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Adaptive Hyperbolic Asymmetric Power ARCH (A-HY-APARCH) model: Stability and Estimation

Charline Uwilingiyimana (1,*) **, Abdou Kâ Diongue** (2)**, Carlos Ogouyandjou** (3)

(1) Institut de Mathématiques et de Sciences Physiques de l'Université d'Abomey-Calavi, BP 613, Porto-Novo, Bénin
(2) Laboratoire d'Etudes et de Recherches en Statistiques et Développement (LERSTAD), UFR de sciences Appliquées et de Technologies, B.P 234, Université Gaston Berger, Saint-Louis, Sénégal
(3) Institut de Mathématiques et de Sciences Physiques de l'Université d'Abomey-Calavi, BP 613, Porto-Novo, Bénin

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Abstract. In this paper, a new asymmetric GARCH type model that generalizes the Hyperbolic Asymmetric Power ARCH (HY-APARCH) process is proposed. The proposed model takes into consideration some characteristics of financial time series data like volatility clustering, long memory and structural changes. The necessary and sufficient conditions for the asymptotic stability of the model are derived and parameter estimation methods are proposed. The Monte Carlo Simulations are done to prove the performance of the estimation method.

Key words: long range dependence; structural changes; HYAPARCH.
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* Corresponding author: Charline Uwilingiyimana (u.charline@gmail.com)
Abdou Kâ Diongue : abdou.diongue@ugb.edu.sn
Carlos Ogouyandjou: ogouyandjou@imsp-uac.org

Résumé. (Abstract in French) Nous proposons un modèle GARCH symmétrique qui généralise le modèle asymétrique hyperbolique Power ARCH. Notre modèle prend en compte des caractéristiques de séries chronologiques financières telles que la volatilité du grappage, les longues mémoires et les changement structurels. Nous obtenons des conditions nécessaires et suffisantes de statilité asymptotiques et procédons à une estimation paramétrique pour valider le modèle. Une étude de simulation vient en appui aux résultats théoriques.

The authors.

Charline Uwilingiyimana, M.Sc., is preparing a Ph.D, under the supervision of the second author at: Institut de Mathématiques et de Sciences Physiques de l'Université d'Abomey-Calavi.

Abdou Kâ Diongue, Ph.D., is full professor at the Department of Applied Mathematics of Gaston Berger University, Saint-Louis, SENEGAL

Carlos Ogouyandjou, Ph.D., is associate professor at: Institut de Mathématiques et de Sciences Physiques de l'Université d'Abomey-Calavi.

1. Introduction

The Autoregressive conditional heteroscedasticity of order p, or ARCH(p), was first introduced by Engle, R. F. (1982) and generalized by Bollerslev, T. (1986) to model the conditional variance or the volatility in financial time series with a heavy tail. Based on their work several models have been developed to capture the main features of financial time series data like leptokurtic, skewness, and Volatility clustering. In past years, there has been an increasing interest in introducing models that have been dealing with the asymmetric effect due to negative and positive shocks on the conditional variance like the EGARCH proposed by Nelson, D. B. (1991), the APARCH of Ding, Z.*et al.*(1993) and the GJR GARCH proposed by Glosten, L. R. *et al.*(1993). Baillie, R. T.*et al.*(1996) extend GARCH model by introducing FIGARCH model to capture long memory features in assets returns. Tse, Y. K. (1998) has introduced an asymmetric long memory volatility model as an extension of FIAPARCH model by adding a function extracted from the APARCH process.

Engle, R. F.*et al.*(2004) found that researchers should take into account other important characteristics like leverage effect and long memory. However, more studies have confirmed the existence of the long memory persistence and the asymmetric effects in the volatility of financial returns (see for instance, Davidson, J. (2004), Conrad, C. (2010), Pérez-Rodríguez, J. V.*et al.*(2019)). The Hyperbolic Asymmetric Power ARCH (HY-APARCH) process was proposed by Schoffer, O. (2003) and generalized by Diongue, A. K.*et al.*(2007) by proposing seasonal Hyperbolic APARCH that take into consideration the long memory and asymmetries in volatility. Dark, J. G. (2010) used HY-APARCH to model the conditional skewness and kurtosis, and this model has failed to capture the structural changes known as a structural break when a time series abruptly changes at a point in time. It is well known that the persistence shocks to the conditional variance is measured by the long range dependence parameter, Diebold, F. X. *et al.*(2001) in their work concluded that structural changes may produce superious long memory effects. Hillebrand, E. (2005) found that ignoring structural changes in volatility may cause the sum of the estimated autoregressive to converge to one.

Recently, some studies have defined the new processes that accounted for structural changes through the Fourier approximation see Baillie, R. T.*et al.*(2009), Li, J *et al.*(2017), Shi, Y. *et al.*(2018). Another study by Choi, K.*et al.*(2010) examined the structural changes in the daily exchange rate and showed that the long memory is due to structural changes so it is difficult to distinguish long memory and structural changes as proved by studies. It has been demonstrated that structural changes can be partly described some extremely persistent volatility models, and may also cause a time series to have long memory characteristic, this has supported by Messow, P. *et al.*(2013) in the study which reveals that structural change is related to long memory and can lead to overestimated the long-range dependence parameter.

Motivated by the studies that examined the existence of both long memory and structural changes, this study introduces a new framework that combines an HY-GARCH structure for returns with an APARCH model and with a time-varying determistic function intercept. The model within our framework is called an Adaptive Hyperbolic Asymmetric Power ARCH (A-HY-APARCH) process. This model extends the HY-APARCH model and it is designed for modeling both long memory and structural changes. This new model generalizes the HY-APARCH of Schoffer, O. (2003) when $w_t = \frac{w_o}{1-\beta(1)}$ and the Seasonal HY-APARCH of Diongue, A. K.*et al.*(2007) when $w_t = \frac{w_o}{1-\beta(1)}$ and $s = 1$. The necessary and sufficient conditions for the stability of the second moment are investigated following the method of Mohammadi, F. *et al.*(2017). The asymptotic behaviours of the Maximum Likelihood Estimations are evaluated by Monte Carlo.
This paper is organized as follows: Section 2 presents the A-HY-APARCH process. Section 3 derives the condition for the second moment to be asymptotically bounded. Section 4 presents the Estimation of the parameters. Section 5 is dedicated to simulation studies and concluding remarks are given in Section 6.

2. The model

2.1. HY-APARCH model

The HYGARCH process of Davidson, J. (2004) has failed to capture the asymmetric effects observed in time series data, to overcome this problem, the HY-APARCH model has been introduced and it combined the characteristics of the HYGARCH model with those of the APARCH model. Assume z_t is a sequence of independent identically distributed random variable (i.i.d) with $\mathbb{E}(z_t) = 0$ and $\mathrm{Var}(z_t) = 1$. The process $(x_t)_{t\in\mathbb{Z}}$ is a HY-APARCH(p,d,q) process if it satisfies, for all t and some strictly positive valued process h_t, the following equations

$$x_t = h_t z_t$$

$$[1-\alpha(B)-\beta(B)]\Big((1-\tau)+\tau(1-B)^d\Big)y_t = w_0 + (1-\beta(B))(v_t), \tag{1}$$

where $\alpha(B) = \sum_{j=0}^{p}\alpha_j B^j$, $\beta(B) = \sum_{j=0}^{q}\beta_j B^j$, $y_t = \Big(|x_t| - \epsilon x_t\Big)^{\phi}$,
and $v_t = y_t - h_t^{\phi}$, for $t \in \mathbb{Z}$, $\phi > 0$, $|\epsilon| < 1$, $w_0 > 0$, $0 < d < 1$ and $\tau \geq 0$.
By rearranging the equation (1), the HYAPARCH(p,d,q) can be written as the following equation:

$$h_t^{\phi} = \tfrac{w_0}{1-\beta(B)} + \Big\{1 - \tfrac{1-\Phi(B)}{1-\beta(B)}\Big((1-\tau)+\tau(1-B)^d\Big)\Big\}\Big(|x_t| - \epsilon x_t\Big)^{\phi}. \tag{2}$$

where $\Phi(B) = 1-\alpha(B)-\beta(B)$
The HYAPARCH reduce to FIGARCH when $\epsilon = 0$, $\phi = 2$ and $\tau = 1$; to HYGARCH model for $\phi = 2$ and $\epsilon = 0$ and to FIAPARCH model for $\tau = 1$. Consider the conditonal variance of the HYAPARCH(p,d,q) model in (2), following Li, M.*et al.*(2015), the process HYAPARCH(1,d,1) can be represented by the following equations:

$$h_t^{\phi} = (1-\tau)h_{1,t}^{\phi} + \tau h_{2,t}^{\phi} \tag{3}$$

where

$$h_{1,t}^{\phi} = \frac{w_0}{1-\beta(1)} + \left(1 - \frac{1-\Phi B}{1-\beta B}\right) y_t$$

and

$$h_{2,t}^{\phi} = \frac{w_0}{1-\beta(1)} + \left(1 - \frac{1-\Phi B}{1-\beta B}(1-B)^d\right) y_t$$

2.2. A-HY-APARCH model

Adaptive Hyperbolic Power Asymmetric ARCH (A-HY-APARCH) model proposed in this study is designed for modeling long memory, asymmetric effects, and structural changes in the conditional variance process. The modification in the HY-APARCH consists of replacing the intercept $\frac{w_0}{1-\beta(1)}$ by a slowly varying deterministics function w_t in equation (2), developed by Gallant, A. R. (1984)'s flexible functional form that models structural changes. There are other non-linear models such as logistic smooth transition Autoregressive and Markov switching GARCH types model that can capture smooth breaks as well but the logistic smooth transition requires that the number of breaks should be known. The advantage of the time-varying intercept is that we do need to assume the number of breaks is to be known and its main purpose is to allow the structural change in the conditional variance.

The A-HY-APARCH(p,d,q,k) extends the HY-APARCH(p,d,q) process by allowing the intercept w_t in the conditional variance equation to be time slowly varying function, but this new process does not allow all the parameters in the conditional variance equation of the HY-APARCH model to be time-dependent. Note that a constant intercept in the HY-APARCH model is replaced by equation (2) as demonstrated by the studies of Baillie, R. T.*et al.*(2009), Pascalau, R. *et al.*(2011), Nasr, A. B.*et al.*(2010) and recently shown by Shi, Y. *et al.*(2018). Let x_t denote a real-valued discrete-time stochastic process and assume z_t is a sequence of independent identically distributed random variable (i.i.d) with $\mathbb{E}(z_t) = 0$ and $\mathrm{Var}(z_t) = 1$. The conditional variance of the A-HY-APARCH(p,d,q,k) is defined as the following equations:

$$x_t = h_t z_t$$

$$\begin{aligned} h_t^{\phi} &= w_t + \left\{1 - \frac{\Phi(B)\Big((1-\tau) + \tau(1-B)^d\Big)}{\beta(B)}\right\}\Big(|x_t| - \epsilon x_t\Big)^{\phi} \\ &= w_t + \Psi(B) y_t \end{aligned}$$

where

$$w_t = w_0 + \sum_{j=1}^{k}\left[n_j \sin(\tfrac{2\pi jt}{T}) + m_j \cos(\tfrac{2\pi jt}{T})\right]$$

and $y_t = \Big(|x_t| - \epsilon x_t\Big)^{\phi}$, subject to the following constraints:

$$\phi > 0, \quad 0 < d < 1, \quad -1 < \epsilon < 1, \quad w_t > 0$$

$$\beta(B) = 1 - \beta_1 B - \beta_2 B^2 - \cdots - \beta_q B^q$$

$$\Phi(B) = 1 - \alpha(B) - \beta(B)$$

and

$$\alpha(B) = 1 - \alpha_1 B - \alpha_2 B^2 - \cdots - \alpha_p B^p$$

provided that the roots of the characteristics polynomials $\Phi(B)$ and $\beta(B)$ lie inside or outside the unit circle.
The A-HYAPARCH reduce to the existing HY-APARCH(p,d,q)
when $w_t = \frac{w_0}{1-\beta(1)}$, or when $n_j = m_j = 0$. The A-HYAPARCH(1,d,1,k) is of the form:

$$h_t^{\phi} = w_t + \left\{1 - \tfrac{1-\Phi B}{1-\beta B}\Big((1-\tau) + \tau(1-B)^d\Big)\right\} y_t. \tag{4}$$

Using the equation (3), the equation (4) can be represented as follows:

$$h_t^{\phi} = (1-\tau)h_{1,t}^{\phi} + \tau h_{2,t}^{\phi} \tag{5}$$

where

$$h_{1,t}^{\phi} = w_t + \left(1 - \frac{1-\Phi B}{1-\beta B}\right) y_t$$

and

$$h_{2,t}^{\phi} = w_t + \left(1 - \frac{1-\Phi B}{1-\beta B}(1-B)^d\right) y_t.$$

It is well known that the hyperbolic memory of the model has the following representation

$$(1-B)^d = 1 - \sum_{i=1}^{\infty} \pi_i B^i \tag{6}$$

where

$$\pi_i = \frac{d\Gamma(i-d)}{\Gamma(1-d)\Gamma(i+1)} = \frac{d}{\Gamma(i-d)} i^{-1-d}.$$

Using equations (5),(6) and by employing the methodology of Mohammadi, F. *et al.*(2017), the A-HYAPARCH(1,d,1,k) model can be rewritten as follows:

$$x_t = h_t z_t$$

$$h_t^{\phi} = (1-\tau)h_{1,t}^{\phi} + \tau h_{2,t}^{\phi}, \tag{7}$$

where

$$h_{1,t}^{\phi} = w_t + \beta\left(h_{1,t-1}^{\phi} - w_{t-1}\right) + (\Phi - \beta) y_{t-1}, \tag{8}$$

$$h_{2,t}^{\phi} = w_t + \beta\left(h_{2,t-1}^{\phi} - w_{t-1}\right) + (\beta - \Phi + \pi_1) y_{t-1} + \sum_{i=0}^{\infty}\left(\pi_{i+2} - \Phi\pi_{i+1}\right) B^i y_{t-2}, \tag{9}$$

and

$$w_t = w_0 + \sum_{j=1}^{k}\left[n_j \sin\left(\frac{2\pi j t}{T}\right) + m_j \cos\left(\frac{2\pi j t}{T}\right)\right], \tag{10}$$

where z_t are iid standard normal variables. To guarantee the positivity of the conditional variance, the condition to the constrants are imposed: $\beta, (\beta - \Phi) > 0$, $0 < \tau < \Phi < \beta < d < 1$ and w_t is a time varying function bounded between 0 and c_0.

3. Stability

In this section the stability of the the A-HY-APARCH model which point out to the asymptotic finiteness of the variance of the series can be imposed by regarding some conditions to ensure the asymptotic boundedness of unconditional second moment. stability results are derived by considering the time varying function intercept $w_t \in [0, c_0]$.

Lemma 1. *If n_j and m_j are the nonnegative numbers with $j \in \{1, 2, \ldots, k\}$ such that $\sum_{j=1}^{k}(n_j + m_j) < \min(1, w_0)$ then*

$$0 \leqslant w_0 + \sum_{j=1}^{k}\Big(n_j \sin(\frac{2\pi jt}{T}) + m_j \cos(\frac{2\pi jt}{T})\Big) \leqslant 1 + w_0 := c_0.$$

Proof. We want to show that there exists a constant c_0 such that for all $t > 0$, $w_0 > 0$,

$$0 \leqslant w_t \leqslant 1 + w_0 := c_0$$

where $w_t = \sum_{j=1}^{k}\left(n_j \sin(\frac{2\pi jt}{T}) + m_j \cos(\frac{2\pi jt}{T})\right)$. Knowing that $-1 \leqslant \cos(\frac{2\pi jt}{T}) \leqslant 1$ and $-1 \leqslant \sin(\frac{2\pi jt}{T}) \leqslant 1$ hence

$$-\sum_{j=1}^{k}(n_j + m_j) \leqslant \sum_{j=1}^{k}\Big(n_j \sin(\frac{2\pi jt}{T}) + m_j \cos(\frac{2\pi jt}{T})\Big) \leqslant \sum_{j=1}^{k}(n_j + m_j)$$

and $\sum_{j=1}^{k}(n_j + m_j) \leqslant \min(1, w_0) \leqslant 1$, therefore

$$w_0 - \sum_{j=1}^{k}(n_j + m_j) \leqslant w_0 + \sum_{j=1}^{k}\Big(n_j \sin(\frac{2\pi jt}{T}) + m_j \cos(\frac{2\pi jt}{T})\Big) \leqslant 1 + w_0.$$

Using the assumption that $\sum_{j=1}^{k}(n_j + m_j) \leqslant \min(1, w_0)$, the $\sum_{j=1}^{k}(n_j + m_j) \leqslant w_0$ and $w_0 - \sum_{j=1}^{k}(n_j + m_j) \geqslant -w_0 + w_0 = 0$.
We conclude that $0 \leqslant w_t \leqslant 1 + w_0 := c_0$.

Lemma 2. *Let $\Big(\mathbb{V}, \|.\|\Big)$ be a normed space such that, $\mathbb{V} = \left\{(y_t)_{t\in\mathbb{Z}} \,/\, \sup_{t\in\mathbb{Z}} \mathbb{E}|y_t| < \infty\right\}$ and let B be a linear operator on $\mathbb{V}$ defined by*

$$\begin{aligned} B \colon \mathbb{V} &\to \mathbb{V} \\ y &\mapsto By = (By_t)_{t\in\mathbb{Z}} = (y_{t-1})_{t\in\mathbb{Z}} \end{aligned}$$

and $\|B\|_\infty = \sup_{t\in\mathbb{Z}} \mathbb{E}|y_t|$, then the delayed operator,

$$\|B^i\|_\infty = 1, \quad \forall i \in \mathbb{N}$$

Proof.

$$\begin{aligned} \|B\| &= \sup_{y\in\mathbb{V}} \frac{\sup_{t\in\mathbb{Z}} \mathbb{E}(|By_t|)}{\sup_{t\in\mathbb{Z}} \mathbb{E}(|y_t|)}. \\ &= \sup_{y\in\mathbb{V}} \frac{\sup_{t\in\mathbb{Z}} \mathbb{E}(|y_{t-1}|)}{\sup_{t\in\mathbb{Z}} \mathbb{E}(|y_t|)}. \end{aligned}$$

Since $\sup_{t\in\mathbb{Z}} \mathbb{E}(|y_{t-1}|) = \sup_{t\in\mathbb{Z}} \mathbb{E}(|y_t|)$, therefore $\|B^i\|_\infty = sup_{y_t\in\mathbb{V}}(1) = 1$.

For the rest of this section we consider n_j, m_j and w_0 are such that,

$$0 \leqslant w_t \leqslant c_0.$$

Theorem 1. *The conditional variance h_t of A-HY-APARCH model satisfies, for all $\delta > 0$,*

$$\begin{aligned}\mathbb{E}(h_t^{\delta}) &\leqslant c_0 + \beta\mathbb{E}(h_{1,t-1}^{\delta}) + C(\beta + \tau\pi_1)\mathbb{E}(h_{t-1}^{\delta})\\ &+ \tau\beta\mathbb{E}(h_{2,t-1}^{\delta}) + C\tau\sum_{i=0}^{\infty}\pi_{i+2}\mathbb{E}(h_{t-2}^{\delta})\end{aligned}$$

where c_0, and C are some constants.

Proof. Under the assumption that the distribution of z_t is symmetric about zero, the leverage effect is still taken into consideration and the second moment of A-HY-APARCH model is computed as follows:

$$\mathbb{E}(y_t^2) = \mathbb{E}(|h_t z_t| - \epsilon h_t z_t)^{2\phi} = C\mathbb{E}(h_t^{2\phi})$$

with $C := \mathbb{E}(|z_t| - \epsilon z_t)^{2\phi}$. Since z_t are iid, $\mathbb{E}(|z_t| - \epsilon z_t)^{2\phi}$ is a constant depending on the distribution of z_t, the leverage parameter ϵ and the parameter ϕ . This constant is specified for the Gaussian distribution, the Generalized Error Distribution (GED) and for standardized Student t distribution with mean zero, variance one and degree of freedom ν (where $\nu \in \mathbb{N} \setminus [0, 2]$), see Ding, Z.*et al.*(1993), Laurent, S.*et al.*(2002), Lambert, P. *et al.*(2001), respectively.

Let $\delta = 2\phi$, using the equations (8),(9) and (10), the expectation of h_t^{δ} in (7) is given by:

$$\begin{aligned}\mathbb{E}(h_t^{\delta}) &= \mathbb{E}(w_t) + \beta\mathbb{E}(h_{1,t-1}^{\delta}) - \beta\mathbb{E}(w_{t-1}) + (\beta - \Phi)\mathbb{E}(Y_{t-1}^2) - \tau\mathbb{E}(w_t) - \tau\beta\mathbb{E}(h_{1,t-1}^{\delta})\\ &+\tau\beta\mathbb{E}(w_{t-1}) - \tau(\beta - \Phi)\mathbb{E}(y_{t-1}^2) + \tau\mathbb{E}(w_t) + \tau\beta\mathbb{E}(h_{2,t-1}^{\delta}) - \tau\beta\mathbb{E}(w_{t-1})\\ &+\tau(\Phi - \beta + \pi_1)\mathbb{E}(y_{t-1}^2) + \tau\sum_{i=0}^{\infty}(\pi_{i+2} - \Phi\pi_{i+1})B^i\mathbb{E}(y_{t-2}^2).\end{aligned} \tag{11}$$

From equation (11), we get

$$\begin{aligned}\mathbb{E}(h_t^{\delta}) &= \mathbb{E}(w_t) + (-\beta)\mathbb{E}(w_{t-1}) + (\beta - \tau\beta)\mathbb{E}(h_{1,t-1}^{\delta})\\ &+ \Big((\beta - \Phi) - \tau(\beta - \Phi) + \tau(\Phi - \beta + \pi_1)\Big)\mathbb{E}(y_{t-1}^2)\\ &+ \tau\beta\mathbb{E}(h_{2,t-1}^{\delta}) + \tau\sum_{i=0}^{\infty}(\pi_{i+2} - \Phi\pi_{i+1})B^i\mathbb{E}(y_{t-2}^2).\end{aligned} \tag{12}$$

Let

$$k_1 = \beta - \tau\beta, \quad k_2 = (1-\tau)(\beta - \Phi) + \tau(\Phi - \beta + \pi_1)$$

and

$$k_3 = \tau\beta.$$

The equation (12) becomes

$$\mathbb{E}(h_t^\delta) = \mathbb{E}(w_t) - \beta\mathbb{E}(w_{t-1}) + k_1\mathbb{E}(h_{1,t-1}^\delta) + k_2\mathbb{E}(y_{t-1}^2) + k_3\mathbb{E}(h_{2,t-1}^\delta) + \tau\sum_{i=0}^{\infty}(\pi_{i+2} - \Phi\pi_{i+1})B^i\mathbb{E}(y_{t-2}^2). \tag{13}$$

Since $\mathbb{E}(y_{t-1}^2) = C\mathbb{E}(h_{t-1}^\delta)$, using Lemma 1 and Lemma 2 , an upper bound of (13) are calculated as follows:

$$\begin{aligned}
&\mathbb{E}(w_t) \leqslant c_0 \\
&k_1\mathbb{E}(h_{1,t-1}^\delta) \leqslant |k_1|\mathbb{E}(h_{1,t-1}^\delta) \leqslant \beta\mathbb{E}(h_{1,t-1}^\delta) \\
&k_2\mathbb{E}(y_{t-1}^2 w_t) \leqslant C|k_2|\mathbb{E}(h_{t-1}^\delta) \leqslant C(\beta + \tau\pi_1)\mathbb{E}(h_{t-1}^\delta) \\
&k_3\mathbb{E}(h_{2,t-1}^\delta w_t) \leqslant |k_3|\mathbb{E}(h_{2,t-1}^\delta) \leqslant \tau\beta\mathbb{E}(h_{2,t-1}^\delta) \\
&\tau\sum_{i=0}^{\infty}(\pi_{i+2} - \Phi\pi_{i+1})B^i\mathbb{E}(y_{t-2}^2) \leqslant C\tau\sum_{i=0}^{\infty}\pi_{i+2}\mathbb{E}(h_{t-2}^\delta)
\end{aligned} \tag{14}$$

By substituting the obtained results above in (13), therefore the $\mathbb{E}(h_t^\delta)$ is attained

$$\mathbb{E}(h_t^\delta) \leqslant c_0 + \beta\mathbb{E}(h_{1,t-1}^\delta) + C(\beta + \tau\pi_1)\mathbb{E}(h_{t-1}^\delta) + \tau\beta\mathbb{E}(h_{2,t-1}^\delta) + C\tau\sum_{i=0}^{\infty}\pi_{i+2}\mathbb{E}(h_{t-2}^\delta) \tag{15}$$

Consider the A-HY-APARCH process given by the the following relations

$$\mathbb{E}(h_t^\delta) \leqslant c_0 + \beta\mathbb{E}(h_{1,t-1}^\delta) + C(\beta + \tau\pi_1)\mathbb{E}(h_{t-1}^\delta) + \tau\beta\mathbb{E}(h_{2,t-1}^\delta) + C\tau\sum_{i=0}^{\infty}\pi_{i+2}\mathbb{E}(h_{t-2}^\delta) \tag{16}$$

$$\mathbb{E}(h_{1,t}^\delta) \leqslant c_0 + \beta\mathbb{E}(h_{1,t-1}^\delta) + C\Phi\mathbb{E}(h_{t-1}^\delta) \tag{17}$$

$$\mathbb{E}(h_{2,t}^\delta) \leqslant c_0 + C(\Phi + \pi)\mathbb{E}(h_{t-1}^\delta) + \beta\mathbb{E}(h_{2,t-1}^\delta) + C\sum_{i=0}^{\infty}\pi_{i+2}\mathbb{E}(h_{t-2}^\delta) \tag{18}$$

Consider the inequalities (16), (17), (18), that can be written in matrix form as

$$H_t \leq M + AH_{t-1},$$

the iterative method is expressed as

$$H_t \leq \sum_{i=0}^{t-1} A^i M + A^t H_0,$$

that converges for each initial H_{-1} if and only if the spectral radius of matrix A is smaller than one. The matrices H_t, M and A are defined as follows :

$$H_t = \begin{pmatrix} \mathbb{E}(h_t^\delta) \\ \mathbb{E}(h_{1,t}^\delta) \\ \mathbb{E}(h_{2,t}^\delta) \\ \mathbb{E}(h_{t-1}^\delta) \end{pmatrix}$$

$$M = \begin{pmatrix} c_0 \\ c_0 \\ c_0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} C(\beta + \tau\pi_1) & \beta & \tau\beta & C\tau\sum_{i=0}^{\infty}\pi_{i+2} \\ C\Phi & \beta & 0 & 0 \\ C(\Phi + \pi_1) & 0 & \beta & C\sum_{i=0}^{\infty}\pi_{i+2} \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Lemma 3. *Let* τ, Φ *and* β *be the parameters of A-HY-APARCH model such that* $0 < \tau < \Phi < \beta < 1$. *If,*

$$\begin{cases} C\left[\beta(1 + (\tau+1)\Phi - \beta) + \tau(\pi_1 + \sum_{i=0}^{\infty}\pi_{i+2})\right] + \beta - 1 < 0 \\ C(\beta + \tau\pi_1) + \beta < 2 \end{cases}$$

then $\rho(A) < 1$.

Proof. Let first show that the spectrum $\Lambda(A)$ is not empty set and its maximum eigenvalue is strictry less than one. Given matrix A

$$A = \begin{pmatrix} C(\beta + \tau\pi_1) & \beta & \tau\beta & C\tau\sum_{i=0}^{\infty}\pi_{i+2} \\ C\Phi & \beta & 0 & 0 \\ C(\Phi + \pi_1) & 0 & \beta & C\sum_{i=0}^{\infty}\pi_{i+2} \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For sake of simplicity, let us rewrite matrix A as : $A = \begin{pmatrix} a & d & e & f \\ b & \beta & 0 & 0 \\ c & 0 & \beta & g \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Note that for $0 < \tau < \phi < \beta < 1$, we have $eg\beta - f\beta^2 = 0$, thus the characteristic polynomial of A is

$$\begin{aligned} P_A(x) &= x^4 - (a + 2\beta)x^3 + (-f - ec + \beta(2a + \beta) - bd)\,x^2 \\ &+ \left(2f\beta - eg + ec\beta - a\beta^2 + bd\beta\right)x \end{aligned}$$

By solving $P_A(x) = 0$, the eigenvalues of matrix A are $x_1 = 0$, $x_2 = \beta$,
$x_3 = \frac{1}{2}\left((a + \beta) - \sqrt{a^2 - 2a\beta + \beta^2 + 4(f + ec + bd)}\right)$
and $x_4 = \frac{1}{2}\left((a + \beta) + \sqrt{a^2 - 2a\beta + \beta^2 + 4(f + ec + bd)}\right)$,
obviously $\left(a^2 - 2a\beta + \beta^2 + 4(f + ec + bd)\right) > 0$.
By definition, the spectral radius of a matrix A is defined by

$$\rho(A) = \sup_{x \in \Lambda(A)} |x|.$$

Since $\max\{0, \beta, |x_3|, |x_4|\} = x_4$ that is,

$$\rho(A) = \frac{1}{2}\left((a+\beta) + \sqrt{a^2 - 2a\beta + \beta^2 + 4(f + ec + bd)}\right).$$

The spectral radius of A is less than one if and only if the following condition is satisfied:

$$\begin{cases} a - a\beta - 1 + \beta + f + ec + bd < 0 \\ \qquad\qquad\qquad a + \beta < 2. \end{cases} \tag{19}$$

We just have to replace a, b, c, d and e by their expressions in (19), where $a = C(\beta + \tau\pi_1)$, $b = C\Phi$, $c = C(\Phi + \pi_1)$, $d = \beta$ and $f = C\tau\sum_{i=0}^{\infty}\pi_{i+2}$ and $e = \tau\beta$, therefore (19) is rewritten as follows

$$\begin{cases} C\left[\beta(1 + (\tau+1)\Phi - \beta) + \tau(\pi_1 + \sum_{i=0}^{\infty}\pi_{i+2})\right] + \beta - 1 & < 0 \\ C(\beta + \tau\pi_1) + \beta & < 2. \end{cases} \tag{20}$$

Theorem 2. *Let τ, Φ and β be the parameters of A-HY-APARCH model such that $0 < \tau < \Phi < \beta < 1$ and*

$$\begin{cases} C\left[\beta(1 + (\tau+1)\Phi - \beta) + \tau(\pi_1 + \sum_{i=0}^{\infty}\pi_{i+2})\right] + \beta - 1 < 0 \\ C(\beta + \tau\pi_1) + \beta < 2, \end{cases}$$

then the time series $\{y_t\}$ following A-HYAPARCH model defined in relation (5)-(8) is asymptotically stable with finite variance.

Proof. The recursive vector form of inequalities (16), (17), (18) can be written as

$$H_t \leq M + AH_{t-1} \qquad t \geqslant 0 \tag{21}$$

The iteration of inequality (21) gives

$$H_t \leq \sum_{i=0}^{t-1} A^i M + A^t H_0 \qquad t \geqslant 0 \tag{22}$$

According to convergence matrice theorem Peter Lancaster, M. T. (1985), the iterating inequality (22) converges if and only if the spectral radius is strictly less than one, by Lemma 3, suppose that the spectral radius $\rho(A) < 1$, Now we want to show that if $(I - A)$ exists, its inverse exists and $\sum_{i=0}^{t-1} A^i$=$(I - A)^{-1}$ as $\lim_{t\to\infty} A^t H_0 = 0$. The eigenvalues of $(I - A)$ are $(1 - x(A))$ where $x(A)$ are the eigenvalues of matrix A. The set of eigenvalues of $(I - A)$ is not empty, hence matrix $(1 - x(A))$ is invertible. Let

$$S_n = I + A + A^2 + \ldots + A^n = \sum_{i=0}^{n-1} A^i$$

$$AS_n = A + A^2 + \ldots + A^{n+1}$$

hence,

$$(I - A)S_n = I - A^{n+1},$$

we can prove that $\lim_{n\to\infty}(I - A^{n+1}) = I$, by using $\lim_{n\to\infty} A^{n+1} = 0$, we get $(I - A)\lim_{n\to\infty} S_n = I$, or $\lim_{n\to\infty} S_n = (I - A)^{-1}$, that is $\sum_{i=0}^{n-1} A^n = (I - A)^{-1}$, as $\lim_{t\to\infty} A^t = 0$, under Lemma 3, we conclude that

$$\lim_{t\to\infty} H_t \leq (I - A)^{-1}M$$

4. Estimation

The parameters of the new volatility model are estimated using the method of Maximum Likelihood Estimation under the assumption of normal distributed standardized innovations. We assume that the second moment of the equation satisfying the A-HY-APARCH model is asymptotically bounded.
The conditional likelihood function of A-HYAPARCH(p, d, q, k) process based on the sample $\{x_1, x_2, \cdots, x_T\}$ of T observations may be written as

$$L_T(x_t|F_{t-1};\theta) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi h_t}} exp(-\frac{1}{2}\frac{x_t^2}{h_t^2}).$$

Under the assumption that the resuduals follows a normal distribution, the Gaussian log-likelihood function may be expressed as

$$\begin{aligned} log(L_T(x_t|F_{t-1};\theta)) &= -\frac{T}{2}log(2\pi) - \frac{1}{2}\sum_{t=1}^{T} log(h_t) - \frac{1}{2}\sum_{t=1}^{T}\frac{x_t^2}{h_t^2} \\ &= -\frac{1}{2}\sum_{t=1}^{T}\Big[log(2\pi) + log(h_t) + z_t^2\Big], \end{aligned}$$

with T denotes the number of observations. The Gaussian log-likelihood is numerically maximized with respect to the vector of the unknown parameters in the model. The procedure simultaneously estimates all the parameters in the model, including those in the flexible functional form of the intercept in the conditional variance process. This study uses numerical techniques to approximate the derivatives of the log-likelihood function with respect to the parameter vector θ, that is $\theta = (w', \Phi, \beta, d')$.

5. Simulation study

In this section, the performance of the A-HY-APARCH(1,d,1,k) model using Monte Carlo Simulation is described for different data generating processes, and comparisons are made with the estimation of corresponding HYAPARCH models. The aim is to investigate whether the A-HYAPARCH model is able to produce the best estimates of the long memory parameter when the structural breaks are accounted. Under the assumption that the residuals follow a normal distribution, the results of simulation with and without structural changes are presented, for the conditional variance process h_t. To be sure that the conditional variance is positive, the conditions on the parameters of the A-HY-APARCH(1,d,1,k) model similar to those holding in HYGARCH(1,d,1) are imposed following Conrad, C. (2010)'s simulation study, that is, for $\Psi_i \geq 0$, $\beta > 0$

$$\beta - \tau d \leq \Phi \leq \frac{2-d}{3}$$

$$\tau d(\Phi - \frac{1-d}{2}) \leq \beta(\Phi - \beta + \tau d).$$

The design of the simulation study follows that of Shi, Y. *et al.*(2018)'s simulation of the A-HYEGARCH model. The time series are generated from the A-HYAPARCH(1,d,1,k) model with the long memory parameter d= (0.35,0.45) and $\Phi = 0.30$ and $\beta = 0.40$. when the GARCH model is applied to financial time series data, the persistence parameter tends to be high and this indicates that the structural breaks are present in data, that is the reason why this study also focus on the impact of ignoring structural changes on the estimated short memory parameters Φ and β. We consider frequencies up to four and the errors are assumed to follow a normal distribution with mean zero and variance one. In each design, simulations are carried out for two different sample $T = 1000$ and $T = 3000$ observations and the first 1000 0bservations are discarded to avoid the errors in simulation. For each $k \in \{0, 1, 2, 3, 4\}$, the 500 Monte Carlo replications were employed in all of the designs. The A-HYAPARCH and the standard HYAPARCH with and without structural changes were estimated. In all simulation, we limited in (p,q)= (1,1) and the model is defined as follows:

$$x_t = h_t z_t \qquad z_t \sim NID(0,1)$$

$$h_t^{\phi} = (1-\tau)h_{1,t}^{\phi} + \tau h_{2,t}^{\phi}$$

where

$$h_{1,t}^{\phi} = w_t + \beta\left(h_{1,t-1}^{\phi} - w_{t-1}\right) + (\Phi - \beta)y_{t-1},$$

$$h_{2,t}^{\phi} = w_t + \beta\left(h_{2,t-1}^{\phi} - w_{t-1}\right) + (\beta - \Phi + \pi_1)y_{t-1} + \sum_{i=0}^{\infty}\left(\pi_{i+2} - \Phi\pi_{i+1}\right)B^i y_{t-2},$$

$$w_t = w_0 + \sum_{j=1}^{k}\left[n_j \sin(\frac{2\pi jt}{T}) + m_j \cos(\frac{2\pi jt}{T})\right]$$

and

$$y_t = \left(|x_t| - \epsilon x_t\right)^{\phi}.$$

The advantage of this new model is that it can capture the essential characteristics of one or more structural breaks by using only a small number of low-frequency components through the deterministic time-varying component (w_t). Three different designs were investigated, for each parametrization, 500 replications are conducted. The difference between the three designs is summarized as follows:

- Design 1 (D1), the first design assumed the time-varying intercept function w_t to be corresponding to constant intercept w_0 that is, $w_t = w_0 = 0.1$.
- Design 2 (D2), the second design imposed the time-varying function to have only one step change, following Shi, Y. *et al.*(2018) in their research we consider one breakpoint that is one step change where the intercept jumping from 0.1 to 0.5. Thus,

$$w_t = \begin{cases} 0.1, & \text{if } t = 1, 2, \cdots, \frac{T}{2} \\ 0.5, & \text{if } t = \frac{T}{2} + 1, \cdots, T. \end{cases}$$

- Design 3 (D3) dealing with two step changes, where in the first step the intercept moving from 0.1 to 0.5 at the first break point and turn back to 0.3 at the second break point. Thus,

$$w_t = \begin{cases} 0.1, & \text{if } t = 1, \cdots, \frac{T}{3} \\ 0.5, & \text{if } t = \frac{T}{3} + 1, \cdots, \frac{2T}{3}. \\ 0.3, & \text{if } t = \frac{2T}{3} + 1, \cdots, T. \end{cases}$$

The performance measures like Monte Carlo bias (Bias), Standard Error(SE) and the Root Means Square Error (RMSE) of the estimated long-range dependence parameter d and short memory parameters (Φ and β) are computed to justify the attainment of maximum likelihood estimators .

5.1. Simulation results

The simulation results are reported in table (1) and table (2) based on two different values of long range dependance parameter(d). In each table D1 correspond to the pure HYAPARCH, D2 and D3 correspond to the A-HYAPARCH model where the intercept varies with time respectively. The values of Bias, standard error (SE) and the root means square error (RMSE) of the estimated long-range dependence parameter d and short memory parameters (Φ and β) are presented in each table.

Table (1) reports the estimation results of the A-HYAPARCH and the pure HYAPARCH model for different data generating processes for all three designs (D1, D2 and D3) where D1 corresponds to the standard HYAPARCH model with the constant intercept and D2 and D3 corresponds to the new model where the intercept is subjected to the structural breaks. Knowing that when $k = 0$ the A-HYAPARCH model reduces to the ordinary HY-APARCH specification.

The Bias, RMSE and Standard Errors are summarized in table (1) for $d = 0.35$, $\Phi = 0.30$ and $\beta = 0.40$. We found that the obtained estimates of the long memory parameter d and short memory parameters have smaller degree of bias , RMSE and standard error and always reduced as the sample increased for all designs. This result is consistent for the moderate persistence value of long memory parameter d=0.35 where the obtained bias in A-HYAPARCH with structural changes is small across all two selected values of T compared to those obtained in the standard HYAPARCH. In general the RMSE and BIAS of the new model is lower from the standard A-HYAPARCH and decreases as the sample size increases, thus even structural change is not present, the A-HYAPARCH performs well than the standard HYAPARCH but it works especially very well when breaks are not ignored.

Table 1. Simulation results for estimation of A-HYAPARCH(1,0.35,1,k) and HYAPARCH(1,0.35,1) models.

		$Bias_\Phi$	SE_Φ	$RMSE_\Phi$	$Bias_\beta$	SE_β	$RMSE_\beta$	$Bias_d$	SE_d	$RMSE_d$
					T=1000					
k=0	D1	0.0406	0.3125	0.3107	0.0351	0.3823	0.2675	0.0545	0.2882	0.2890
	D2	0.0171	0.2560	0.2584	0.0300	0.1574	0.1446	0.0410	0.2550	0.2567
	D3	0.0159	0.2491	0.2462	0.0321	0.1663	0.1517	0.0352	0.2407	0.2416
k=1	D1	0.0376	0.1920	0.1896	0.0256	0.2043	0.1841	0.0235	0.2161	0.1975
	D2	0.0232	0.1874	0.1852	0.0211	0.1930	0.1700	0.0112	0.1923	0.1769
	D3	0.0299	0.1832	0.1792	0.0193	0.1937	0.1709	0.0227	0.1928	0.1742
k=2	D1	0.0137	0.2667	0.2589	0.0195	0.2201	0.1857	0.0363	0.2847	0.2346
	D2	0.0044	0.1851	0.1853	0.0170	0.1847	0.1494	0.0279	0.2478	0.1702
	D3	0.00118	0.2461	0.2402	0.0106	0.2025	0.1066	0.0124	0.1752	0.1517
k=3	D1	0.0131	0.2033	0.1748	0.0320	0.2718	0.1539	0.0269	0.2426	0.1740
	D2	0.0093	0.2030	0.1684	0.0195	0.2510	0.1206	0.0048	0.2046	0.1322
	D3	0.0063	0.2019	0.1714	0.0298	0.2649	0.1429	0.0188	0.2106	0.1508
k=4	D1	0.0309	0.2506	0.2046	0.0449	0.3070	0.1662	0.02673	0.2684	0.1840
	D2	0.0159	0.1918	0.1824	0.0262	0.2061	0.1441	0.0115	0.1744	0.1477
	D3	0.0113	0.2460	0.2036	0.0319	0.2810	0.1295	0.0114	0.2162	0.1226
					T=3000					
k=0	D1	0.0318	0.2422	0.2418	0.0244	0.3539	0.2370	0.0429	0.2829	0.2826
	D2	0.0090	0.1969	0.1925	0.0154	0.1501	0.1365	0.0331	0.1714	0.1716
	D3	0.0006	0.1394	0.1307	0.0144	0.1223	0.1030	0.0069	0.1180	0.1170
k=1	D1	0.0295	0.1505	0.1451	0.0168	0.1781	0.1441	0.0239	0.1832	0.1591
	D2	0.0158	0.1432	0.1403	0.0089	0.1565	0.1191	0.0229	0.1744	0.1473
	D3	0.0175	0.1479	0.1436	0.0162	0.1696	0.1337	0.0139	0.1634	0.1397
k=2	D1	0.0096	0.2234	0.1914	0.0115	0.2137	0.1276	0.0227	0.1979	0.1404
	D2	0.0003	0.1494	0.1317	0.0063	0.1971	0.1197	0.0199	0.1936	0.1381
	D3	0.00621	0.2036	0.1880	0.0045	0.2127	0.1182	0.0009	0.1535	0.0981
k=3	D1	0.0172	0.1893	0.1559	0.0150	0.2415	0.1151	0.0256	0.2281	0.1637
	D2	0.0026	0.1797	0.1528	0.0139	0.2361	0.1132	0.0107	0.2057	0.1291
	D3	0.0008	0.1858	0.1444	0.0116	0.2306	0.1046	0.0024	0.1885	0.1019
k=4	D1	0.0136	0.2018	0.1526	0.0161	0.2579	0.1119	0.0124	0.2154	0.1079
	D2	0.0099	0.1868	0.1310	0.0042	0.2444	0.0916	0.0014	0.1507	0.1000
	D3	0.0056	0.192	0.1432	0.0048	0.2498	0.0950	0.0106	0.2033	0.0989

Table (2) presents simulation results for estimates of the HY-APARCH(1, 0.45, 1) and A-HYAPARCH(1, 0.45, 1, k) models without and with structural changes in all designs with d = 0.45, it can be seen that when the degree of persistence increases and the structural breaks have acounted the A-HY-APARCH model performs well than HY-APARCH model. Note that a constraint of d ¡ 0.5 was imposed in the maximum likelihood estimation to ensure stationarity. When the true value of d is approaching 0.5 (the case d = 0.45) it is shown that the A-HYAPARCH model has small RMSE, Standard Error and Bias especially when sample size is $T = 3000$ that is the HYAPARCH is significantly outperformed by the A- HY-APARCH model, hence our model needs a large enough sample more than $T = 3000$ and provide us meaningfully better results.

In general, the Bias and RMSE computed for design 2 and design 3 are very small compared to the results gotten when the value of $k = 0$. In summary, A-HYAPARCH models outperform HYAPARCH when structural changes are accounted. The impact of allowing the structural changes in the conditional variance is clear, with the data generating process having a downwards bias however all bias in all samples are smaller than 0.05. The results of the numerical simulation indicate that the long memory parameter estimate is robust to this form of model specification. Finally, the simulation results have confirmed that RMSE, BIAS and SE of long memory, short memory autoregressive and short memory moving average parameters reduced when we take into account the possible structural breaks. Therefore, Bias, standard error and root mean square error are small and reduced as the sample size increased. As result the A-HYAPARCH model seems to perform well than HYAPARCH model.

Table 2. Simulation results for estimation of A-HY-APARCH(1,0.45,1,k) and HY-APARCH(1,0.45,1) models.

		$Bias_{\Phi}$	SE_{Φ}	$RMSE_{\Phi}$	$Bias_{\beta}$	SE_{β}	$RMSE_{\beta}$	$Bias_d$	SE_d	$RMSE_d$
					T=1000					
k=0	D1	0.0406	0.2931	0.2896	0.0330	0.2848	0.2794	0.0515	0.2922	0.2695
	D2	0.0233	0.2833	0.2830	0.0301	0.1747	0.1668	0.0348	0.2864	0.2624
	D3	0.0230	0.2359	0.2324	0.0253	0.1641	0.1549	0.0270	0.2112	0.1839
k=1	D1	0.0325	0.1797	0.1787	0.0237	0.1964	0.1810	0.0291	0.2845	0.2179
	D2	0.0249	0.1725	0.1714	0.0163	0.1958	0.1799	0.0201	0.2488	0.1759
	D3	0.0253	0.1715	0.1705	0.0114	0.1746	0.1570	0.0254	0.2391	0.1688
k=2	D1	0.0364	0.2363	0.2010	0.0352	0.2426	0.1626	0.0246	0.3067	0.2092
	D2	0.0164	0.2106	0.1982	0.0250	0.2055	0.1552	0.0094	0.2500	0.1259
	D3	0.0247	0.2039	0.1828	0.0183	0.2343	0.146	0.0242	0.2534	0.1670
k=3	D1	0.0162	0.2971	0.2226	0.0303	0.2803	0.1460	0.0141	0.2478	0.1603
	D2	0.0061	0.2466	0.1866	0.0283	0.2517	0.1407	0.0106	0.2011	0.1551
	D3	0.0147	0.2542	0.1885	0.0145	0.2267	0.1256	0.0074	0.1073	0.1433
k=4	D1	0.0243	0.2394	0.2148	0.0387	0.2683	0.1778	0.0194	0.2698	0.2570
	D2	0.01980	0.2178	0.1724	0.0265	0.2376	0.1255	0.0100	0.2398	0.1680
	D3	0.0060	0.2189	0.1846	0.0029	0.2630	0.1449	0.0151	0.2097	0.1571
					T=3000					
k=0	D1	0.0381	0.2834	0.2781	0.0262	0.2556	0.2457	0.0349	0.2912	0.2683
	D2	0.0160	0.1824	0.1765	0.0244	0.1337	0.1222	0.0206	0.1959	0.1662
	D3	0.0107	0.1823	0.1748	0.0244	0.1329	0.1207	0.0132	0.1764	0.1462
k=1	D1	0.0220	0.1782	0.1770	0.0283	0.1873	0.1643	0.0242	0.2338	0.1462
	D2	0.0138	0.1564	0.1544	0.0138	0.1731	0.1447	0.0156	0.2196	0.1325
	D3	0.01780	0.1488	0.1475	0.0258	0.1737	0.1440	0.00411	0.2120	0.1346
k=2	D1	0.0354	0.1993	0.1690	0.0199	0.2013	0.1153	0.0126	0.2511	0.1185
	D2	0.0102	0.1794	0.1626	0.0166	0.1995	0.1127	0.0042	0.2403	0.1092
	D3	0.0063	0.1770	0.1580	0.0072	0.1289	0.1029	0.0009	0.1105	0.1049
k=3	D1	0.0180	0.2229	0.1804	0.0127	0.2306	0.1265	0.0110	0.1905	0.1407
	D2	0.0032	0.1865	0.1618	0.0073	0.2011	0.1140	0.0008	0.1524	0.1370
	D3	0.0109	0.1922	0.1217	0.0099	0.1421	0.1049	0.0003	0.1060	0.1182
k=4	D1	0.0062	0.2122	0.1527	0.0420	0.2276	0.1487	0.0345	0.1640	0.2198
	D2	0.0035	0.1828	0.1412	0.0199	0.2151	0.1241	0.0140	0.1205	0.1425
	D3	0.00019	0.1595	0.1297	0.0064	0.2087	0.0792	0.0017	0.1107	0.1055

6. Conclusion

In this work, the extension of Hyperbolic APARCH to Adaptive Hyperbolic APARCH process is considered. The proposed model nests a large of models in the literature and therefore facilitates in general to specify methodology. The necessary condition of the A-HY-APARCH(1,d,1,k) to be asymptotically stable are derived and the parameter estimation method was proposed to prove the performance of the new model. Numerical experiments prove that Maximum Likelihood Estimation for the A-HYAPARCH performs reasonably well for sample size=3000 observations with the estimate of fractional differencing parameter improving as the structural breaks are accounted. Clearly, Bias, Standard Error(SE) and Root Mean Square Error(RMSE) are generally small and decrease as the sample size increases thus the A-HYAPARCH model with and without the structural changes outperform the standard HYAPARCH. The A-HYAPARCH model proposed in this study is capable to capture the main characteristics observed in financial time series data like volatility clustering, leptokurtosis, asymmetric effects, long memory as well as structural changes in the conditional variance. We plan to study in our future work, the consistency and normality asymptotic of the A-HY-APARCH model as well as its application to the real data and demonstrate its potential performance with other existing Asymmetric long memory volatility models in the literature.

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