have some absolutely continuous distribution function $F(x)$. We also assume that $F(0)=0$ and $F(x)>0$ for all $x>0$. Then X and $(X_1+X_2+...X_n)/n^2$ are identically distributed if and only if their pdf $f(x)$ is given as*

$$f(x)=\frac{1}{\sqrt{2\pi}}\frac{1}{x^{3/2}}e^{-\frac{1}{2}x},\ x\geq 0.$$

Proof. We will prove here only if condition. Let the characteristic function of X be $\phi(t)$. The condition that X and $(X_1+X_2+...X_n)/n^2$ are identically distributed implies that

$$\phi(t)=\left(\phi\left(\frac{t}{n^2}\right)\right)^n. \tag{6}$$

Denote $\Psi(t)=(\ln\phi(t))^2$, then we obtain from Equation (6) that

$$\begin{aligned}\Psi(t) &= n^2\Psi\left(\frac{t}{n^2}\right) &(7)\\ &= \cdots \\ &= n^{2k}\Psi\left(\frac{t}{n^{2k}}\right), k=1,2,\cdots, &(8)\end{aligned}$$

The solution of Equations (8) is given as

$$\Psi(t)=ct,$$

where c is any constant. Thus, we can express $\phi(t)$ as

$$\phi(t)=e^{-\sqrt{ct}},$$

where c is some constant.

Note that $\phi(-t)$ is the complex conjugate of $\phi(t)$. Using that $\phi(-t)$ is the complex conjugate of $\phi(t)$, we can take $c=-2i$

$$\phi(t)=e^{-\sqrt{2it}}$$

which is the characteristic function of the standard Lévy distribution.

Ahsanullah and Nevzorov (2019) proved the following characterization theorem.

Theorem 3. *Suppose that X is an absolutely continuous random variables with cdf $F(x)$ such that $F(0) = 0$ and $F(x) > 0$ for all $x > 0$.Let X_1 and X_2 be two independent copies of X and α_1 and α_2 be two positive numbers.*

Then X and the sum $\frac{\alpha_1^2}{(\alpha_1+\alpha_2)^2}X_1 + \frac{\alpha_2^2}{(\alpha_1+\alpha_2)^2}X_2$ are identically distributed if and only if X has the Lévy distribution with pdf $f(x)$ as

$$f(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{x^{3/2}}e^{-\frac{1}{2}x}, x \geq 0$$

The following theorem is a generalization of Theorem 3.

Theorem 4. *Suppose that X is an absolutely continuous random variables with cdf $F(x)$ such that $F(0) = 0$ and $F(x) > 0$ for all $x > 0$. Let X_1,X_2, ..., X_n be n independent copies of X and let $\alpha_1, \alpha_2, ..., \alpha_n$ be n non negative numbers such at least two of them are not zero. In this situation , X and the sum*

$$\frac{\alpha_1^2}{(\alpha_1+\alpha_2+\cdots+\alpha_n)^2}X_1 + \frac{\alpha_2^2}{(\alpha_1+\alpha_2+\cdots+\alpha_n)^2}X_2 + ... + \frac{\alpha_n^2}{(\alpha_1+\alpha_2+\cdots+\alpha_n)^2}X_n$$

are identically distributed if and only if X has the standard Lévy distribution.

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