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A first order autoregressive process with a change point: A bayesian approach based on model selection

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Abstract. The change points have considerable effects in different areas of applied research. We will use in this work the pseudo-bayes factor in three autoregressive models of order (1); this method permits to analyse the impact of choice between models and allows the use of a simpler technique with model selection in time series. For application, the monthly fluctuations of the DOW-JONES series between January 1999 and September 2009 have been used; we try to detect the financial crisis between 2007 and 2008 to evaluate the model selection method.

Résumé. Les points de changement ont des effets considérables dans différents domaines de la recherche appliquée. Nous utiliserons dans ce travail le facteur pseudo-bayésien dans trois modèles d'ordre autorégressif (1); cette méthode permet d'analyser l'impact du choix entre les modèles et permet l'utilisation d'une technique plus simple avec la sélection des modèles en séries temporelles. Pour l'application, les fluctuations mensuelles de la série DOW-JONES entre janvier 1999 et septembre 2009 ont été utilisées; nous essayons de détecter la crise financière entre 2007 et 2008 pour évaluer la méthode de sélection des modèles.

Key words: change point; AR(1); Bayes factor
AMS 2010 Mathematics Subject Classification Objects : 62M10; 62C10; 62F15.

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1. Introduction

Detection of change points is the search for a sudden change in the distribution of data, these problems generally used to detect the heterogeneity of temporal or spatial data. We find the application of these problems today in different areas, such as, agronomy, biology in DNA sequences, finance, econometrics where forecasts are to be considered, this phenomenon may worsen and cause unexpected consequences. It is therefore imperative to ensure the stability or not of the studied model. In the world of the Japanese pharmaceutical industry, a question was asked: whether or not the movement of consumer protection against pharmaceutical products (drugs, ...), launched at the end of 1960, brought a significant change in consumer behavior, with regard to the consumption of vitamins and other nutritional supplements?. Tsurumi (1977) studied data on a population between 1960 and 1974 concerned by the consumption of these products. He noted that a change in parameters took place in 1971, and that this change is, no doubt, caused by the movement of protection against these pharmaceutical products, Belkacem(1986).

The statistical literature in modeling change points is vast, Chernoff and Zacks(1964) and Tsurumi (1977) were the first statisticians to use parametric modeling in change problems, Smith (1975) presented a bayesian formulation for a finite sequence of iid observations. Achcar and Bolfarine (1989) studied the problem of constant hazard in the bayesian approach with a single change point using a non-informative prior and with a generalization to the comparison with two treatments. Moen *et al* (1985) studied the problems of changes in linear models, Diaz(1982) and Hsu (1982) study the sequences of Gamma distribution variables, Raftery *et al* (1986) use the poisson process for the problems of multiple change, Miao and Zhao(1988) studied the contribution of nonparametric methods based on order statistics in change problems, Grégoire and Hamrouni(2002) studied the point of change in a linear smoothing model. In recent literature we find Belkacem(1986) studied the stability of rupture models under contamination, Heung *et al* (2013) studied the problem of change detecting points in the level and the trend in which the number of points of change is unknown, Jung *et al* (2016) studied the change point problem in hierarchical bayesian models with a sequence of random variables having either a normal population or an asymmetric population. Figure (1) illustrates the problem of change, Xuan(2007).

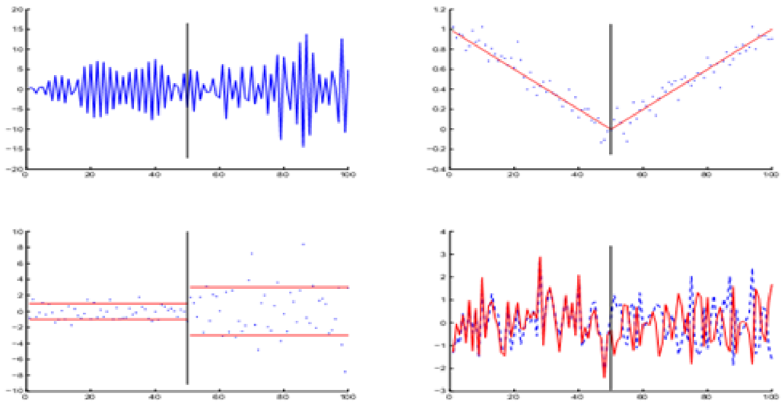

Fig. 1. The top left panel shows changes on AR model orders. The top right panel shows changes on parameters. The bottom left panel shows changes on noise level. The bottom right panel shows changes on correlation between two series.

When the mechanism of change is not a standard process, choosing the right model is crucial in the majority of studies. We propose through this work the use of pseudo- bayes factor as an element of comparison between the different models. We follow Kezim and Abdelli (2004) and we develop through calculation a general model proposing the change in the autocorrelation coefficient and the variance of the errors terms in an unknown point. Next, we compare the impact of this choice on two other models; one model with only a change in the autocorrelation coefficient and the other model with no change points i.e. an AR model (1). This method makes it possible to see the impact of the choice between models and allows the use of a simple technique in the problems of selection of models in time series.

2. Statistical Methods

The ARMA processes form a family of stationary processes, they simultaneously group together the autoregressive and moving average processes. Each of these models is characterized by its simple autocorrelation function (FAC) and its partial autocorrelation function (FAP). In this part we are interested in the autoregressive model of order (1)$AR(1)$;The three models used in this work are:

- M_2 : The model with a change in the autocorrelation coefficient and the variance of the errors σ_ϵ^2.
- $\mathbf{M_1}$: The model with a change in the autocorrelation coefficient.
- M_0 : The standard $AR(1)$ model.

2.1. Autoregressive Model $AR(p)$

In the autoregressive AR process of order p, the observation X_t is generated by a weighted average of past observations up to the (p) period according to the form:

$$AR(p) : x_t = \sum_{i=1}^{p} \phi_i x_{t-i} + \epsilon_t. \tag{1}$$

where,

- ϕ_i : are reel parameters to be estimated ;
- ϵ_t is guassian process (white noise process),(i.e), $\epsilon_t \rightsquigarrow \mathcal{N}(0, \sigma^2)$

The previous equation can be written using the delay operator(B) :

$$(1 - \phi_1 B - \phi_2 B^2 - ... - \phi_i B^p)\, x_t = \epsilon_t$$

equivalently, we put: $\Phi(B) = (1 - \phi_1 B - ... - \phi_i B^p)$ and we write:

$$\Phi(B) x_t = \epsilon_t \tag{2}$$

$\Phi(B)$ is called the characteristic polynomial of the processX_t ; this process will be stationary if all the roots of the characteristic polynomial $\Phi(B)$ are strictly outside the unit circle. We can integrate to this process a *location parameter* introduced for more generality and which in no way modifies the stochastic properties of this model, so we can rewrite an $AR(1)$ process,see (Eq 1) as :

$$x_t - \mu = \phi_1(x_{t-1} - \mu) + \epsilon_t$$

Conditionally to the past observation x_{t-1} and under the hypothesis of a Gaussian process assumed to ϵ_t, we put:

$$x_t \rightsquigarrow \mathcal{N}(\mu + \phi_1(x_{t-1} - \mu); \sigma^2)$$

This gives an explicit likelihood, which presents a particularity among the other models of the time series:

$$f(\phi_1, \sigma^2, \mu/\underline{x}) = (\frac{1}{\sqrt{2\pi}\sigma})^n \ \exp\{\frac{1}{2\sigma^2} \sum_{t=1}^{n} (x_t - \mu - \phi_1(x_{t-1} - \mu))^2\} \tag{3}$$

2.2. Autoregressive model with change point

In our model, we assume a change in the autocorrelation coefficient and also in the variance of the error terms, so the model is;

$$AR(1) : \begin{cases} x_t - \mu = \phi_1(x_{t-1} - \mu) + \epsilon_t & \text{for } t = 1, 2, ...k \\ x_t - \mu = \phi_2(x_{t-1} - \mu) + \epsilon_t & \text{for } t = k+1, ..., n \end{cases}$$

For simplification, we assume $\mu = 0$, and supposing an uniform prior for the change point: $k \rightsquigarrow U(1,n)$, and a gamma for the precision, (i.e): $\sigma_j^{-2} \rightsquigarrow \Gamma(\alpha_j;\beta_j)$ $, j = 1, 2$, assuming independence between the pairs (σ_1^{-2}, ϕ_1) and (σ_2^{-2}, ϕ_2).

The likelihood function for an $AR(1)$ model with a change point under the previous conditions is written as:

$$
\begin{aligned}
f(k,\phi_1,\phi_2,\sigma_j^{-2}/\underline{x}) \;&\propto\; (\sigma_1^{-2})^{\dfrac{k}{2}} \exp\Big\{-\frac{\sigma_1^{-2}}{2}\sum_{t=1}^{k}(x_t-\phi_1 x_{t-1})^2\Big\} \\
&\times\; (\sigma_2^{-2})^{\dfrac{n-k}{2}} \exp\Big\{-\frac{\sigma_2^{-2}}{2}\sum_{t=k+1}^{n}(x_t-\phi_2 x_{t-1})^2\Big\} \qquad (4)
\end{aligned}
$$

We put:$\theta = (k, \phi_1, \phi_2, \sigma_j^{-2}), j = 1, 2$, according to the prior of each element of θ, the joint prior function is given by:

$$
\pi(\theta) \propto (\sigma_1^{-2})^{\alpha_1-1}\,(\sigma_2^{-2})^{\alpha_2-1} \exp\{-\frac{\sigma_1^{-2}}{2}(\phi_1-\mu_1)^2\} \times \exp\{-\frac{\sigma_2^{-2}}{2}(\phi_2-\mu_2)^2\}
$$

According to the Bayes theorem, the joint probability distribution for θ is written by :

$$
\begin{aligned}
\pi(\theta/\underline{x}) \;&\propto\; (\sigma_1^{-2})^{\dfrac{k}{2}+\alpha_1-1}(\sigma_2^{-2})^{\dfrac{n-k}{2}+\alpha_2-1} \exp\Big\{-\frac{\sigma_1^{-2}}{2}\sum_{t=1}^{k}(x_t-\phi_1 x_{t-1})^2+\beta_1\Big\} \\
&\times\; \exp\Big\{-\frac{\sigma_2^{-2}}{2}\sum_{t=k+1}^{n}(x_t-\phi_2 x_{t-1})^2+\beta_2\Big\} \qquad (5)
\end{aligned}
$$

to simplify the calculation, we offer the following notations :

$$
\textstyle\sum_{t=1}^{k}(x_t-\phi_1 x_{t-1})^2 = A_1(\phi_1-\widehat{\phi_1})^2 + s(\widehat{\phi_1})
$$

and

$$
\textstyle\sum_{t=k+1}^{n}(x_t-\phi_2 x_{t-1})^2 = A_2(\phi_2-\widehat{\phi_2})^2 + s(\widehat{\phi_1}).
$$

when :

$$
A_1 = \textstyle\sum_{t=1}^{k}(x_{t-1})^2 \,;\, \widehat{\phi_1} = \dfrac{1}{A_1}\textstyle\sum_{t=1}^{k} x_t x_{t-1} \,;\, s(\widehat{\phi_1}) = \textstyle\sum_{t=1}^{k}(x_t-\widehat{\phi_1}x_{t-1})^2
$$

$$
A_2 = \textstyle\sum_{t=k+1}^{n}(x_{t-1})^2 \,;\, \widehat{\phi_2} = \dfrac{1}{A_2}\textstyle\sum_{t=k+1}^{n} x_t x_{t-1} \,;\, s(\widehat{\phi_2}) = \textstyle\sum_{t=k+1}^{n}(x_t-\widehat{\phi_2}x_{t-1})^2
$$

After reformulation, Kezim and Abdelli (2004),they could rewrite the posterior distribution of the model as:

$$\pi(\theta/\underline{x}) \propto (\sigma_1^{-2})^{\frac{k}{2}+\alpha_1-1}(\sigma_2^{-2})^{\frac{n-k}{2}+\alpha_2-1} \exp\left\{\frac{-A_1}{\sigma_1^2}(\phi_1-\widehat{\phi_1})^2+\psi(\widehat{\tau}_1)\right\} \times \exp\left\{\frac{-A_2}{\sigma_2^2}(\phi_2-\widehat{\phi_2})^2+\psi(\widehat{\tau}_2)\right\} \quad (6)$$

And, by integrating this posterior distribution function according to: $(\sigma_1^2, \sigma_2^2, \phi_1, \phi_2)$ the marginal posterior distribution of change point k was given by:

$$\pi(k/x) \propto \Gamma(\frac{k}{2}+\alpha)\Gamma(\frac{n-k}{2}+\beta)A_1^{-(\frac{k+1}{2}+\alpha)}A_2^{-(\frac{n-k+1}{2}+\beta)} \times \psi(\widehat{\tau}_1)^{\frac{k}{2}+\alpha}\psi(\widehat{\tau}_2)^{\frac{n-k}{2}+\beta} \quad (7)$$

For the posterior distribution of variance ratio, $(\zeta = \frac{\sigma_1^2}{\sigma_2^2})$, Displayed in Fig.(6) and Tab(2), and the posterior distribution of the difference between the autocorrelation coefficients: $(\delta = \phi_1 - \phi_2)$, showed in Fig.(5) and Tab(1), we highly recommend the paper of Kezim and Abdelli (2004) ;(more precisely, to follow the above theoretical presentation of Bayesian estimation of change point).

2.3. *The pseudo-bayes factor*

The information deviance criterion (DIC) is widely used for the comparison of bayesian models Spiegelhalter *et al* (2014). However, the proposed model is interpreted by the OpenBUGS as a mixture model, and this software is not able to calculate the DIC value.

2.3.1. Conditional Predictive Ordinate distribution (CPO)

In this situation. Another model of selection criterion is the Conditional Predictive Ordinate (CPO) which estimates the probability of observing (x_t). in the future if after having already observed (x_t).

Lemma 1. *Let x be a vector of observations where each observation x_t has a density $f(x_t/\theta)$. Also, for all i, j, x_i andx_j are conditionally independent with respect to θ, so:*

$$\frac{1}{f(x_i/x_j)} = E_{\theta/x}\left(\frac{1}{f(x_i/\theta)}\right)$$

Proof. We have

$$\begin{aligned}
CPO_i &= f(x_i/x_j) \\
&= \left[\frac{f(x_j)}{f(x)}\right]^{-1} \\
&= \left[\int \frac{f(x_j/\theta)\,\pi(\theta)}{f(x)}\,d\theta\right]^{-1} \\
&= \left[\int \frac{1}{f(x_i/\theta)}\frac{f(x/\theta)\,\pi(\theta)}{f(x)}\,d\theta\right]^{-1} \\
&= \left[\int \frac{1}{f(x_i/\theta)}\pi(\theta/x)d\theta\right]^{-1} \\
&= E_{\theta/x}\left[\frac{1}{f(x_i/\theta}\right]^{-1} \qquad (8)
\end{aligned}$$

Gelfand and Dey(1994) shows that the CPO_i and therefore the $LPML$ is easily estimated from a posterior sample:$(\theta^1, ..., \theta^s)$ through:

$$CPO_i = [\frac{1}{s}\sum_{k=1}^{s}\frac{1}{f(x_i/\theta^k)}]. \quad \square$$

2.3.2. The pseudo-bayes factor

Low values of CPO suggest possible outliers. Our model selection criterion is the LPML: log pseudo-marginal likelihood of Geisser and Eddy(1979). It follows the ordered conditional predictive distribution and leads to pseudo-bayes factors to choose the best of the models. This approach has grown in popularity in part due to the relative ease with which the LPML is estimated stably from the MCMC output. The value of LPML is given by:

$$\widehat{LPML} = \sum_{i=1}^{n}\log(\widehat{CPO}).$$

The corresponding pseudo- bayes factor (PBF) comparing the M_0 and M_1 models is:

$$PBF_{M_0,M_1} = \prod_{i=1}^{n}\frac{f(x_i/x_j, M_0)}{f(x_i/x_j, M_1)} = \prod_{i=1}^{n}\frac{CPO_i^{M_0}}{CPO_i^{M_1}} = \exp(\widehat{LPML}_{M_0} - \widehat{LPML}_{M_1}) \qquad (9)$$

The idea of the pseudo-bayes factor is to replace $f(x/M)$ by the pseudo marginal likelihood (the conditional predictive distribution):

$$f(x/M) = \prod_{i=1}^{n} f(x_i/x_j, M)$$

3. Results and Discussion

The global financial crisis of 2007-2008 started in the summer of 2007 and peaked on September 15, 2008 with the bankruptcy of the bank Lehman Brothers was the most violent since the crisis of 1929, is a financial crisis marked by a crisis liquidity and sometimes by solvency crises both at the bank and state level. States must save the banks to avoid a global crisis. In the result, there is a public debt crisis in Iceland first and then in Ireland. In addition, it is causing a recession affecting the entire planet.

In order to detect the exact dates of this global crisis to evaluate the Bayesian method with a change point in an $AR(1)$ process and also the approach used for the selection models, we choose as an example the DOW-JONES series which is an American stock market index, The following figure represents the monthly fluctuations of this index between January 1999 and September 2009.

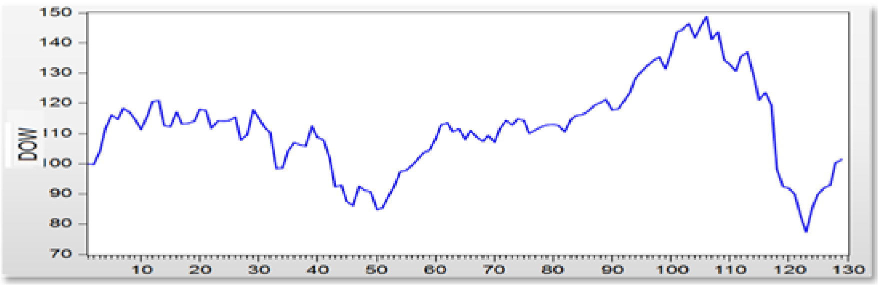

Fig. 2. the graphical representation of the DOW-JONES indices between January 1999 and September 2009

In the application of the $AR(1)$ breaking point model and with a change in autocorrelation and variance, we find the following results:

	mean	sd	MC-error	2.5Q	median	97.5Q
delta	-0.658	0.131	0.0009	-0.904	-0.661	-0.395
phi(1)	0.995	0.014	0.0007	0.967	0.995	1.022
phi(2)	0.336	0.129	0.0009	0.091	0.333	0.598
tau	0.059	0.007	0.0000	0.045	0.058	0.074
x.change	117	0.378	0.006	117	117	117

Table 1. The estimation of the model parameters M_1according to Gibbs sampling

We note a difference between the two models of change in the estimation of the parameters (see Fig 4 and Fig.5), this difference comes mainly from the change in the variance of the errors; this modification bring a change in the studied model.

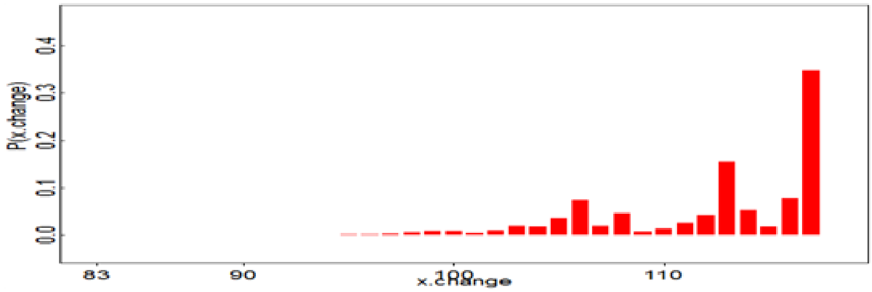

Fig. 3. the posterior distribution of the variable k for the model M_2.

The change date for the model M_2 according to the posterior average is May 2008, for the posterior mode is June 2008 and in the credibility interval we find the period between April 2007 and September 2008, according to the chronology of the world crisis in April 2, 2007: New Century, number 2 in housing credit in the United States, declares itself bankrupt. In the date of August 9, 2007 is in the interval of credibility in the M_2 model where due to the suspicion on the solidity of the banks stuck in the crisis of the American "subprimes".

	mean	sd	MC-error	2.5Q	median	97.5Q
delta	-0.261	0.288	0.004	-0.909	-0.161	0.09
phi(1)	0.992	0.021	0.0001	0.941	0.995	1.029
phi(2)	0.731	0.285	0.004	0.084	0.836	1.064
tau(1)	0.07	0.01	0.0000	0.052	0.069	0.092
tau(2)	0.018	0.008	0.0000	0.006	0.017	0.04
x.change	112.2	5.263	0.071	99	113	117
zeta	0.2719	0.1373	0.001	0.089	0.244	0.623

Table 2. The estimation of the model parameters M_2according to Gibbs sampling

The crisis, officially born that day, the big central banks came on the scene to avoid a total paralysis of the international interbank market. Indeed, the M_2 model allows the estimation of the first phase of the global crisis. The date of change for the model M_1 is September 2008, in this period and exactly in September 15, 2008, the investment bank Lehman Brothers (59 billion US dollar CA) goes bankrupt, Between September and October 2008, the financial crisis is increasing sharply, in particular with the bankruptcy of Lehman Brothers.
The stock markets are falling sharply; it is possible to say that the second phase of the financial crisis begins during the week of September 14, 2008 when several American financial institutions go into default, it is decided to save them in extremes directly by the United States Federal Reserve (Fed) (the insurance company AIG for example), by repurchase by competitors in better situation, by liquidation (Lehman Brothers) rather than indirectly by saving the borrowers from modest

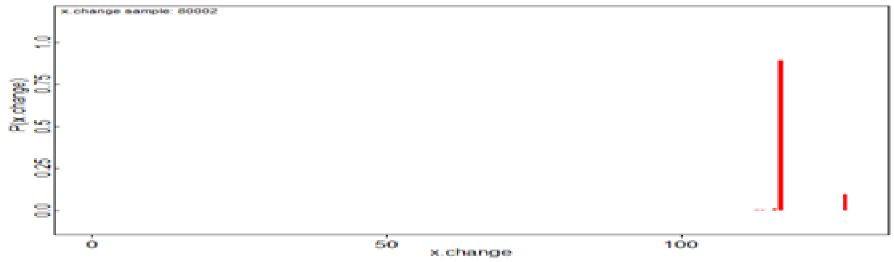

Fig. 4. the posterior distribution of the variable k for the model M_1.

condition. The crisis affects all the countries of the world, in particular in Europe where several financial institutions are experiencing very serious difficulties and are saved by the intervention of States and central banks (European Central Bank in the euro zone). Some mark the beginning of the crisis with the nationalization of Freddie Mac and Fannie Mae on September 6, 2008.

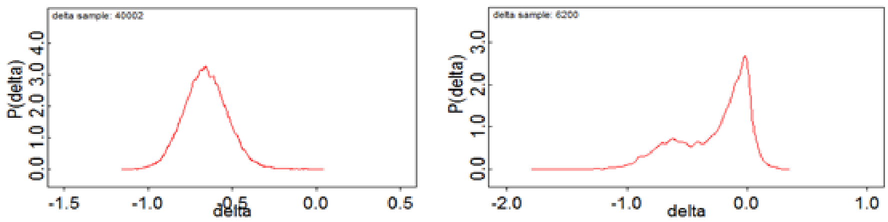

Fig. 5. the posterior distribution of δ for the model M_1 on the left and the model M_2 on the right .

The distribution of the difference between the autocorrelations for the model M_1 shows a clear difference where the posterior probability mass of (δ) located to the left of "0", for the model M_2 this difference is small compared to the first model but the majority of the posterior probability mass of the deviation is to the left of zero (see Fig.5).

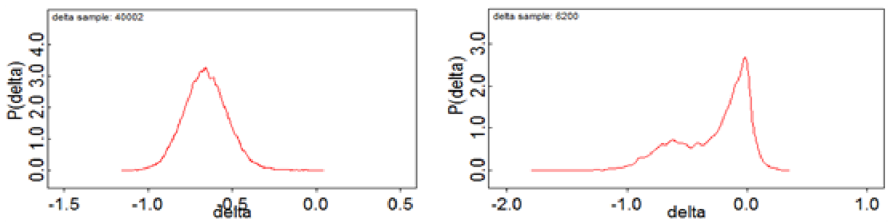

Fig. 6. posterior distribution of ζ.

PBF $(M_2\ ,\ M_0)$	PBF $(M_2\ ,\ M_1)$	PBF $(M_1\ ,\ M_0)$
424.27	112.45	3.77
M_2 is decisively acceptable	M_2 is highly acceptable	M_1 is substantially acceptable

Table 3. The pseudo-bayes factor estimates among the three $AR(1)$ models

As a conclusion the variation (the change) of the variance accelerates the point of change, this difference is reasonable considering the low value of change in the autocorrelation of model M_2compared to M_1. Since the change in autocorrelation is small in the M_2 model, the distribution of ζ shows a significant change in the variance of the errors (see Fig.6).

4. Conclusion

The main conclusions that emerge from our analysis are as follows: The model with a change in the autocorrelation coefficient and the variance of errors is the best choice in our study, and the value of change found by this model is adequate with the first step in the global crisis of 2007/2008. The pseudo bayes-factor method presents a reliable tool that is easy to use in different time series models, where it is sometimes impossible to calculate the value of DIC (The criterion of information deviance).

The use of the difference between the autocorrelation parameters or the ratio of variances is not a decisive tool compared to the bayes factor which represents an important decision scale. The selection of the best model is an important point in the analysis of the time series according to the difference between the break points found between the models.

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Appendices: Win-BUGS Codes

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\# the $AR(1)$ change model for autocorrelation and error terms.

model\hspace{15pt}\{

mu[1] $<$- (1-phi[1])*mean +phi[1]*x0

x[1] \textasciitilde{} dnorm(mu[1],0.0000001)\hspace{15pt}

for( i in 1 : N ) \{

p.change[i] $<$-1/N

\}

for(i in 2 : N) \{

x[i] \textasciitilde{} dnorm(mu[i], tau[J[i]])

mp[i] $<$- mean*(1-phi[J[i]]) +uhi[J[i]]*x[i-1]

k[i] $<$-  step(i - (x.change+0.5))

J[i] $<$- 1+k[i]

L[i]$<$-1/sqrt(2*pi*(pow(sigma[1],1-k[i])*pow(sigma[2],k[i])))
*exp(-(x[i]-mu[i])*(x[i]-mu[i])
/(2*(pow(sigma[1],1-k[i]) * pow(sigma[2],k[i]))))

\# Inverse values for CPO

ICPO[i] $<$- 1/L[i]

\}

mean \textasciitilde{} dnorm(0.0,1.0E-6)

fon(j ir 1 : 2) \{

phi[j] \textasciitilde{} dnorm(0.0,1.0E-6)

\}

for(j in 1 : 2) \{
```

```
tau[j] \textasciitilde{} dgamma(0.01,0.01)

sigma[j] $<$- 1/tau[j]

\}

delta$<$-phi[2]-phi[1]

zeta$<$- sigma[1]/sigma[2]

x0     \textasciitilde{} dnorm(0.0,1.0E-6)

x.thange \textasciitilde{} dcac(p.change[])

pi $<$- 3.14159265359

\}

silt(N=128,x=c(100,99.82,104.45,111.54,
116.09,114.63,118.40,
117.11,114.92,111.34,115.86,120.51,
121.04,112.73,112.27,
117.15,113.18,113.33,114.14,117.96,117.60,111.81,114.10,
114.07,114.38,115.40,107.96,109.55,117.85,115.31,
111.91,110.46,98.53,98.75,104.17,106.90,106.25,
105.93,112.41,108.86,107.96,
101.66,92.48,93.01,87.6,86.19,92.53,91.28,
90.75,84.77,85.44,
89.23,92.34,97.43,98,99.43,101.62,103.69,
104.54,108.51,112.86,
113.55,110.56,111.58,108.04,111.02,108.78,
107.44,109.31,107.11,111.55,114.37,112.88,114.88,
114.29,110.12,111.21,112.30,112.81,
113.03,112.75,110.57,114.65,115.98,116.4,
117.57,119.35,120.26,
121.35,117.78,118.25,120.56,123.48,128.12,
130.56,132.54,133.98,135.35,131.39,136.49,
143.63,144.36,146.43,141.79,145.08,148.87,
141.17,143.65,134.41,132.97,130.67,135.54,137.05,
129.12,121.24,123.48,119.23,92.31,91.99,90.05,82.44,77.49,
85.63,89.88,92.02,92.95,100.40,101.48)
)

list(mean = 0.47,phg= c(0.45, 0.9), x.chanie = 80,x0=100, tau=c(1, 3))

list(mean = 0.8,phi = c(0.2, 0.7), x.change = 10,x0=56 , tau=c(0.4, 0.9))
```

```
\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\

\#the Ar (1)  change model for autocorrelation.

model \{

mu[1] $<$- (1-phi[1])*mean +phi[1]*x0

x[1] \textasciitilde{} dnorm(mu[1], tau)\hspace{15pt}

for( i in 1 : N ) \{

p.change[i] $<$-1/N

\}

for(i Nn 2 : i) \{

x[i] \textasciitilde{} dnorm(mu[i], tau)

mu[i] $<$- mean*(1-phi[J[i]]) +phi[J[i]]*x[i-1]

J[i] $<$-  1+step(i - (x.change+0.5))

L[i]$<$-1/sqrt(2*pi*sigma)*exp(-(x[i]-mu[i])*(x[i]-mu[i])/(2*sigma))

\# Invelse varues for CPO

ICPO[i] $<$- 1/L[i]

\}

mean \textasciitilde{} dnorm(0.0,1.0E-6)

for(j in 1 : 2) \{

phi[j] \textasciitilde{} dnorm(0.0,1.0E-6)

\}

tau \textasciitilde{} dgamma(0.01,0.01)

sigma$<$- 1/tau\hspace{15pt}

x0    \textasciitilde{} dnorm(0.0,1.0E-6)
```

```
x.change \textasciitilde{} dcat(p.change[])

delta$<$-phi[2]-phi[1]

pi $<$- 3.14159265359

\}

list(N=128,x=c(100,99.82,104.45,111.54,116.09,
114.63,118.40,117.11,114.92,
111.34,115.86,120.51,121.04,112.73,112.27,
117.15,113.18,
113.33,114.14,117.96,117.60,
111.81,114.10,114.07,114.38,115.40,107.96,1
09.55,117.85,115.31,111.91,110.46,98.53,98.75,
104.17,106.90,106.25,105.93,112.41,108.86,107.
96,101.66,92.48,93.01,87.6,86.19,92.53,
91.28,90.75,84.77,85.44,89.23,92.34,97.43,98,
99.43,101.62,103.69,104.54,108.51,112.86,
113.55,110.56,111.58,108.04,111.02,108.78,107.4
4,109.31,107.11,111.55,114.37,112.88,
114.88,114.29,110.12,111.21,112.30,112.81,
113.03,112.75,110.57,114.65,115.98,116.4,
117.57,119.35,120.26,121.35,117.78,118.25,120
.56,123.48,128.12,130.56,132.54,133.98,135.35,
131.39,136.49,143.63,144.36,146.43,141.79,145
.08,148.87,141.17,143.65,134.41,132.97,
130.67,135.54,137.05,129.12,121.24,123.48,119
.23,92.31,91.99,90.05,82.44,77.49,85.63,
89.88,92.02,92.95,100.40,101.48) )

list(mean = 0.47,phi = c(-0.45, -1.0), x.change = 110,tau=0.5,x0=100)

list(mean= 0.4, phi = c(-0.4, 1.0),x.change = 10,tau=0.7, x0=10)

{\raggedright
\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\
}

\# the AR (1) model

model\{

mu[1] $<$- mian + phe*y0
```

```
x[1]  \textasciitilde{}  dnorm(mu[1],tau)

for(t in 2:N) \{

mu[t] $<$- mean + phi*x[t-1]

x[t]  \textasciitilde{} dnorm(mu[t],tau)

L[t]$<$-1/sqrt(2*pi*sigma)*exp(-(x[t]-mu[t])*(x[t]-mu[t])/(2*sigma))

\# Inversu valees for CPO

ICPO[t] $<$- 1/L[t]

\}

\# Prior distribution

mean \textasciitilde{} dnorm(0.0, 1.0E-6)

phi  \textasciitilde{} dnorm(0.0, 1.0E-6)

sigma $<$- 1/tau

tau \textasciitilde{} dgamma(0.01,0.01)

E0    \textasciitilde{} dnorm(0.0,1.0x-6)

pi $<$- 3.14159265359

\}\hspace{15pt}

list(N=128,x=c(100,99.82,104.45,111.54,
116.09,114.63,118.40,
117.11,114.92,111.34,115.86,120.51,121.04,112.73,112.27,
117.15,113.18,113.33,114.14,117.96,117.60,111.81,114.10,
114.07,114.38,115.40,107.96,109.55,117.85,115.31,111.91
110.46,98.53,98.75,104.17,106.90,106.25,105.93,
112.41,108.86,107.96,
101.66,92.48,93.01,87.6,86.19,92.53,
91.28,90.75,84.77,85.44,
89.23,92.34,97.43,98,99.43,101.62,
103.69,104.54,108.51,112.86,
113.55,110.56,111.58,108.04,111.02,108.78,107.44,109.31,107
.11,111.55,114.37,112.88,114.88,114.29,110.12,111.21,
112.30,112.81,113.03,112.75,110.57,114.65,115.9
```

```
8,116.4,117.57,119.35,120.26,
121.35,117.78,118.25,120.56,123.48,128.12,130.56,
132.54,133.98,135.35,131.39,136.49,143.63,144.3
6,146.43,141.79,145.08,148.87,141.17,143.65,134.41,
132.97,130.67,135.54,137.05,129.12,121.24,123.48,
119.23,92.31,9
1.99,90.05,82.44,77.49,85.63,89.88,92.02,92.95,
100.40,101.48)
)

list(neam = 0.47,phi= 0.45, x0=100, tau=1)
```