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A New Gamma Generalized Lindley-Log-logistic Distribution with Applications

Thatayaone Moakofi [(1)], **Broderick Oluyede**[(1)] **and Boikanyo Makubate** [(1)]

[(1)] Botswana International University of Science and Technology, Boseja Ward, Palapye, Botswana

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Abstract. A new distribution called the gamma exponentiated Lindley Log-logistic (GELLLoG) distribution is developed. Some properties of the new distribution including hazard function, quantile function, moments, conditional moments, mean and median deviations, Bonferroni and Lorenz curves, distribution of the order statistics and Rényi entropy are derived. Maximum likelihood estimation technique is used to estimate the model parameters. We conduct a simulation study to examine the bias and mean square error of the maximum likelihood estimators. Finally, applications to real datasets to illustrate the usefulness of the proposed distribution are presented.

Key words: Gamma distribution; Lindley distribution; Exponentiated distribution; Log-logistic distribution; Generalized distribution; Maximum likelihood estimation.

Mathematics Subject Classifications: 62E99; 60E05.

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[1]Thatayaone Moakofi: thatayaone.moakofi@studentmail.biust.ac.bw
[1]Broderick Oluyede : oluyedeb@biust.ac.bw, boluyede@georgiasouthern.edu
[1]Boikanyo Makubate : makubateb@biust.ac.bw

Résumé. Abstract in French.

The authors.

Thatayaone Moakofi is a preparing a M.Sc dissertation under the supervision of the second author at the Department of Mathematics and Statistical Sciences at Botswana International University of Science and Technology.

Broderick Oluyede, Ph.D., is professor of Mathematics and Statistics, Botswana International University of Science and Technology.

Boikanyo Makubate, Ph.D., is professor of Statistics at Botswana International University of Science and Technology.

1. Introduction

There are very useful and important generalizations of the Lindley distribution in the literature that are suitable for modeling data with different types of hazard rate functions: increasing, decreasing, bathtub and unimodal. Lindley(1958) used a mixture of exponential and length-biased exponential distributions to illustrate the difference between fiducial and posterior distributions. The resulting mixture is called the Lindley (L) distribution. Oluyede and Yang(2015) developed an extension of the Lindley distribution called the beta generalized Lindley distribution. A generalization of the Lindley distribution called Kumaraswamy Lindley distribution with applications to lifetime data was presented by Oluyede *et al.*(2015). Ghitany *et al.*(2008) investigated the properties of Lindley distribution. Nadarajah *et al.*(2011) studied the mathematical and statistical properties of the exponentiated or generalized Lindley (GL) distribution. The cumulative distribution function (cdf) and probabilty density function (pdf) of the GL distribution are given by

$$G_{GL}(x;\alpha,\lambda)=\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\alpha}, \tag{1}$$

and

$$g_{GL}(x;\alpha,\lambda)=\frac{\alpha\lambda^2}{1+\lambda}(1+x)\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\alpha-1}\exp(-\lambda x), \tag{2}$$

for $x>0$, $\lambda>0$, and $\alpha>0$. This distribution is the exponentiated Lindley distribution. Ghitany *et al.*(2013) presented results on a two-parameter Lindley distribution referred to as power-Lindley distribution. Zakerzadeh and Dolati(2009) looked at a different generalization of the Lindley distribution.
Lindley distribution is a mixture of exponential and gamma distributions, that is $f(x;\lambda)=(1-p)f_G(x;\lambda)+pf_E(x;\lambda)$ with $p=\frac{1}{1+\lambda}$, where $f_G(x;\lambda)\equiv GAM(2,\lambda)$, and $f_E(x;\lambda)\equiv EXP(\lambda)$.

1.1. Zografos and Balakrishnan Model

We consider the family of distributions with the pdf $f(x)$ and cdf $F(x)$ given as:

$$f(x)=\frac{1}{\Gamma(\delta)\psi^{\delta}}\left[-log(1-G(x))\right]^{\delta-1}(1-G(x))^{\frac{1}{\psi}-1}g(x), x\in R, \delta>0, \tag{3}$$

and

$$F(x)=\frac{1}{\Gamma(\delta)\psi^{\delta}}\int_0^{-log(1-G(x))}t^{\delta-1}e^{-t/\psi}dt=\frac{\gamma\left(\delta,-\psi^{-1}log(1-G(x))\right)}{\Gamma(\delta)}, \tag{4}$$

respectively, where $\gamma(x,\delta)=\int_0^x t^{\delta-1}e^{-t}dt$ is the incomplete gamma function and take the cdf G(x) to be the exponentiated Lindley-log-logistic distribution Oluyede *et al.*(2020). The corresponding hazard rate function is

$$h_F(x)=\frac{\left[-log(1-G(x))\right]^{\delta-1}f(x)(1-G(x))^{1/\psi-1}}{\psi^{\delta}\left(\Gamma(\delta)-\gamma\left(-\psi^{-1}log(1-G(x)),\delta\right)\right)}.$$

When ψ=1, this distribution is referred to as the Zografos and Balakrishian-G (ZB-G) family of distributions Zografos and Balakrishnan(2009).

This paper employs exponentiation, competing risk transformation and ZB-G formulation to obtain a new distribution involving both the Lindley and log-logistic distributions. The new distribution called the gamma exponentiated Lindley log-logistic (GELLLoG) distribution is quite useful, generalizes the Lindley, generalized Lindley and log-logistic distributions, and is more flexible distribution for the description of reliability and lifetime data. The combined distribution of Lindley and log-logistic is obtained from the product of the reliability or survival functions of the Lindley and log-logistic distributions via competing risk model. A motivation for developing this model is the advantages presented by this extended distribution with respect to having a hazard function that exhibits increasing, decreasing and bathtub shapes, as well as the versatility and flexibility of exponentiated distributions in general, as well as the Lindley and log-logistic distributions in modeling lifetime data.

This paper is organized as follows. In section 2, some basic results, the GELLLoG distribution and its sub-models, hazard function and the quantile function are presented. The moments and moment generating function, mean and median deviations are given in section 3. Section 4 contains some additional useful results on the distribution of order statistics and Rényi entropy. In section 5, results on the estimation of the parameters of the GELLLoG distribution via the method of maximum likelihood are presented. A Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators in section 6. Applications are given in section 7, followed by some concluding remarks.

2. The Model, Series Expansion of Density Function, Sub-models, Hazard and Quantile Functions

In this section, we derive some properties of the new gamma exponentiated Lindley log-Logistic (GELLLoG) distribution including expansion of the density, hazard function, quantile function, sub-models, moments, conditional moments and maximum likelihood estimation of model parameters.

The cdf, survival function (sf) and pdf of the exponentiated Lindley log-logistic (ELLLoG) distribution Oluyede *et al.*(2020) are given by

$$G(x;\lambda,c,\alpha)=\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^{c})}\right]^{\alpha}, \tag{5}$$

$$\overline{G}(x;\lambda,c,\alpha)=1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^{c})}\right]^{\alpha}, \tag{6}$$

and

$$\begin{aligned} g(x;\lambda,c,\alpha) &= \alpha\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^{c})}\right]^{\alpha-1} \\ &\times \frac{(1+x^{c})^{-1}}{1+\lambda}e^{-\lambda x}\left[\lambda^{2}(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^{c}}\right], \end{aligned}$$

respectively, for $\lambda,c,\alpha>0$. If a random variable X has the ELLLoG distribution, we write $X\sim ELLLoG(\lambda,c,\alpha)$.

The cdf and pdf of the proposed gamma exponentiated Lindley log-logistic (GELLLoG) distribution are given by

$$\begin{aligned} F_{GELLLoG}(x;\lambda,c,\alpha,\delta) &= \frac{1}{\Gamma(\delta)}\int_{0}^{-\log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^{c})}\right)^{\alpha}\right)} t^{\delta-1}e^{-t}dt \\ &= \frac{\gamma\left(-\log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^{c})}\right)^{\alpha}\right),\delta\right)}{\Gamma(\delta)}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} f_{GELLLoG}(x;\lambda,c,\alpha,\delta) &= \frac{1}{\Gamma(\delta)}\left[-\log\left(1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^{c})}\right]^{\alpha}\right)\right]^{\delta-1} \\ &\times \alpha\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^{c})}\right]^{\alpha-1} \\ &\times \frac{(1+x^{c})^{-1}}{1+\lambda}e^{-\lambda x}\left[\lambda^{2}(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^{c}}\right], \end{aligned} \tag{8}$$

respectively, for $\lambda,c,\alpha,\delta>0$, where $\gamma(x,\delta)=\int_{0}^{x}t^{\delta-1}e^{-t}dt$ is the lower incomplete gamma function. If a random variable X has the GELLLoG distribution, we write $X\sim GELLLoG(\lambda,c,\alpha,\delta)$.

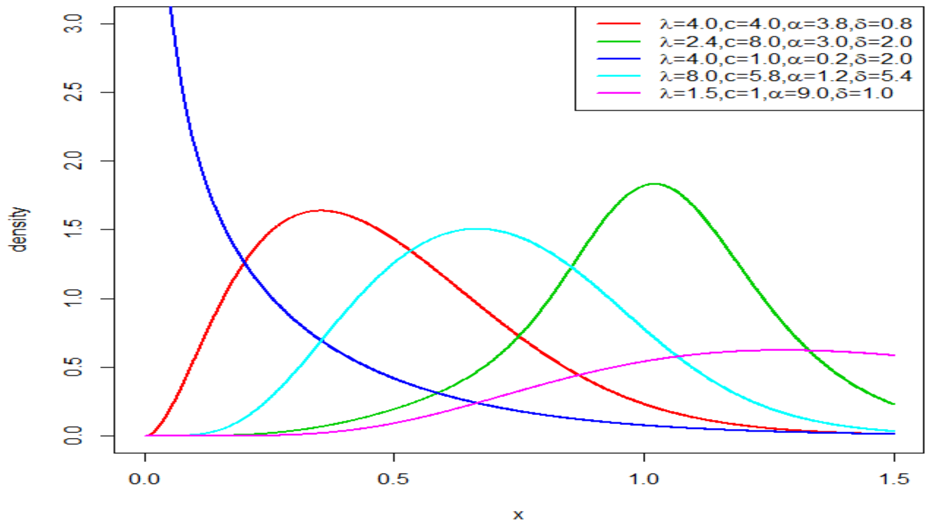

Fig. 1: Plots of GELLLoG Density Function

2.1. Series Expansion of Density Function

In this section, series expansion of the GELLLoG density function is presented. The results allows for the mathematical and statistical properties of the model to be readily obtained.

Let $y = \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}$, $0 < y < 1$, $\alpha, \lambda, c > 0$, then using the series representation $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, we have

$$\left[-\log(1-y)\right]^{\delta-1} = y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right],$$

and applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty} a_s y^s\right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \tag{9}$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, Gradshteyn and Ryzhik(2000), the GELLLoG pdf can be written as

$$\begin{aligned} f_{GELLLoG}(x) &= \frac{\alpha}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}y^{m+s+\delta-1}\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha-1} \\ &\times \frac{(1+x^c)^{-1}}{1+\lambda}e^{-\lambda x}\left[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right] \\ &= \frac{\alpha}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}\frac{(m+s+\delta)}{(m+s+\delta)} \\ &\times \left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha(m+s+\delta)-1} \\ &\times \frac{(1+x^c)^{-1}}{1+\lambda}e^{-\lambda x}\left[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right] \\ &= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}\frac{\alpha b_{s,m}}{(m+s+\delta)\Gamma(\delta)}g_{ELLLoG}(x;c,\lambda,\alpha^*), \end{aligned} \tag{10}$$

where $g_{ELLLoG}(x;c,\lambda,\alpha^*)$ is the exponentiated Lindley-log-logistic (ELLLoG) pdf with parameters c, λ, and $\alpha^* = \alpha(m+s+\delta) > 0$. Let $D = \{(m,s) \in \mathbf{Z}_+^2\}$, then the weights in the GELLLoG pdf are

$$\omega_\nu = \binom{\delta-1}{m}\frac{\alpha b_{m,s}}{(m+s+\delta)\Gamma(\delta)}, \tag{11}$$

and

$$f_{GELLLoG}(x) = \sum_{\nu\in D}\omega_\nu g_{ELLLoG}(x;c,\lambda,\alpha(m+s+\delta)). \tag{12}$$

It follows therefore that the GELLLoG density is an infinite linear combination of the ELLLoG pdfs. The statistical and mathematical properties of the GELLLoG distribution can be readily obtained from those of the ELLLoG distribution.

2.2. Sub-models of GELLLoG Distribution

In this subsection, some useful and important sub-models are presented.

- When $\lambda \to 0^+$, the resulting distribution is the gamma exponentiated log-logistic (GELLoG) distribution.
- When $\lambda \to 0^+$, and $\alpha = 1$, we obtain the gamma log-logistic (GLLoG) distribution.
- We obtain gamma Lindley log-logistic (GLLLoG) distribution with $\alpha = 1$.
- When $\delta = 1$, we obtain the baseline exponentiated Lindley log-logistic (ELLLoG) distribution.
- When $\delta = \alpha = 1$, we obtain the Lindley log-logistic (LLLoG) distribution.
- When $\lambda \to 0^+$, and $\delta = 1$, we obtain the exponentiated log-logistic (ELLoG) distribution.
- If $\lambda \to 0^+$ and $\alpha = \delta = 1$, we obtain log-logistic (LLoG) distribution.

- If $\delta = c = 1$, and $\lambda \to 0^+$, we obtain one parameter distribution denoted by $GELLLoG(1, 1, \alpha, 1)$, with the cdf

$$F(x;\alpha) = \left[1 - \frac{1}{(1+x)}\right]^{\alpha}, \quad \alpha > 0. \tag{13}$$

- If $\delta = c = 1$, we obtain the two parameter distribution denoted by $GELLLoG(\lambda, 1, \alpha, 1)$, with the cdf

$$F(x;\lambda,\alpha) = \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x)}\right]^{\alpha}, \quad \lambda, \alpha > 0. \tag{14}$$

- If $\delta = c = \alpha = 1$, we obtain the one parameter distribution denoted by $GELLLoG(\lambda, 1, 1, 1)$, with the cdf

$$F(x;\lambda) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x)}, \quad \lambda > 0. \tag{15}$$

- If $\alpha = c = 1$, we obtain the two parameter distribution denoted by $GELLLoG(\lambda, 1, 1, \delta)$, with the cdf

$$F(x;\lambda,\delta) = \frac{1}{\Gamma(\delta)}\gamma\left(-\log\left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{1+x}\right)\right), \delta\right), \quad \lambda, \delta > 0. \tag{16}$$

- If c=1, we obtain the three parameter distribution denoted by $GELLLoG(\lambda, 1, \alpha, \delta)$, with the cdf

$$F(x;\lambda,\alpha,\delta) = \frac{1}{\Gamma(\delta)}\gamma\left(-\log\left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{1+x}\right)^{\alpha}\right), \delta\right), \quad \lambda, \delta > 0. \tag{17}$$

2.3. Hazard and Quantile Functions

In this section, we present the hazard and quantile functions of the GELLLoG distribution. Plots of the hazard function for selected values of the model parameters are presented in Figure 2. The hazard rate function of the GELLLoG distribution is given by

$$\begin{aligned}
h_{F_{GELLLoG}}(x) &= \frac{f_{GELLLoG}(x)}{\overline{F}_{GELLLoG}(x)} \\
&= \left[-\log\left(1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha}\right)\right]^{\delta-1} \\
&\times\ \alpha\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha-1} \\
&\times\ \frac{(1+x^c)^{-1}}{1+\lambda}e^{-\lambda x}\left[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right] \\
&\times\ \left[\Gamma(\delta)-\gamma\left(-\log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right),\delta\right)\right]^{-1}.
\end{aligned}$$

Plots of the GELLLoG hazard below shows different shapes including decreasing, increasing, bathtub followed by upside down, upside down bathtub, and bathtub shapes.

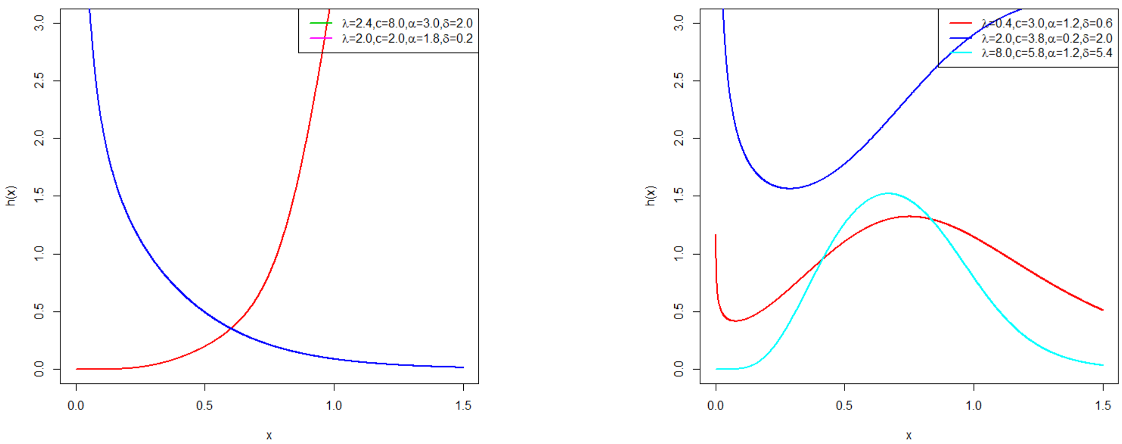

Fig. 2: Plots of GELLLoG Hazard Function

The quantile function of the GELLLoG distribution is obtained by solving the non-linear equation:

$$\frac{\gamma\left(-\log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right),\delta\right)}{\Gamma(\delta)}=u, \tag{18}$$

$0 \leq u \leq 1$, that is,

$$\left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha} = 1 - e^{-\gamma^{-1}(u\Gamma(\delta),\delta)}, \tag{19}$$

so that

$$\frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)} = 1 - \left(1 - e^{-\gamma^{-1}(u\Gamma(\delta),\delta)}\right)^{1/\alpha}. \tag{20}$$

Consequently, random numbers can be generated for the GELLLoG distribution by numerically solving the nonlinear equation

$$\lambda x + \log(1+x^c) - \log\left(1 + \frac{\lambda x}{1+\lambda}\right) + \log\left(1 - \left(1 - e^{-\gamma^{-1}(u\Gamma(\delta),\delta)}\right)^{1/\alpha}\right) = 0. \tag{21}$$

Table 1 presents quantiles of the GELLLoG distribution for selected values of the model parameters λ, c, α and δ.

Table 1: **Table of Quantiles for GELLLoG Distribution**

	$(\lambda, c, \alpha, \delta)$				
u	$(1.2, 1.2, 1, 1.8))$	$(0.2, 1.5, 2.2, 2)$	$(0.8, 1.2, 2, 1)$	$(2, 1, 1, 2.2)$	$(1, 1.8, 2.6, 2)$
0.1	0.2943	1.4788	0.3456	0.2970	0.9711
0.2	0.4582	2.0185	0.5276	0.4557	1.2086
0.3	0.6130	2.5516	0.7021	0.6062	1.4150
0.4	0.7726	3.1391	0.8868	0.7636	1.6199
0.5	0.9460	3.8300	1.0944	0.9395	1.8390
0.6	1.1430	4.6905	1.3409	1.1496	2.0886
0.7	1.3780	5.8378	1.6538	1.4253	2.3939
0.8	1.6776	7.5312	2.0931	1.8588	2.8079
0.9	2.1043	10.6012	2.8509	3.4573	3.4967

3. Moments, Conditional Moments, Mean and Median Deviations

In this section, we present the moments, moment generating function, mean and median deviations for the GELLLoG distribution. Moments are very important and necessary in any statistical analysis, especially in applications. Moments can be used to study the most important features and characteristics of a distribution (e.g., central tendency, dispersion, skewness and kurtosis). These measures (moments, moment generating function, mean and median deviations) can be readily obtained for the sub-models given in section 2.

3.1. Moments and Moment Generating Function

Let $\alpha^* = \alpha(m+s+\delta)$, and $Y \sim ELLLoG(c, \lambda, \alpha^*)$. Note that the k^{th} moment of the ELLLoG random variable Y is obtained as follows. The k^{th} raw moment, μ'_k of the

ELLLoG distribution is given by:

$$\begin{aligned}
E(Y^k) &= \int_0^\infty y^k g_{ELLLoG}(y;\lambda,c,\alpha(m+s+\delta))dy \\
&= \int_0^\infty y^k \alpha(m+s+\delta)\left[1-\frac{1+\lambda+\lambda y}{1+\lambda}\frac{e^{-\lambda y}}{(1+y^c)}\right]^{\alpha(m+s+\delta)-1} \\
&\times \frac{(1+y^c)^{-1}}{1+\lambda}e^{-\lambda y}\left[\lambda^2(1+y)+\frac{(1+\lambda+\lambda y)cy^{c-1}}{1+y^c}\right]dy \\
&= \sum_{t=0}^\infty \binom{\alpha(m+s+\delta)-1}{t}(-1)^t\alpha(m+s+\delta)\int_0^\infty y^k\left[\frac{1+\lambda+\lambda y}{1+\lambda}\frac{e^{-\lambda y}}{(1+y^c)}\right]^t \\
&\times \frac{(1+y^c)^{-1}}{1+\lambda}e^{-\lambda y}\left[\lambda^2(1+y)+\frac{(1+\lambda+\lambda y)cy^{c-1}}{1+y^c}\right]dy \\
&= \sum_{t,p=0}^\infty \frac{\alpha(m+s+\delta)(-1)^{t+p}[\lambda(t+1)]^p}{(1+\lambda)^{t+1}p!}\binom{\alpha(m+s+\delta)-1}{t} \\
&\times \left[\lambda^2\sum_{q=0}^\infty\binom{t}{q}\lambda^q(1+\lambda)^{t-q}\int_0^\infty y^{k+p+q}(1+y)(1+y^c)^{-t-1}dy\right. \\
&+ \left. c\sum_{q=0}^\infty\binom{t+1}{q}\lambda^q(1+\lambda)^{t+1-q}\int_0^\infty y^{k+p+c+q-1}(1+y^c)^{-t-2}dy\right]. \qquad (22)
\end{aligned}$$

We note that by applying $(1+\lambda+\lambda y)^{t+1} = \sum_{q=0}^\infty\binom{t+1}{q}(\lambda y)^q(1+\lambda)^{t+1-q}$, $(1+\lambda+\lambda y)^t = \sum_{q=0}^\infty\binom{t}{q}(\lambda y)^q(1+\lambda)^{t-q}$, and the substitution $w=(1+y^c)^{-1}$, we have

$$\begin{aligned}
E(Y^k) &= \sum_{t,p=0}^\infty \frac{\alpha(m+s+\delta)(-1)^{t+p}[\lambda(t+1)]^p}{(1+\lambda)^{t+1}p!}\binom{\alpha(m+s+\delta)-1}{t} \\
&\times \left[\sum_{q=0}^\infty\binom{t}{q}\lambda^{q+2}\frac{(1+\lambda)^{t-q}}{c}\int_0^1 w^{t+1-\frac{k+p+q+1}{c}-1}(1-w)^{\frac{k+p+q+1}{c}-1}dw\right. \\
&+ \int_0^1 w^{t+1-\frac{k+p+q+2}{c}-1}(1-w)^{\frac{k+p+q+2}{c}-1}dw \\
&+ \left. c\sum_{q=0}^\infty\binom{t+1}{q}\lambda^q(1+\lambda)^{t+1-q}\int_0^1 w^{t+2-\frac{k+p+c+q}{c}-1}(1-w)^{\frac{k+q+p+c+q}{c}-1}dt\right]. \qquad (23)
\end{aligned}$$

Consequently,

$$\begin{aligned} E(Y^k) &= \sum_{t,p=0}^{\infty} \frac{\alpha(m+s+\delta)(-1)^{t+p}[\lambda(t+1)]^p}{(1+\lambda)^{t+1}p!} \binom{\alpha(m+s+\delta)-1}{t} \\ &\times \left[\sum_{q=0}^{\infty} \binom{t}{q} \lambda^{q+2} \frac{(1+\lambda)^{t-q}}{c} \Bigg(B\Big(t+1-\frac{k+p+q+1}{c}, \frac{k+p+q+1}{c}\Big) \right. \\ &+ B\Big(t+1-\frac{k+p+q+2}{c}, \frac{k+p+q+2}{c}\Big)\Bigg) \\ &+ c\sum_{q=0}^{\infty} \binom{t+1}{q} \lambda^q (1+\lambda)^{t+1-q} \\ &\times \left. B\Big(t+2-\frac{k+p+c+q}{c}, \frac{k+q+p+c+q}{c}\Big)\right]. \end{aligned} \tag{24}$$

Thus, the k^{th} moments of the GELLLoG distribution is given by

$$E(X^k) = \sum_{\nu \in D} \omega_\nu \Delta(t,p,q,c,\lambda,k), \tag{25}$$

where $\Delta(t,p,q,c,\lambda,k)$ is given by equation (24). The moment generating function of the GELLLoG class of distribution is given by $E(e^{tX}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k)$, where $E(X^k)$ is given by the equation (25). The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD=σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}},$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively.

Some moments for selected parameters values are given in Table 2 and plots of CS and CK versus the shape parameters, α, c and δ are presented in Figure 3, Figure 4 and Figure 5. Plots of skewness and kurtosis for choices of the model parameters reveal that skewness and kurtosis depend on the parameters α, c, and δ.

Table 2: Table of Moments for Selected Parameters Values of GELLLoG Distribution

	(0.1,0.2,0.2,0.5)	(1.8,1.5,2.2,0.5)	(0.8,1.0,2.2,1.0)	(2.0,2.2,0.2,1.8)	(0.1,1.0,2.0,0.5)
E(X)	0.0080	0.2415	0.2861	0.1690	0.3272
E(X^2)	0.0039	0.1321	0.1611	0.0727	0.2062
E(X^3)	0.0025	0.0889	0.1088	0.0422	0.1474
E(X^4)	0.0019	0.0665	0.0811	0.0285	0.1135
E(X^5)	0.0015	0.0530	0.0642	0.0210	0.0918
E(X^6)	0.0012	0.0439	0.0529	0.0164	0.0769
SD	0.0011	0.0375	0.0450	0.0133	0.0660
CV	0.0009	0.0327	0.0390	0.0112	0.0577
CS	0.0008	0.0290	0.0344	0.0096	0.0513
CK	0.0007	0.02607	0.0308	0.0084	0.0461

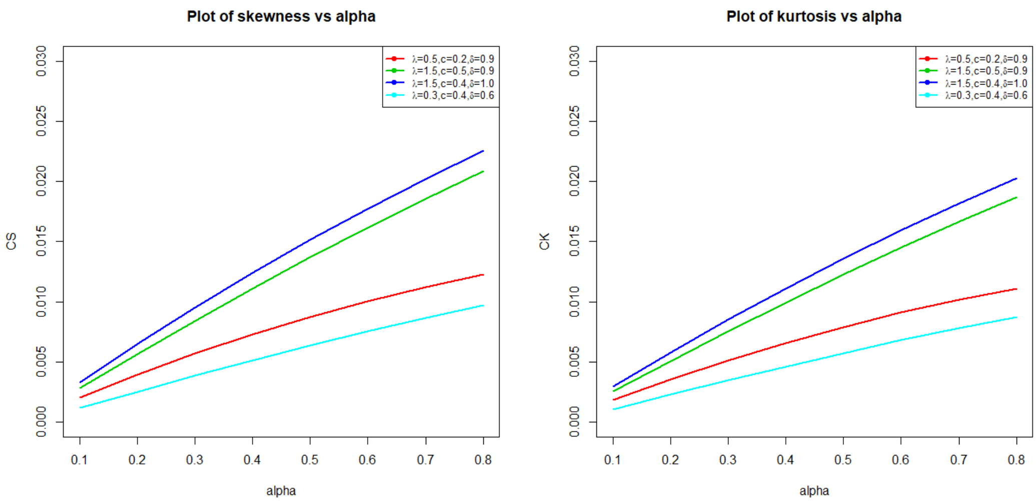

Fig. 3: Plots of Skewness and Kurtosis for parameter alpha

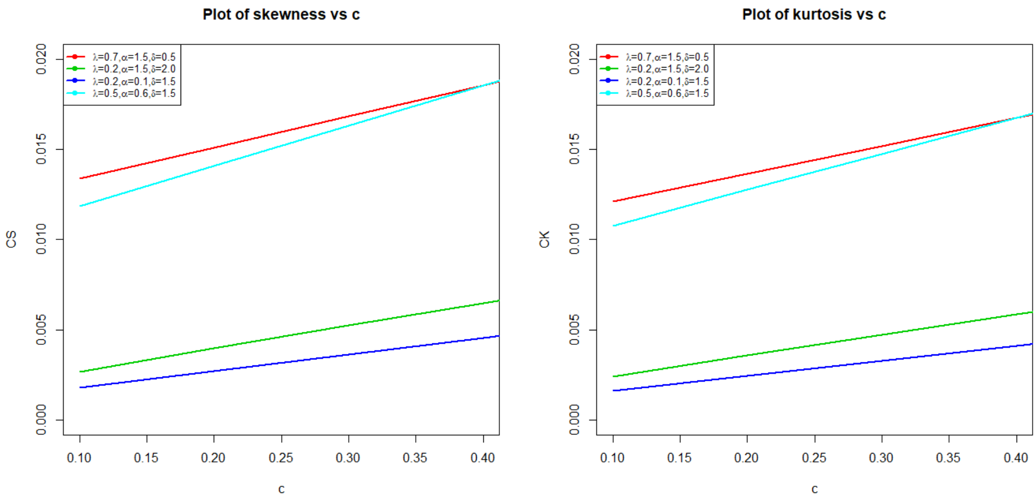

Fig. 4: Plots of Skewness and Kurtosis for parameter c

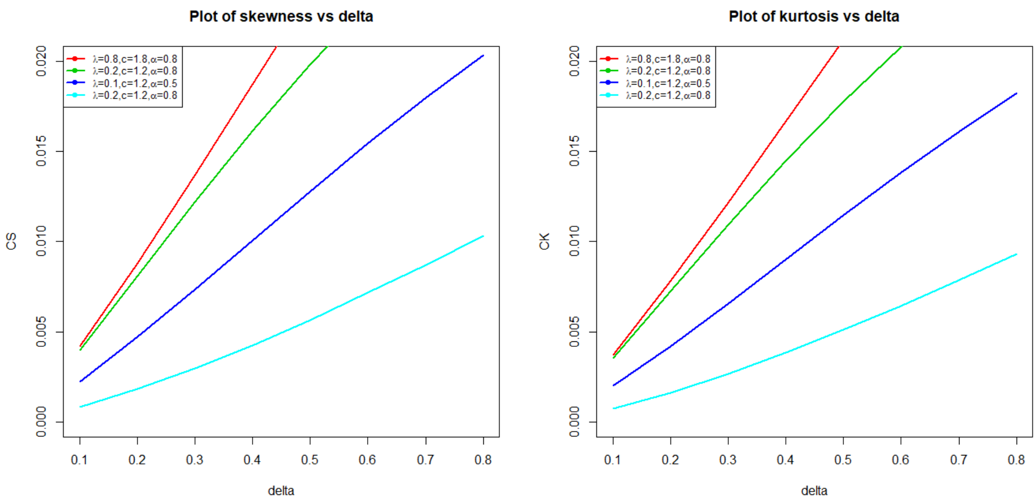

Fig. 5: Plots of Skewness and Kurtosis for parameter delta

3.2. Conditional Moments

The mean residual life function, vitality function and related reliability measures can be readily obtained from the conditional moments of a distribution.
The k^{th} conditional moments for the GELLLoG distribution is given by

$$\begin{aligned}
E(X^k|X>a) &= \frac{1}{\overline{F}_{GELLLoG}(a)}\int_t^\infty x^k f_{GELLLoG}(x;c,\lambda,\alpha,\delta)dx \\
&= \frac{1}{\overline{F}_{GELLLoG}(a)}\sum_{\nu\in D}\sum_{t,p=0}^{\infty}\omega_\nu \frac{\alpha(m+s+\delta)(-1)^{t+p}[\lambda(t+1)]^p}{(1+\lambda)^{t+1}p!} \\
&\times \binom{\alpha(m+s+\delta)-1}{t}\left[\sum_{q=0}^{\infty}\binom{t}{q}\lambda^{q+2}\frac{(1+\lambda)^{t-q}}{c}\right. \\
&\times \left(B_{(1+a^c)^{-1}}\left(t+1-\frac{k+p+q+1}{c},\frac{k+p+q+1}{c}\right)\right. \\
&+ \left.B_{(1+a^c)^{-1}}\left(t+1-\frac{k+p+q+2}{c},\frac{k+p+q+2}{c}\right)\right) \\
&+ c\sum_{q=0}^{\infty}\binom{t+1}{q}\lambda^q(1+\lambda)^{t+1-q} \\
&\times \left.B_{(1+a^c)^{-1}}\left(t+2-\frac{k+p+c+q}{c},\frac{k+q+p+c+q}{c}\right)\right],
\end{aligned}$$

where $B_{(1+a^c)^{-1}}(a,b)$ is the incomplete beta function.

3.3. Mean Deviation, Lorenz and Bonferroni Curves

Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the GELLLoG distribution are presented in this subsection.

3.3.1. Mean Deviations

The mean deviation about the mean and the mean deviation about the median are defined by

$$\delta_1(x)=\int_0^\infty |x-\mu|f_{GELLLoG}(x)dx \quad \text{and} \quad \delta_2(x)=\int_0^\infty |x-M|f_{GELLLoG}(x)dx, \tag{26}$$

respectively, where $\mu=E[X]$ and $M=Median(X)$ denotes the median. We note that $\delta_1(x)$ and $\delta_2(x)$ can be expressed as $\delta_1(x)=2\mu F_{GELLLoG}(\mu)-2\mu+2\int_\mu^\infty f_{GELLLoG}(x)dx$ and $\delta_2(x)=-\mu+2\int_M^\infty xf_{GELLLoG}(x)dx$, respectively, that is,

$$\delta_1(x)=2\mu F_{GELLLoG}(\mu)-2\mu+2T(\mu) \quad \text{and} \quad \delta_2(x)=2T(M)-\mu, \tag{27}$$

where

$$\begin{aligned}
T(\mu) &= \int_{\mu}^{\infty} x f_{GELLLoG}(x)dx \\
&= \sum_{\nu \in D} \sum_{t,p=0}^{\infty} \omega_{\nu} \frac{\alpha(m+s+\delta)(-1)^{t+p}[\lambda(t+1)]^{p}}{(1+\lambda)^{t+1}p!} \\
&\times \binom{\alpha(m+s+\delta)-1}{t} \left[\sum_{q=0}^{\infty} \binom{t}{q} \lambda^{q+2} \frac{(1+\lambda)^{t-q}}{c} \right. \\
&\times \left(B_{(1+\mu^{c})^{-1}} \left(t+1-\frac{1+p+q+1}{c}, \frac{1+p+q+1}{c} \right) \right. \\
&+ \left. B_{(1+\mu^{c})^{-1}} \left(t+1-\frac{1+p+q+2}{c}, \frac{1+p+q+2}{c} \right) \right) \\
&+ c \sum_{q=0}^{\infty} \binom{t+1}{q} \lambda^{q}(1+\lambda)^{t+1-q} \\
&\times \left. B_{(1+\mu^{c})^{-1}} \left(t+2-\frac{1+p+c+q}{c}, \frac{1+q+p+c+q}{c} \right) \right]. \qquad (28)
\end{aligned}$$

3.3.2. Lorenz and Bonferroni Curves

Lorenz and Bonferroni curves are applicable to economics for the study of income and poverty, and are also usefull in other areas such as reliability, demography, insurance and medicine. Bonferroni and Lorenz curves for the GELLLoG distibution are given as

$$B(p) = \frac{1}{p\mu} \int_{0}^{q} x f_{GELLLoG}(x)dx = \frac{1}{p\mu}[\mu - T(q)],$$

and

$$L(p) = \frac{1}{\mu} \int_{0}^{q} x f_{GELLLoG}(x)dx = \frac{1}{\mu}[\mu - T(q)],$$

respectively, where $T(q) = \int_{q}^{\infty} x f_{GELLLoG}(x)dx$ is given by equation (28), $q = F^{-1}_{GELLLoG}(p)$, $0 \le p \le 1$.

4. Order Statistics and Rényi Entropy

Order statistics play an important role in probability and statistics, particularly in reliability and lifetime data analysis. The concept of entropy plays a vital role in information theory. In this section, we present the distribution of the i^{th} order statistics and Rényi entropy for the GELLLoG distribution.

4.1. Order Statistics

In this subsection, the pdf of the i^{th} order statistic and the corresponding moments are presented. Let $X_1, X_2, ..., X_n$ be independent and identically distributed GELLLoG random variables. Using the binomial expansion $(1 - G_{GELLLoG}(x))^{n-i} = \sum_{j=0}^{n-i} \binom{n-i}{j}(-1)^j [G_{GELLLoG}(x)]^j$, the pdf of the i^{th} order statistic from the GELLLoG pdf $f_{GELLLoG}(x)$ can be written as

$$\begin{aligned} f_{i:n}(x) &= \frac{n! f_{GELLLoG}(x)}{(i-1)!(n-i)!}[F_{GELLLoG}(x)]^{i-1}[1 - F_{GELLLoG}(x)]^{n-i} \\ &= \frac{n! f_{GELLLoG}(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} \\ &\times \left[\frac{\gamma\left(-\log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right),\delta\right)}{\Gamma(\delta)}\right]^{i+j-1}. \end{aligned} \tag{29}$$

Now, let $0 < y = \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha} < 1$, $x > 0$, $c, \lambda, \alpha > 0$. Using the fact that $\gamma(x, \delta) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{m+\delta}}{(m+\delta)m!}$, and setting $c_m = (-1)^m/((m+\delta)m!)$, we can write the pdf of the i^{th} order statistic from the GELLLoG distribution as follows:

$$\begin{aligned} f_{i:n}(x) &= \frac{n! f_{GELLLoG}(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{[\Gamma(\delta)]^{i+j-1}} \\ &\times \left[-\log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right)\right]^{\delta(i+j-1)} \\ &\times \left[\sum_{m=0}^{\infty}\frac{(-1)^m\left(\log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right)\right)^m}{(m+\delta)m!}\right]^{i+j-1} \\ &= \frac{n! f_{GELLLoG}(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{[\Gamma(\delta)]^{i+j-1}} \\ &\times \left[-\log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right)\right]^{\delta(i+j-1)} \\ &\times \sum_{m=0}^{\infty} d_{m,i+j-1}\left[-\log\left(1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right)\right]^{m}, \end{aligned} \tag{30}$$

where $d_0 = c_0^{(i+j-1)}$, $d_{m,i+j-1} = (mc_0)^{-1}\sum_{l=1}^{m}[(i+j-1)l - m + l]c_l d_{m-l,i+j-1}$. We note that the pdf of the i^{th} order statistic from the GELLLOG distribution can be written as

$$
\begin{aligned}
f_{i:n}(x) &= \frac{n! f_{GELLLoG}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j-1}} \\
&\times \left[-\log\left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right)\right]^{\delta(i+j-1)+m} \\
&= \frac{n! \left[-\log\left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right)\right]^{\delta-1}}{(i-1)!(n-i)!\Gamma(\delta)} \alpha \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha-1} \\
&\times \frac{(1+x^c)^{-1}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right] \\
&\times \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j-1}} \\
&\times \left[-\log\left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right)\right]^{\delta(i+j-1)+m} \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j-1}} \\
&\times \frac{\Gamma(\delta(i+j-1)+m+\delta)}{\Gamma(\delta(i+j-1)+m+\delta)} \frac{\left[-\log\left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right)\right]^{\delta(i+j-1)+m+\delta-1}}{\Gamma(\delta)} \\
&\times \alpha \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha-1} \\
&\times \frac{(1+x^c)^{-1}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right] \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \\
&\times \frac{(-1)^j d_{m,i+j-1} \Gamma(\delta(i+j-1)+m+\delta)}{[\Gamma(\delta)]^{i+j}} f_{GELLLoG}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
f_{GELLLoG}(x) &= \frac{\left[-\log\left(1 - \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}\right)^{\alpha}\right)\right]^{\delta(i+j-1)+m+\delta-1}}{\Gamma(\delta(i+j-1)+m+\delta)} \\
&\times \alpha \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha-1} \\
&\times \frac{(1+x^c)^{-1}}{1+\lambda} e^{-\lambda x} \left[\lambda^2(1+x) + \frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right] \qquad (31)
\end{aligned}
$$

is the GELLLoG pdf with parameters $c, \lambda, \alpha > 0$, and shape parameter $\delta^* = \delta(i+j) + m > 0$. It follows therefore that the t^{th} moment of the i^{th} order statistic from the

GELLLoG density is given by

$$E(X_{i:n}^t) = \sum_{\nu\in D}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\omega_\nu \ell_{i,j,m}E(X^t), \tag{32}$$

where $E(X^t)$ is the t^{th} moment of the GELLLoG distribution given by (25) with the parameters c, α, λ and $\delta(i+j)+m>0$,

$$\ell_{i,j,m} = \frac{n!}{(i-1)!(n-i)!}\frac{(-1)^j d_{m,i+j-1}\Gamma(\delta(i+j)+m)}{[\Gamma(\delta)]^{i+j}}.$$

We note that these moments are often used in several areas including reliability, survival analysis, biometry, engineering, insurance and quality control for the prediction of future failures times from a set of past or previous failures.

4.2. Rényi Entropy

Rényi entropy Rényi(1960) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^\infty [f_{GELLLoG}(x;c,\alpha,\lambda,\delta)]^v dx\right), v\neq 1, v>0. \tag{33}$$

Rényi entropy tends to Shannon entropy as $v\to 1$. Note that

$$\begin{aligned}\int_0^\infty f_{GELLLoG}^v(x)dx &= \left(\frac{1}{\Gamma(\delta)}\right)^v\int_0^\infty\left[-\log\left(1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^\alpha\right)\right]^{v(\delta-1)}\\ &\times\ \alpha^v\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^{v(\alpha-1)}\\ &\times\ \frac{(1+x^c)^{-v}}{(1+\lambda)^v}e^{-\lambda v x}\left[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right]^v dx.\end{aligned} \tag{34}$$

Let $0<y=\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^\alpha<1$. Note that

$$\begin{aligned}\left(\lambda^2(1+x)+(1+\lambda+\lambda x)(1+x^c)^{-1}cx^{c-1}\right)^v &= \sum_{p=0}^\infty\binom{v}{p}\lambda^{2(v-p)}(1+x)^{v-p}c^p x^{cp-p}\\ &\times\ \frac{(1+\lambda+\lambda x)^p}{(1+x^c)^p},\end{aligned}$$

and

$$\left[-\log\left(1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^\alpha\right)\right]^{v(\delta-1)} = \sum_{m,s=0}^\infty\binom{v\delta-v}{m}d_{s,m}y^{m+s+\delta-1},$$

by applying the result on power series raised to a positive integer, with $c_s=(s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^\infty c_s y^s\right)^m = \sum_{s=0}^\infty d_{s,m}y^s, \tag{35}$$

where $d_{s,m} = (sc_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]c_l d_{s-l,m}$, and $d_{0,m} = c_0^m$, Gradshteyn and Ryzhik(2000), so that

$$\begin{aligned}\int_0^{\infty} f^v_{GELLLoG}(x)dx &= \left(\frac{1}{\Gamma(\delta)}\right)^v \sum_{m,s,p,k,q,t,w=0}^{\infty} d_{s,m}\binom{v\delta-v}{m}\binom{v}{p}\binom{v-p}{t}\binom{k+p}{w}\\ &\times \frac{\Gamma(\alpha(m+s+\delta+v-1)-v+1)}{\Gamma(\alpha(m+s+\delta+v-1)-v+1-k)k!}\\ &\times \frac{c^p\lambda^{2(v-p)+w}(-1)^q[\lambda(k+v)]^q}{q!(1+\lambda)^{v-p+w}}\\ &\times \int_0^{\infty} x^{cp-p+q+w+t}(1+x^c)^{-v-k-p}dx.\end{aligned}$$

Now, with $y=(1+x^c)^{-1}$, Rényi entropy for the GELLLoG distribution reduces to

$$\begin{aligned}I_R(v) &= \frac{1}{1-v}\log\left[\left(\frac{1}{c\Gamma(\delta)}\right)^v \sum_{m,s,p,k,q,t,w=0}^{\infty} d_{s,m}\binom{v\delta-v}{m}\binom{v}{p}\binom{v-p}{t}\binom{k+p}{w}\right.\\ &\times \frac{\Gamma(\alpha(m+s+\delta+v-1)-v+1)}{\Gamma(\alpha(m+s+\delta+v-1)-v+1-k)k!}\frac{c^p\lambda^{2(v-p)+w}(-1)^q[\lambda(k+v)]^q}{q!(1+\lambda)^{v-p+w}}\\ &\left.\times \mathscr{B}\left(v+k+p-\frac{cp+q+w+t-p+1}{c},\frac{cp+q+w+t-p+1}{c}\right)\right],\end{aligned}$$

for $v>0$, $v\neq 1$, where $\mathscr{B}(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}$ is the beta function.

5. Maximum Likelihood Estimation

Let $X \sim GELLLoG(c,\alpha,\lambda,\delta)$ and $\mathbf{\Delta}=(c,\alpha,\lambda,\delta)^T$ be the parameter vector. The log-likelihood $\ell_n=\ell_n(\mathbf{\Delta})$ based on a random sample of size n from the GLLoGW distribution is given by

$$\begin{aligned}\ell_n(\mathbf{\Delta}) &= -n\ln\Gamma(\delta)+(\delta-1)\sum_{i=1}^{n}\ln\left[-\ln\left(1-\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha}\right)\right]\\ &+ n\ln(\alpha)+(\alpha-1)\sum_{i=1}^{n}\ln\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]-\sum_{i=1}^{n}\ln(1+x_i^c)\\ &- n\ln(1+\lambda)-\sum_{i=1}^{n}\lambda x_i+\sum_{i=1}^{n}\ln\left[\lambda^2(1+x_i)+\frac{(1+\lambda+\lambda x_i)cx_i^{c-1}}{(1+x_i^c)}\right]. \qquad (36)\end{aligned}$$

The first derivative of the log-likelihood function with respect to each component of the parameter vector $\mathbf{\Delta}=(c,\alpha,\lambda,\delta)^T$ can be readily obtained. The equations obtained by setting the partial derivatives to zero are not in closed form and the values of the parameters $c,\alpha,\lambda,$ and δ must be found by using iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial\ell_n}{\partial c},\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\lambda},\frac{\partial\ell_n}{\partial\delta})^T=\mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by

$\mathbf{I}(\mathbf{\Delta}) = [\mathbf{I}_{\theta_i,\theta_j}]_{4X4} = E(-\frac{\partial^2 \ell_n}{\partial\theta_i \partial\theta_j})$, $i, j = 1, 2, 3, 4$ can be numerically obtained by MATLAB, SAS or R software. The total Fisher information matrix $n\mathbf{I}(\mathbf{\Delta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Delta}}) \approx \left[-\frac{\partial^2 \ell_n}{\partial\theta_i \partial\theta_j}\bigg|_{\mathbf{\Delta}=\hat{\mathbf{\Delta}}} \right]_{4X4}, \quad i, j = 1, 2, 3, 4. \tag{37}$$

For a given set of observations, the matrix given in equation (37) is obtained after the convergence of the Newton-Raphson procedure. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\Delta} = (\hat{c}, \hat{\alpha}, \hat{\lambda}, \hat{\delta})$ be the maximum likelihood estimate of $\mathbf{\Delta} = (c, \alpha, \lambda, \delta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Delta}} - \mathbf{\Delta}) \xrightarrow{d} N_4(\underline{0}, I^{-1}(\mathbf{\Delta}))$, where $I(\mathbf{\Delta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Delta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Delta}}$, that is $J(\hat{\mathbf{\Delta}})$. The multivariate normal distribution $N_4(\underline{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0, 0, 0, 0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for c, α, λ and δ are given by:

$$\widehat{c} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{cc}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\lambda} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\lambda\lambda}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\delta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\delta\delta}^{-1}(\widehat{\mathbf{\Delta}})},$$

respectively, where $\mathbf{I}_{cc}^{-1}(\widehat{\mathbf{\Delta}})$, $\mathbf{I}_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Delta}})$, $\mathbf{I}_{\lambda\lambda}^{-1}(\widehat{\mathbf{\Delta}})$, and $\mathbf{I}_{\delta\delta}^{-1}(\widehat{\mathbf{\Delta}})$ are the diagonal elements of $\mathbf{I}_n^{-1}(\hat{\mathbf{\Delta}}) = (n\mathbf{I}(\hat{\mathbf{\Delta}}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

We maximize the likelihood function using NLmixed in SAS as well as the function nlm in R rdevelopmentcoreteam(2011). These functions were applied and executed for wide range of initial values. This process often results or lead to more than one maximum, however, in these cases, we take the MLEs corresponding to the largest value of the maxima. In a few cases, no maximum was identified for the selected initial values. In these cases, a new initial value was tried in order to obtain a maximum.The issues of existence and uniqueness of the MLEs are theoretical interest and has been studied by several authors for different distributions including Seregin(2010), Santos and Tenreyro(2010), Zhou(2009), and Xia *et al.*(2009). At this point we are not able to address the theoretical aspects (existence, uniqueness) of the MLE of the parameters of the GELLLoG distribution.

The maximum likelihood estimates (MLEs) of the GELLLoG parameters c, α, λ, and δ are computed by maximizing the objective function via the subroutine NLmixed in SAS and the function nlm in R. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic $(-2\ln(L))$, Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), and Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\mathbf{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented. In order to compare the models, we use the criteria stated above. Note that for the value of the log-likelihood function at its maximum (ℓ_n), larger value is good and preferred, and for AIC, AICC and BIC, smaller values are preferred. The GELLLoG distribution is fitted to the data sets and these fits are compared to the fits of the nested gamma exponentiated log-logistic (GELLoG), Lindley-log-logistic (LLLoG), and log-logistic distributions (LLoG), and several non-nested distributions given in section 7.

The likelihood ratio (LR) test is applied to compare the fit of the GELLLoG distribution with its sub-models for a given data set. For example, to test $\delta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{c}, \hat{\alpha}, \hat{\lambda}, \hat{\delta})) - \ln(L(\tilde{c}, \tilde{\alpha}, \tilde{\lambda}, 1))]$, where $\hat{c}$, $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\delta}$ are the unrestricted estimates, and $\tilde{c}$, $\tilde{\alpha}$, and $\tilde{\lambda}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 1 degree of freedom.

6. Simulation Study

In this section, we examine the performance of the GELLLoG distribution by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800) via the R package. We simulate $N = 2000$ samples for the true parameters values given in the Table 3. The table lists the mean MLEs of the four model parameters along with the respective root mean squared errors (RMSEs). From the results, we can verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero. The bias and RMSE for the estimated parameter $\hat{\theta}$, say, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{n} \hat{\theta}_i}{n} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{n} (\hat{\theta}_i - \theta)^2}{n}},$$

respectively.

Table 3: **Monte Carlo Simulation Results**

		(1.0,2.5,1.0,2.0)			(1.5,2.0,1.0,2.0)			(2.0,1.0,3.0,1.0)		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
λ	35	1.6155	1.3790	0.6155	2.0458	1.3208	0.5458	2.7723	1.4651	0.7723
	50	1.4916	1.1719	0.4916	1.9560	1.2516	0.4560	2.5528	1.2286	0.5528
	100	1.3178	0.9065	0.3178	1.8398	1.0126	0.3397	2.4102	0.9590	0.4102
	200	1.1820	0.6118	0.1820	1.71647	0.7932	0.2164	2.2497	0.6901	0.2497
	400	1.1122	0.4292	0.1122	1.6470	0.6060	0.1470	2.1503	0.5352	0.1503
	800	1.0550	0.2932	0.0550	1.5893	0.4428	0.0893	2.0678	0.4055	0.0678
c	35	2.0212	1.2877	-0.4787	1.6426	1.1469	-0.3573	1.1900	0.9087	0.1900
	50	2.0901	1.2225	-0.4098	1.6338	1.0348	-0.3661	1.1487	0.6768	0.1487
	100	2.1640	1.0116	-0.3351	1.6559	0.9061	-0.3440	1.0954	0.4732	0.0954
	200	2.2644	0.7613	-0.2355	1.6734	0.7595	-0.3265	1.0722	0.3848	0.0722
	400	2.3433	0.5544	-0.1566	1.7840	0.6140	-0.2159	1.0628	0.3169	0.0628
	800	2.4305	0.3781	-0.0694	1.8432	0.4621	-0.1567	1.0459	0.2826	0.0459
α	35	2.0618	2.4833	1.0618	1.8239	2.0295	0.8239	2.4286	2.1427	-0.5713
	50	1.9298	2.3119	0.9298	1.7459	1.7429	0.7459	2.5856	2.1084	-0.4143
	100	1.8306	2.0975	0.8306	1.6569	1.6916	0.6569	2.6287	1.9137	-0.3712
	200	1.6500	1.7575	0.6500	1.6298	1.5404	0.6298	2.7726	1.6800	-0.2273
	400	1.4112	1.4684	0.4112	1.3464	1.0195	0.3464	2.9444	1.6574	-0.0555
	800	1.1481	0.7029	0.1481	1.2392	0.9252	0.2392	3.0005	1.3830	0.0005
δ	35	2.6765	2.1522	0.6765	2.5907	2.0466	0.5907	2.1088	2.0959	1.1088
	50	2.5350	1.7651	0.5350	2.4733	1.9415	0.4733	1.8387	1.7348	0.8387
	100	2.2934	1.4306	0.2934	2.3002	1.5411	0.3002	1.6543	1.3991	0.6543
	200	2.1168	1.0218	0.1168	2.0839	1.1625	0.0839	1.4012	0.9425	0.4012
	400	2.0405	0.7137	0.0405	2.0596	0.8639	0.0596	1.2569	0.7230	0.2569
	800	2.0349	0.4869	0.0349	2.0282	0.6477	0.0282	1.1451	0.5387	0.1451

7. Applications

In this section, we present examples to illustrate the flexibility and usefulness of the GELLLoG distribution and its sub-models for data modeling. We also compare GELLLoG distribution to the non-nested new modified Weibull (NMW) distribution introduced by Doostmoradi *et al.*(2014), a four parameter beta generalized exponential (BGE) distribution introduced by Barreto-Souza *et al.* (2010), beta generalized Lindley (BGL) distribution by Oluyede and Yang(2015) and exponentiated modified Weibull distribution by Elbatal(2011). The pdf of four parameter NMW, BGE, BGL and EMW distributions are given in equation equation (38), (39),(40) and (41), respectively, that is,

$$g_{NMW}(x) = \left(\alpha\gamma x^{\gamma-1}e^{\alpha x^{\gamma}} + \lambda\beta x^{\lambda-1}e^{-\beta x^{\lambda}}\right)e^{-e^{\alpha x^{\gamma}}+e^{-\beta x^{\lambda}}}, \quad x > 0, \tag{38}$$

$$g_{BGE}(x) = \frac{\alpha\lambda}{B(a,b)}e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha a-1}\left(1-\left(1-e^{-\lambda x}\right)^{\alpha}\right)^{b-1}, \quad x > 0. \tag{39}$$

$$\begin{aligned} g_{BGL}(x) &= \frac{\alpha\lambda^2}{B(a,b)(1+\lambda)}(1+x)e^{-\lambda x}\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right]^{a\alpha-1} \\ &\times \left[1-\left(1-\frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right)^{a}\right]^{b-1}, \quad x > 0, \end{aligned} \tag{40}$$

and

$$g_{EMW}(x) = \gamma\left[\delta+\lambda\theta^{\alpha}x^{\lambda-1}\right]e^{-(\delta x+(\theta x^{\lambda})}\left[1-e^{-(\delta x+(\theta x)^{\lambda})}\right]^{\delta-1}, \quad x > 0. \tag{41}$$

Plots of the fitted densities, the histogram of the data and probability plots Chambers *et al.*(1983) are given in Figure 6 and Figure 7 for the two datasets considered in this section. For the probability plot, we plotted

$F_{GELLLoG}(x_{(j)}; \hat{c}, \hat{\alpha}, \hat{\lambda}, \hat{\delta})$ against $\frac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares (SS)

$$SS = \sum_{j=1}^{n} \left[F_{GELLLoG}(x_{(j)}) - \left(\frac{j-0.375}{n+0.25} \right) \right]^2 .$$

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan(1995) as well as Kolmogorov-Smirnov (KS) statistic, its P-value and SS are also presented in the tables. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit.

7.1. Lifetime data

Gross and Clark(1975) presented the following data for lifetime data. The data are: 1.1,1.4,1.3,1.7,1.9,1.8,1.6,2.2,1.7,2.7,4.1,1.8,1.5,1.2,1.4,3,1.7,2.3,1.6,2.
Estimates of the parameters of GELLLoG distribution and its related sub-models (standard error in parentheses), AIC, BIC, and the goodness-of-fit statistics W*, A*, KS and its P-value as well as SS are given in Table 4. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 6.

Table 4: Estimates of Models for Lifetime Data

	Estimates				Statistics								
Model	λ	c	α	δ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
GELLLoG	0.1243	5.1200	3.4885	2.0262	30.8287	38.8287	41.4954	42.8117	0.0262	0.1512	0.0960	0.9928	0.0216
	(1.6299)	(6.8941)	(18.2812)	(7.8205)									
GELLoG	0	1.1287	1.2996×10^{-04}	1.0504	109.5000	115.4986	116.9986	118.4858	0.0623	0.3667	1.0000	2.2×10^{-16}	6.6216
	-	(2.4417×10^{-01})	(8.1022×10^{-05})	(3.8729×10^{-03})									
ELLLoG	2.3230×10^{-09}	2.3964	0.5000	1	63.0817	69.0818	70.5818	72.0690	0.0548	0.3218	0.7462	4.241×10^{-10}	3.9409
	(1.4863×10^{-01})	(4.4268×10^{-01})	(3.1192×10^{-01})	-									
LLLoG	4.7922×10^{-09}	2.4917	1	1	65.6049	69.6049	70.3100	71.5964	0.0492	0.2878	0.5616	6.621×10^{-06}	2.3313
	(1.5407×10^{-01})	(4.4792×10^{-01})	-	-									
LLoG	0	2.4916	1	1	65.6049	67.6049	67.8271	68.6007	0.0492	0.2878	0.5616	6.6210×10^{-06}	2.3313
	-	(0.4479)	-	-									
	α	λ	a	b									
BGE	28.4888	5.4032	28.9646	0.2951	30.8779	38.8779	41.5446	42.8609	0.0348	0.1958	0.1012	0.9865	0.0262
	(215.8902)	(3.7126)	(217.1584)	(0.2620)									
	α	γ	λ	β									
NMW	0.0090	3.4110	6.6101	0.0169	37.8087	406.7058	409.3725	410.6888	1.1163	5.6637	0.6348	1.992×10^{-07}	1.9879
	(0.0163)	(1.2816)	(1.4907)	(0.0156)									
	α	θ	a	b									
BGL	1.2026×10^{-01}	2.9054×10^{-07}	3.0161×10^{-01}	1.0100×10^{01}	162.3258	170.33	172.9967	174.313	0.0872	0.5162	0.4939	0.0001	1.5551
	(1.2389×10^{-02})	(4.2756×10^{-06})	(1.1499×10^{-03})	(3.5923×10^{-05})									
	γ	δ	λ	θ									
EMW	3.6683×10^{01}	2.2352	1.7001×10^{01}	1.0000×10^{-04}	32.5212	40.5212	43.1878	44.5041	0.0542	0.3184	0.1343	0.8633	0.0431
	(2.5254×10^{01})	(4.3602×10^{-01})	(2.6643×10^{-13})	(4.5297×10^{-08})									

The Likelihood ratio (LR) test statistic for testing H_0: GELLoG against H_a: GELLLoG, H_0: LLLoG against H_a: GELLLoG and H_0: ELLLoG against H_a: GELLLoG are 78.6712 (p-value < 0.0001), 34.7761 (p-value < 0.0001) and 32.2529 (p-value < 0.0001). We can conclude that there are significant differences between GELLoG and GELLLoG distributions, LLLoG and GELLLoG distributions as well between GELLLoG and ELLLoG distributions, respectively based on the LR tests at 5% level. The values of AIC and BIC are smallest for the GELLLoG distribution, when compared to the corresponding values for the non-nested BGE, NMW, BGL and EMW distributions. The values of the goodness-of-fit-statistics W*, A*, KS and its p-value show that the GELLLoG distribution is the “best” fit for the lifetime data.

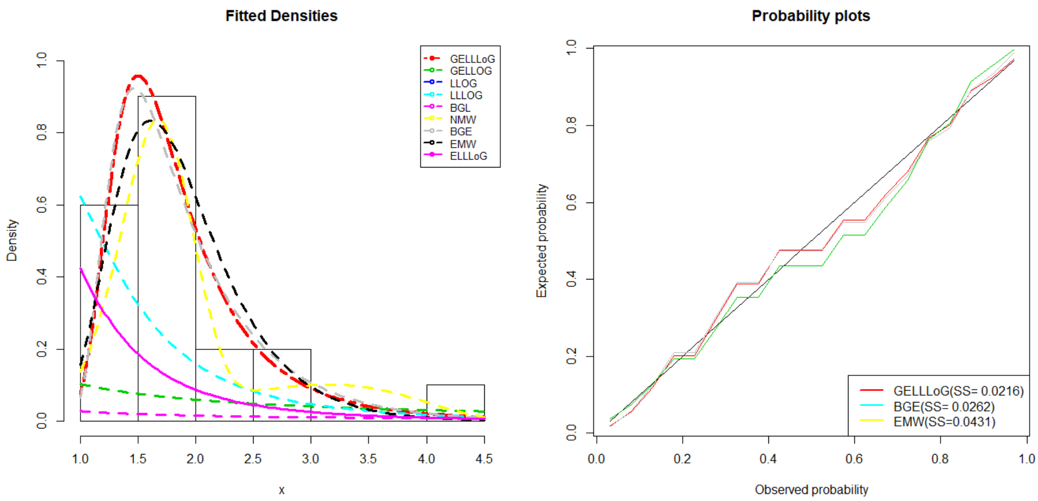

Fig. 6: Fitted Densities and Probability Plots of the Lifetime Data

7.2. Repair lifetimes of an airborne transceiver

These data correspond to maintenance on active repair times (in hours) for an airborne communication transceiver with size n=46 from Leiva *et al.*(2009) and Chhikara and Folks(1977). These data are:
0.2,0.3,0.5,0.5,0.5,0.5,0.6,0.6,0.7,0.7,0.7,0.8,0.8,1.0,1.0,1.0,1.0,1.1,1.3,1.5,1.5, 1.5,
1.5,2.0,2.0,2.2,2.5,2.7,3.0,3.0,3.3,3.3,4.0,4.0,4.5,4.7,5.0,5.4,5.4,7.0,7.5, 8.8,9.0,10.3,
22.0,24.5.
Estimates of the parameters of GELLLoG distribution and its related sub-models (standard error in parentheses), AIC, BIC, W*, A* , KS and its P-value as well as SS are given in Table 5. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 7.

Table 5: Estimates of Models for repair lifetimes of an airborne transceiver Data

	Estimates				Statistics								
Model	λ	c	α	δ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
GELLLoG	0.0909	1.2382	1.8733	1.0064	199.7542	207.7542	208.7298	215.0688	0.0489	0.3168	0.0929	0.8216	0.0546
	(0.0754)	(0.3801)	(3.2566)	(1.3790)									
GELLoG	0	1.1267	1.4823×10^{-04}	1.0504×10^{01}	266.6504	272.6513	273.2227	278.1372	0.0676	0.4027	1.0000	2.2×10^{-16}	15.2901
	-	(1.5154×10^{-01})	(6.2240×10^{-05})	(3.1145×10^{-03})									
ELLLoG	0.0454	1.3410	0.5000	1	202.0737	208.0737	209.3084	214.2229	0.0593	0.3655	0.4893	5.419×10^{-10}	4.5084
	(0.0576)	(0.1853)	(0.1417)	-									
LLLoG	0.0525	1.3205	1	1	214.5014	218.5014	218.7805	222.1587	0.0564	0.3441	0.2435	0.0085	1.1386
	(0.0591)	(0.1826)	-	-									
LLoG	0	1.3643	1	1	214.9587	216.9587	217.0496	218.7873	0.0624	0.3725	0.2362	0.0117	1.1276
	-	(0.1670)	-	-									
	α	λ	a	b									
BGE	10.9759	1.2799	0.1848	0.1855	201.8082	209.8083	210.7839	217.1229	0.0563	0.4558	0.1109	0.6228	0.0890
	(20.9289)	(0.7340)	(0.3193)	(0.0620)									
	α	γ	λ	β									
NMW	0.1280	0.3343	1.2957	0.1810	237.2661	245.2661	246.2417	252.5807	0.1341	0.8364	0.2340	0.0129	0.6827
	(0.0680)	(0.2015)	(0.1966)	(0.0619)									
	α	θ	a	b									
BGL	1.1793×10^{-01}	1.7823×10^{-06}	3.0140×10^{-01}	1.0100×10^{01}	386.7372	394.7331	395.7087	402.0476	0.0769	0.4980	0.5073	1.0370×10^{-10}	3.3521
	(7.3103×10^{-03})	(2.8513×10^{-06})	(4.6377×10^{-04})	(2.5191×10^{-05})									
	γ	δ	λ	θ									
EMW	9.5828×10^{-01}	2.6937×10^{-01}	1.7001×10^{01}	1.0000×10^{-04}	209.9658	217.9658	218.9414	225.2804	0.1441	1.0004	0.1519	0.2385	0.1889
	(1.8975×10^{-01})	(5.4358×10^{-02})	(5.0648×10^{-19})	(8.6130×10^{-14})									

The LR test statistic for testing H_0: GELLoG against H_a: GELLLoG, H_0: LLLoG against H_a: GELLLoG and H_0: ELLLoG against H_a: GELLLoG are 66.8962 (p-value < 0.00001), 14.2598 (p-value < 0.000801) and 2.3195 (p-value=0.1277). We can conclude that are significant differences between GELLoG and GELLLoG distributions, as well as between LLLoG and GELLLoG distributions, respectively based on the LR tests. There is no significant difference between GELLLoG and ELLLoG distributions based on the LR test. The GELLLoG distribution is significantly better than the sub-models considered above. The values of the statistics: AIC, AICC, and BIC are smallest for the GELLLoG distribution. Also, the goodness-of-fit statistics W^* and A^* are the smallest and definitely points to the GELLLoG distribution as the "best"fit for the Repair lifetimes of an airborne transceiver data when compared to the corresponding values for the sub-models. The goodness-of-fit statistics W^* and A^* are also better for the GELLLoG distribution when compared to the values for the non-nested BGE, MMW, BGL and EMW distributions. Thus, there is indeed convincing evidence that the GELLLoG distribution is the "best" fit for the repair lifetimes of an airborne transceiver data.

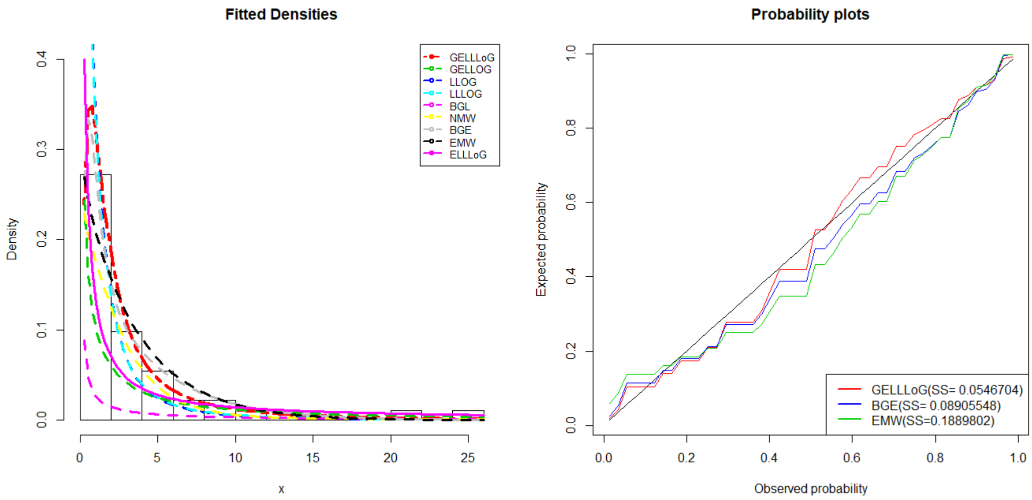

Fig. 7: Fitted Densities and Probability Plots of the Repair Lifetimes of an Airborne Transceiver Data

8. Concluding Remarks

A new generalized distribution called the gamma exponentiated Lindley log-logistic (GELLLoG) distribution is presented. The GELLLoG distribution has several new and known distributions as special cases or sub-models. The density of this new distribution can be expressed as a linear combination of ELLLoG density functions. The GELLLoG distribution possesses hazard function with flexible behavior. We also obtain closed form expressions for the moments, mean and median deviations, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. The performance of the GELLLoG distribution was examined by conducting Monte Carlo simulations for different sizes. Finally, the GELLLoG distribution is fitted to real data sets to illustrate the applicability and usefulness of the new generalized distribution.

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9. Appendix A

Elements of the score vector are given by

$$\frac{\partial \ell_n}{\partial c} = (\delta-1)\sum_{i=1}^{n}\frac{\left(1-\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha}\right)^{-1}}{\ln\left(1-\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha}\right)}\alpha\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha-1}$$
$$\times\left[\frac{(1+\lambda+\lambda x_i)e^{-\lambda x_i}}{1+\lambda}+\frac{x_i^c\ln x_i}{(1+x_i^c)^2}\right]+(\alpha-1)\sum_{i=1}^{n}\frac{\left[\frac{(1+\lambda+\lambda x_i)e^{-\lambda x_i}}{1+\lambda}+\frac{x_i^c\ln x_i}{(1+x_i^c)^2}\right]}{\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]}$$
$$-\sum_{i=1}^{n}\frac{x_i^c\ln x_i}{(1+x_i^c)}+\sum_{i=1}^{n}\frac{\frac{(1+\lambda+\lambda x_i)}{(1+x_i^c)^2}\left((x_i^{c-1}+cx_i^{c-1}\ln x_i)(1+x_i^c)-cx_i^{c-1}x_i^c\ln x_i\right)}{\left[\lambda^2(1+x_i)+\frac{(1+\lambda+\lambda x_i)cx_i^{c-1}}{(1+x_i^c)}\right]},$$

$$\frac{\partial \ell_n}{\partial \alpha} = (\delta-1)\sum_{i=1}^{n}\frac{\left(1-\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha}\right)^{-1}}{\ln\left(1-\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha}\right)}\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha}$$
$$\times\ln\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]+\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right],$$

$$\frac{\partial \ell_n}{\partial \lambda} = (\delta-1)\sum_{i=1}^{n}\frac{\left(1-\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha}\right)^{-1}}{\ln\left(1-\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha}\right)}\alpha\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha-1}$$
$$\times\frac{-e^{-\lambda x_i}}{(1+x_i^c)(1+\lambda)}\left(\frac{(1+x_i)(1+\lambda)-(1+\lambda+\lambda x_i)}{1+\lambda}+(1+\lambda+\lambda x_i)(x_i)\right)$$
$$+(\alpha-1)\sum_{i=1}^{n}\frac{\frac{e^{-\lambda x_i}}{(1+x_i^c)}\left[\frac{(1+x_i)(1+\lambda)-(1+\lambda+\lambda x_i)}{(1+\lambda)^2}+\frac{(1+\lambda+\lambda x_i)x_i}{1+\lambda}\right]}{\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]}-\frac{n}{1+\lambda}-\sum_{i=1}^{n}x_i$$
$$+\sum_{i=1}^{n}\frac{2\lambda x_i+\frac{cx_i^{c-1}(1+x_i)}{(1+x_i^c)}}{\left[\lambda^2(1+x_i)+\frac{(1+\lambda+\lambda x_i)cx_i^{c-1}}{(1+x_i^c)}\right]},$$

and

$$\frac{\partial \ell_n}{\partial \delta} = -\frac{n\Gamma'(\delta)}{\Gamma(\delta)}+\sum_{i=1}^{n}\ln\left[-\ln\left(1-\left[1-\frac{1+\lambda+\lambda x_i}{1+\lambda}\frac{e^{-\lambda x_i}}{(1+x_i^c)}\right]^{\alpha}\right)\right].$$

References

Barreto-Souza, W., Santos, A. H. S. and Cordeiro, G. M. (2010) The Beta Generalized Exponential Distribution, *Journal of Statistical Computation and Simulation*, 80(2), (2010), 159-172.

Chambers, J., Cleveland, W., Kleiner, B., and Tukey, J.: *Graphical Methods for Data Analysis*, Chapman and Hall, London, (1983).

Chen, G. and Balakrishnan, N.: *A General Purpose Approximate Goodness-of-fit Test*, Journal of Quality Technology, 27, (1995), 154-161.

Chhikara R.S., and Folks J. I.: *The Inverse Gaussian Distribution as a Lifetime Model*, Technometrics, 19, (1977), 461-8.

Doostmoradi, A., Zadkarami, M. R., and Roshani Sheykhabad, A.: *A New Modified Weibull Distribution and Its Application*, Journal of Statistical Research and Training Center, Iran, 11, (2014), 97-118.

Elbatal, I.: *Exponentiated Modified Weibull Distribution.* Economic Quality Control, 26, (2011), 189-200.

Ghitany, M.E., Al-Mutairi, D.K., Balakrishnan, N., and Al-Enezi, L.J.: *Power Lindley Distribution and Associated Inference*, Comput. Stat. Data Anal, 64, (2013), 20-33.

Ghitany, M.E., Atieh, B., and Nadarajah, S.: *Lindley Distribution and Its Applications*, Math. Comput. Simul, 78(4), (2008), 493-506.

Gradshteyn, I. S., and Ryzhik, I., M.: *Table of Integrals, Series and Products*, Academic Press, San Diego, (2000).

Gross, A.J., and Clark, V.A.: *Survival distributions: reliability applications in the biomedical sciences*, NewYork, JohnWiley and Sons, (1975).

Leiva V., Barros M., and Paula G. A.: *Generalized Birnbaum-Sanders Models using R*, Brazil, Brazilian Statistical Association, 53, (2009), 1511-1528.

Lindley, D.V.: *Fiducial Distributions and Bayes Theorem*, J. Royal Stat. Soc. Ser. B, 20, (1958), 102-107.

Nadarajah, S., Bakouch, H. S., and Tahmasbi, R.: *A Generalized Lindley Distribution*, Sankhya B, 73, (2011), 331-359.

Oluyede, B. O., Raheem, S., and Wanduku, D.: *The beta Lindley Log-logistic Distribution with Applications to Lifetime Data, submitted,* (2020).

Oluyede, B.O, and Yang, T.: *A New Class of Generalized Lindley Distributions with Applications*, Journal of Statistical Computation and Simulation, 85(10), (2015), 2072-2100, DOI: 10.1080/00949655.2014.917308

Oluyede, B. O., Yang, T., and Omolo, B.: *A Generalized Class of Kumaraswamy Lindley Distribution with Applications to Lifetime Data*, Journal of Computations & Modelling, 5(1), (2015), 27-70.

R Development Core Team.: *A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria, (2011).

Rényi, A.: *On Measures of Entropy and Information,* Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, 1, (1960), 547-561.

Santos Silva, J. M. C., and Tenreyro, S.: *On the Existence of Maximum Likelihood Estimates in Poisson Regression*, Econ. Lett, 107, (2010), 310-312.

Seregin, A.: *Uniqueness of the Maximum Likelihood Estimator for k-Monotone Densities*, Proc. Amer. Math. Soc, 138, (2010), 4511-4515.

Xia, J., Mi, J., and Zhou, Y. Y.: *On the Existence and Uniqueness of the Maximum Likelihood Estimators of Normal and Log-normal Population Parameters with Grouped Data*, J. Probab. Statist., Article id 310575, (2009), 16 pages.

Zakerzadeh, H., and Dolati, A.: *Generalized Lindley Distribution*, Journal of Mathematical Extension, 3, 2, (2009), 13-25.

Zhou, C.: *Existence and Consistency of the Maximum Likelihood Estimator for the Extreme Index*, J. Multivariate Analysis, 100, (2009), 794-815.

Zografos, K., and Balakrishnan, N.: *On Families of beta- and Generalized Gamma-Generated Distribution and Associated Inference*, Stat. Method, 6, (2009), 344-362.