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CS-PALT Combined With Different Censoring Techniques On Gompertz Distribution: Some Inferences

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Abstract. The major objective of the present article is to study the effect on Bayes risks for different censoring techniques combined with constant-stress partially accelerated life test. The underlying distribution of this study was considered as two-parameter Gompertz distribution. The Bayes risk under invariant LINEX loss function has been obtained for the parameters under study. The censoring techniques have been used here, Type-I progressive hybrid, Progressive Type-II, Type-I progressive and Type-II censoring. Numerical illustration based on real and simulated data has been carried out. The Metropolis-Hastings algorithm has also been combined with the simulation study for improvement in precision of inferences.

Key words: Constant-Stress Partially Accelerated Life Test (CS-PALT), Type-I Progressive Hybrid (T-IPH) Censoring, Progressive Type-II (PT-II) Censoring, Type-I Progressive (T-IP) Censoring, Type-II (T-II) Censoring; Invariant LINEX Loss function (ILLF).
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Résumé. L'objectif principal du présent article est d'étudier l'effet de différentes techniques de censure sur les risques bayésiens lorsqu'elles sont combinées dans un test de survie partiellement accéléré sous contrainte constante. La distribution sous-jacente de cette étude a été considérée comme une distribution de Gompertz à deux paramètres. Le risque Bayes la sous fonction de perte LINEX invariante a été determimé pour les paramètres étudiés. Les techniques de censure utilisées ici sontM: la hybridation progressive de type I, la censure progressive de type II, la censure progressive de type I et la censure de type II. Une illustration numérique basée sur des données réelles et simulées a été réalisée. L'algorithme de Metropolis-Hastings a également est utilisé dans l'étude de simulation pour l'amélioration de la précision des inférences.

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1. Introduction

Gompertz (1875) introduced the distribution named after him. That is used as an enduring model in reliability and survival analysis. It also plays an important role in modeling human mortality and fitting actuarial tables. This distribution is a uni-model and has an increasing hazard rate function with positive skewness. Present distribution does not seem to have received enough consideration for a long period of time. However, in recent years, many authors have considered Gompertz distribution for the studies of statistical methodology and characterization.

Few important and recent studies are included here only. Some applications and surveys for the Gompertz model was discussed by Al-Hussaini et *al.* (2000). Wu *et al.* (2003) was developed an exact joint confidence region along with exact confidence intervals for the parameters of the Gompertz distribution by using first-failure censoring plan. Wu *et al.* (2004) discussed about the estimation of parameters by using least square method, whereas Soliman *et al.* (2012) used the progressive first-failure censored data for estimating the parameters.

Ismail (2010) proposed some Bayes estimation for unknown parameters under the partially accelerated life tests on Type-I censoring for Gompertz distribution. Prakash (2015) presents a discussion on one-sample Bayes prediction bound length under different censoring plans for the Gompertz distribution. Prakash (2016) presents some inference on progressive Type-II censored Gompertz data by using random removal scheme. Singh *et al.* (2016) presents different methods of estimation for Gompertz distribution when the available data are in the form of fuzzy numbers. Al-Omari (2016) studied the Gompertz Distribution under interval censored data by using MCMC.

If we assumed that, the parameters θ and σ are denoted as shape and scale parameter respectively. Then the probability density function of the two-parameter Gompertz distribution is defined as

$$f\left(x;\theta,\sigma\right)=\sigma e^{\theta x}exp\left(-\frac{\sigma}{\theta}\left(e^{\theta x}-1\right)\right)\ ;\ x>0,\sigma>0,\theta>0. \tag{1}$$

It is noted here that, for $\theta\rightarrow 0$, the Gompertz distribution is a particular case of an Exponential distribution.
Although the T-IPH censoring scheme is relevant, most of the preceding work under CS-PALT were deliberated using the usual time and failure censoring schemes and no consideration has been provided in investigating hybrid censored data in a Bayesian framework. The focus of this article is on combining these two different approaches and study there fruitfulness in terms of Bayes risks. The Bayes risks of unknown parameters have been investigated under the invariant LINEX loss function. Type-II progressive, Type-I progressive and Type-II censoring cases are also considered here in combination with CS-PALT for comparison of the Bayes risks. Numerical illustration based on real and simulated data has been carried out. For the improvement in precision of the inferences, the Metropolis-Hastings (M-H) algorithm has also been combined with simulation study.

2. Type-I Progressive Hybrid censoring

The most common censoring schemes discussed in literature are Type-I and Type-II censoring. The experiment lasts up to a pre-specified time $t(>0)$ in traditional Type-I censoring and any failures that happen after $t(>0)$ are not observed. The end point $t(>0)$ of the experiment is independent of the failure times of the items. Whereas in Type-II censoring, the experimenter finishes the experiment after a pre-identified number of units $m(\leq n)$ fail. In this censoring, only the lowest lifetimes are noticed. The major difference in both censoring is, the number of failures witnessed is random and the end point of the experiment is fixed in Type-I censoring whereas in Type-II censoring, the number of failures is fixed and the termination time is random.

These censoring schemes are not a good choice in life testing of highly reliable products or in case where the needs of the units for other services during the experiment. A better choice is progressive Type-II (PT-II) censoring, which provides the flexibility of removal of units at each intermediate failure under Type-II censoring framework. In PT-II censoring, total of n test units are placed on a life test and the trial stops at the occurrences of $m^{th}(\leq n)$ pre-assumed failure, with a pre-fixed censoring pattern $R\equiv(r_1,r_2,...,r_m)$ follows $n-m=r_1+r_2+...+r_m$ (See Prakash (2015) for more details on PT-II censoring).

Our concerned is of Type-I progressive hybrid (T-IPH) censoring, which is the combination of traditional Type-I progressive (T-IP) and Progressive Type-II (PT-II) censoring. In this censoring, the test ends either at $m^{th}(\leq n)$ failure or at time t with a number of failures X_j, which one occurred earlier. The number of failure m and end time t both are pre-fixed in advance. In case, the experiment stops at time t with a number of failures X_j, then they must satisfy the condition $X_j < t < X_{j+1}$, and remaining live test units $n - R_1 - R_2 - ...R_j - j = R*$ (say) are removed from the test. So, the observed sample may be one of the following two types:

$$\begin{cases} I: & X_1, X_2, ..., X_\epsilon, X_{\epsilon+1}, ..., X_m \,;\; if\; X_m < t \\ II: & X_1, X_2, ..., X_\epsilon, X_{\epsilon+1}, ..., X_j \,;\; if\; X_j < t < X_{j+1}. \end{cases} \tag{2}$$

Several literature available on hybrid censoring, but none of study has found on the combination of constant-stress partially accelerated life test with Type-I hybrid progressive censoring. Selective important and recent studies on hybrid censoring are deliberated here. Kundu (2006) discussed the behaviors of the maximum likelihood estimators and Bayes estimators of unknown parameters under the hybrid censoring for the Exponential distribution.Balakrishnan and Cramer (2014) have discussed more informative literature on hybrid censoring in his book. Adaptive Type-I progressive hybrid censoring on Exponential distribution was considered by Lin and Huang (2012) and study about the exact distribution of the MLE, asymptotic distribution, confidence intervals for failure rate using exact distribution and bootstrap resampling methods.

Some Bayesian inferences on Type-I progressive hybrid censoring have discussed by **?** for Burr Type-XII distribution. Some maximum likelihood and Bayes estimation on Type-II progressive hybrid censoring scheme for the Pareto distribution was suggested by Mohie El-Din *et al.* (2016). Recently, Abushal (2018) has studied the problem of estimation of unknown parameters of the Weibull-Exponential distribution under Type-I hybrid censored data. Bayes prediction bound lengths for Pareto Type-II data ware recently discussed by Prakash (2019.) by using Type-I progressive hybrid censoring.

3. Constant-Stress Partially Accelerated Life Test

The usual life testing methods of high reliability products required a long period to gain sufficient failure data to make inferences. In reliability analysis, the accelerated life test is the common way to measure such products life. Under accelerated life test settings, products are tested at higher than usual levels of stress to bring failures speedily and economically. Applying an accelerated life test depends on a life and stress relationship. The parameters under study can be estimated through this relationship from the failure data obtained under the accelerated conditions. The partially accelerated life test is a good application to implement the life test in some test situations where such relationship can't be assumed or known.

The stress can be applied in different ways, common techniques are constant-stress and step-stress. The present study is dedicated to constant-stress under partially accelerated life test. In CS-PALT, run each unit at either normal (use) condition or accelerated condition only, i.e. each unit is run at a constant-stress level until the test is finished. The accelerated stresses include higher than normal power, load, temperature, pressure etc. (Nelson (1990)). Numerous literature available on CS-PALT scenario. See Prakash (2017) and Prakash (2018) for more important and recent studies on CS-PALT.

In the present article, we deal with Type-I progressive hybrid censored CS-PALT's when the lifetime of test unit follows Gompertz distribution. We first assume, total of n test units is used for the life test, in which n_1 test units are selected randomly from total test units, run at the normal test condition with lifetimes $X_{1i}; \forall\, i = 1, 2, ..., n_1$, follows Gompertz distribution having probability density function

$$f_1 = \sigma e^{\theta x_1} exp\left(-\frac{\sigma}{\theta}\left(e^{\theta x_1} - 1\right)\right). \tag{3}$$

Remaining $n_2 (= n - n_1)$ test units are kept on accelerated test condition with lifetimes $X_{2i}; \forall\, i = 1, 2, ..., n_2$, follows Gompertz distribution. Substituting $X_1 = \beta X_2$, the probability density function under the accelerated test condition is obtained as

$$f_2 = \sigma\beta e^{\theta\beta x_2} exp\left(-\frac{\sigma}{\theta}\left(e^{\theta\beta x_2} - 1\right)\right). \tag{4}$$

Here, $\beta (> 1)$ is the acceleration factor. Since, T-IPH censoring is the combination of Type-I Progressive (T-IP) and progressive Type-II (PT-II) censoring. Hence, total of n_1 items has been kept on normal test condition and the test terminated either at m_1^{th} failure or at the time t_1 with number of failure item j_1 which one occurred earlier. Similarly, n_2 items kept under the accelerated condition and the test terminated either at m_2^{th} failure or at the time t_2 with number of failure item j_2 which one occurred earlier.

Case I: For Progressive Type-II (PT-II) censoring:

If the test will terminate at m_1^{th} failure of the normal test condition and at m_2^{th} failure for the accelerated condition, then the joint probability function under CS-PALT by using PT-II censoring is given as

$$\left\{ I:\ L \propto \prod_{i=1}^{m_1}\left(f_1\left(1 - F_1\right)^{R_{1i}}\right) \times \prod_{i=1}^{m_2}\left(f_2\left(1 - F_2\right)^{R_{2i}}\right) \right. \tag{5}$$

Here, $R_{1i} \cong (r_1, r_2, ..., r_{m_1})$ and $R_{2i} \cong (r_1, r_2, ..., r_{m_2})$ be the different progressive censoring patterns for n_1 and n_2 items respectively, and are pre-fixed in advance.

Case II: For Type-I Progressive (T-IP) censoring:

In this case the test terminated at the time t_1 and t_2 with the number of failures j_1 and j_2 respectively for the normal and accelerated test condition. Hence, the joint probability function on CS-PALT by using T-IP censoring is given as

$$\begin{cases} II: \ L \propto \prod_{i=1}^{j_1} \left(f_1 \left(1 - F_1\right)^{R_{1i}} \left(1 - F_1(t_1)\right)^{R_1 *} \right) \\ \qquad \times \prod_{i=1}^{j_2} \left(f_2 \left(1 - F_2\right)^{R_{2i}} \left(1 - F_2(t_2)\right)^{R_2 *} \right). \end{cases} \tag{6}$$

Here, $R_1 * = n_1 - \sum_{i=1}^{j_1} r_i - j_1$ and $R_2 * = n_2 - \sum_{i=1}^{j_2} r_i - j_2$.

Case III: For PT-II & T-IP censoring both:

If the test will terminate at the m_1^{th} failure of the normal test condition and at the time t_2 with the number of failures j_2 for the accelerated test condition. Then the joint probability function under CS-PALT is given as

$$\begin{cases} III: \ L \propto \prod_{i=1}^{m_1} \left(f_1 \left(1 - F_1\right)^{R_{1i}} \right) \times \prod_{i=1}^{j_2} \left(f_2 \left(1 - F_2\right)^{R_{2i}} \left(1 - F_2(t_2)\right)^{R_2 *} \right). \end{cases} \tag{7}$$

Case IV: For T-IP & PT-II censoring both:

If the test will terminate at the time t_1 with the number of failures j_1 of normal test condition and at m_2^{th} failure for accelerated test condition, then the joint probability function under CS-PALT is given as

$$\begin{cases} IV: \ L \propto \prod_{i=1}^{j_1} \left(f_1 \left(1 - F_1\right)^{R_{1i}} \left(1 - F_1(t_1)\right)^{R_1 *} \right) \times \prod_{i=1}^{m_2} \left(f_2 \left(1 - F_2\right)^{R_{2i}} \right). \end{cases} \tag{8}$$

Hence, the joint probability density (likelihood) function is now rewritten under CS-PALT by combining T-IPH censoring, as

$$\begin{cases} I: & L \propto \prod_{i=1}^{m_1} \left(f_1 \left(1 - F_1\right)^{R_{1i}} \right) \times \prod_{i=1}^{m_2} \left(f_2 \left(1 - F_2\right)^{R_{2i}} \right) \\ II: & L \propto \prod_{i=1}^{j_1} \left(f_1 \left(1 - F_1\right)^{R_{1i}} \left(1 - F_1(t_1)\right)^{R_1 *} \right) \\ & \qquad \times \prod_{i=1}^{j_2} \left(f_2 \left(1 - F_2\right)^{R_{2i}} \left(1 - F_2(t_2)\right)^{R_2 *} \right) \\ III: & L \propto \prod_{i=1}^{m_1} \left(f_1 \left(1 - F_1\right)^{R_{1i}} \right) \times \prod_{i=1}^{j_2} \left(f_2 \left(1 - F_2\right)^{R_{2i}} \left(1 - F_2(t_2)\right)^{R_2 *} \right) \\ IV: & L \propto \prod_{i=1}^{j_1} \left(f_1 \left(1 - F_1\right)^{R_{1i}} \left(1 - F_1(t_1)\right)^{R_1 *} \right) \times \prod_{i=1}^{m_2} \left(f_2 \left(1 - F_2\right)^{R_{2i}} \right) \end{cases} \tag{9}$$

Simplifying Eq. (9), the joint density function on CS-PALT under T-IPH censoring is obtained as

$$L \propto \beta^{\delta}\sigma^{d}e^{\theta\omega_0}e^{-\frac{\sigma}{\theta}\omega_1}\,; \tag{10}$$

where $d = \begin{cases} I: & m_1+m_2 \\ II: & j_1+j_2 \\ III: & m_1+j_2 \\ IV: & j_1+m_2 \end{cases}$, $T_{0l} = \sum_{i=1}^{v_1} x_{1i} + \beta\sum_{i=1}^{v_2} x_{2i}$,

$W_1 = \sum_{i=1}^{m_1}(1+R_{1i})\left(e^{\theta x_{1i}}-1\right)$, $\delta = \begin{cases} I: & m_2 \\ II: & j_2 \\ III: & j_2 \\ IV: & m_2 \end{cases}$, $\omega_0 = \begin{cases} I: & T_{01} \\ II: & T_{02} \\ III: & T_{03} \\ IV: & T_{04} \end{cases}$,

$v_1 = \begin{cases} I: & m_1 \\ II: & j_1 \\ III: & m_1 \\ IV: & j_1 \end{cases}$, $W_2 = \sum_{i=1}^{m_2}(1+R_{2i})\left(e^{\theta\beta x_{2i}}-1\right)$, $W_3 = \sum_{i=1}^{j_1}(1+R_{1i})\left(e^{\theta x_{1i}}-1\right) + R_1 * \left(e^{\theta t_1}-1\right)$, $v_2 = \begin{cases} I: & m_2 \\ II: & j_2 \\ III: & j_2 \\ IV: & m_2 \end{cases}$, $W_4 = \sum_{i=1}^{j_2}(1+R_{2i})\left(e^{\theta\beta x_{2i}}-1\right) + R_2 * \left(e^{\theta\beta t_2}-1\right)$,

$\omega_1 = \begin{cases} I: & W_1+W_2 \\ II: & W_3+W_4 \\ III: & W_1+W_4 \\ IV: & W_3+W_2 \end{cases}$ and $l = 1, 2, 3, 4$.

4. Bayes Estimation under T-IPH Censoring

Prakash (2018) considered one parameter Gamma distribution as the natural family of conjugate prior for Gompertz distribution by considering one of the parameter of the Gompertz distribution as a random variable. Hence, the prior distributions are

$$\pi_\theta \propto \theta^{\alpha-1}e^{-\theta}\,; \alpha > 0 \tag{11}$$

and

$$\pi_\sigma \propto \sigma^{\lambda-1}e^{-\sigma}\,; \lambda > 0. \tag{12}$$

For the acceleration factor β, a vague prior is assumed, because the vague prior do not play any significant role in the analyses. Thus the joint prior distribution is given as

$$\pi_{(\theta,\sigma,\beta)} \propto \theta^{\alpha-1}\sigma^{\lambda-1}\beta^{-1}e^{-\theta-\sigma}. \tag{13}$$

The resultant joint and marginal posterior densities corresponding to the parameters θ, σ and β are obtained and given as

$$\pi^*_{(\theta,\sigma,\beta)} = \Omega\sigma^{d+\lambda-1}\theta^{\alpha-1}\beta^{\delta-1}e^{\theta(\omega_0-1)}exp\left\{-\sigma\left(\frac{\omega_1}{\theta}+1\right)\right\},$$

$$\pi^*_{(\theta)} = \Omega\theta^{\alpha-1}e^{-\theta}\int_\sigma\left\{\int_\beta \beta^{\delta-1}e^{\theta\omega_0}\, exp\left\{-\sigma\left(\frac{\omega_1}{\theta}+1\right)\right\}d\beta\right\}\sigma^{d+\lambda-1}d\sigma, \tag{14}$$

$$\pi^*_{(\beta)} = \Omega\beta^{\delta-1}\int_\sigma\left\{\int_\theta \theta^{\alpha-1}e^{\theta(\omega_0-1)}\, exp\left\{-\sigma\left(\frac{\omega_1}{\theta}+1\right)\right\}d\theta\right\}\sigma^{d+\lambda-1}d\sigma \tag{15}$$

and

$$\pi^*_{(\sigma)} = \Omega\sigma^{d+\lambda-1}e^{-\sigma}\int_\theta\left\{\int_\beta \beta^{\delta-1}e^{\theta\omega_0}\, exp\left\{-\sigma\frac{\omega_1}{\theta}\right\}d\beta\right\}\theta^{\alpha-1}e^{-\theta}d\theta \tag{16}$$

where $\Omega = \left\{\int_\sigma\left\{\int_\theta \frac{\int_\beta \beta^{\delta-1}e^{\theta(\omega_0-1)}exp\left\{-\sigma\left(\frac{\omega_1}{\theta}+1\right)\right\}d\beta}{\theta^{1-\alpha}}d\theta\right\}\sigma^{d+\lambda-1}d\sigma\right\}^{-1}$.

In case where the positive and negative errors have different consequences and/or when the overestimation is more serious than underestimation, or vice-versa, the use of symmetric loss function (squared error loss function) may not be appropriate in Bayesian estimation. Hence, in the present section, we considered a useful and flexible class of asymmetric loss function known as invariant LINEX loss function (Singh *et al.* (2007)). The ILLF is defined for any parameter Θ with its any estimate $\hat{\Theta}$ as

$$L(\partial*) = e^{a\partial*} - a\partial * -1\,; a \neq 0, \partial* = \frac{\hat{\Theta}-\Theta}{\Theta}. \tag{17}$$

Here, $'a'$ is the shape parameter of ILLF. (Follow Prakash and Singh (2009) for more details on ILLF). The Bayes estimator corresponding to the parameter $\Theta\,(=\theta,\sigma,\beta)$ under ILLF is obtained by simplifying following equality for each parameter respectively

$$\int_\Theta\left\{\frac{1}{\Theta}exp\left(-a\frac{\hat{\Theta}_l}{\Theta}\right)\right\}\pi^*_\Theta = e^a\int_\Theta\frac{1}{\Theta}\pi^*_\Theta\,;\Theta = \theta,\sigma,\beta. \tag{18}$$

The theoretical solution of the above equation for each parameter separately does not exist. The expressions for Bayes estimators and corresponding Bayes risks under ILLF are also not possible to solvable theoretical. For an analysis of the proposed methods a numerical method is applied here.

5. Others Censoring Patterns

5.1. Progressive Type-II Censoring (PT-II)

The joint probability function on CS-PALT under PT-II censoring is obtained by substituting $j_1 = 0 = j_2$ and $t_1 \to \infty \leftarrow t_2$ in Eq. (10),

$$L_p \propto \beta^{m_2} \sigma^{m_1+m_2} e^{\theta T_{01}}\, exp\left(-\frac{\sigma}{\theta}\left(W_1 + W_2\right)\right). \tag{19}$$

The joint and marginal posterior densities on PT-II censoring under CS-PALT, corresponding to the parameters θ, σ and β are obtained and given as

$$\pi^*_{(\theta,\sigma,\beta)} = \Omega_p \sigma^{m_1+m_2+\lambda-1} \theta^{\alpha-1} \beta^{m_2-1} e^{\theta(T_{01}-1)} exp\left\{-\sigma\left(\frac{W_1+W_2}{\theta} + 1\right)\right\},$$

$$\pi^*_{(\theta P)} = \Omega_p \theta^{\alpha-1} e^{-\theta} \int_\sigma \left\{ \frac{\int_\beta \beta^{m_2-1} e^{\theta T_{01}} exp\left\{-\sigma\left(\frac{W_1+W_2}{\theta}+1\right)\right\} d\beta}{\sigma^{-m_1-m_2-\lambda+1}} \right\} d\sigma, \tag{20}$$

$$\pi^*_{(\beta P)} = \Omega_p \beta^{m_2-1} \int_\sigma \left\{ \frac{\int_\theta \theta^{\alpha-1} e^{\theta(T_{01}-1)} exp\left\{-\sigma\left(\frac{W_1+W_2}{\theta}+1\right)\right\} d\theta}{\sigma^{-m_1-m_2-\lambda+1}} \right\} d\sigma \tag{21}$$

and

$$\pi^*_{(\sigma P)} = \Omega_p \sigma^{m_1+m_2+\lambda-1} e^{-\sigma} \int_\theta \left\{ \frac{\int_\beta \beta^{m_2-1} e^{\theta T_{01}} exp\left\{-\sigma\left(\frac{W_1+W_2}{\theta}\right)\right\} d\beta}{\theta^{1-\alpha} e^{\theta}} \right\} d\beta \tag{22}$$

where $\Omega_p = \left(\int_\sigma \frac{\left\{\int_\theta \left\{\int_\beta \beta^{m_2-1} e^{\theta(T_{01}-1)} exp\left\{-\sigma\left(\frac{W_1+W_2}{\theta}+1\right)\right\} d\beta\right\} \theta^{\alpha-1} d\theta\right\}}{\sigma^{-m_1-m_2-\lambda+1}} d\sigma \right)^{-1}$.

The Bayes estimator corresponding to parameter $\Theta\,(=\theta,\sigma,\beta)$ under ILLF is obtained by simplifying following equality for each parameter respectively

$$\int_\Theta \left\{ \frac{1}{\Theta} exp\left(-a\frac{\hat{\Theta}_{lP}}{\Theta}\right)\right\} \pi^*_{\Theta P} d\Theta = e^a \int_\Theta \frac{1}{\Theta} \pi^*_{\Theta P} d\Theta \;; \Theta\,(=\theta,\sigma,\beta). \tag{23}$$

5.2. Type-I Progressive Censoring (T-IP)

In this censoring pattern, the test terminated at time t_1 and t_2 with the number of failures j_1 and j_2 respectively for the normal and accelerated test condition respectively. The joint probability function by using T-IP censoring on CS-PALT is obtained by substituting $m_1 = n_1$ and $m_2 = n_2$ as

$$L_T \propto \beta^{j_2} \sigma^{j_1+j_2} e^{\theta T_{02}} exp\left(-\frac{\sigma}{\theta}\left(W_3 + W_4\right)\right). \tag{24}$$

The joint and marginal posterior densities on T-IP censoring under CS-PALT are obtained similarly for the parameters θ, σ and β as

$$\pi^*_{(\theta,\sigma,\beta)} = \Omega_T\, \sigma^{j_1+j_2+\lambda-1}\theta^{\alpha-1}\beta^{j_2-1}e^{\theta(T_{02}-1)}exp\left\{-\sigma\left(\frac{W_3+W_4}{\theta}+1\right)\right\},$$

$$\pi^*_{(\theta T)} = \Omega_T\, \theta^{\alpha-1}e^{-\theta}\int_{\sigma}\frac{\int_{\beta}\beta^{j_2-1}e^{\theta T_{02}}exp\left\{-\sigma\left(\frac{W_3+W_4}{\theta}+1\right)\right\}d\beta}{\sigma^{-j_1-j_2-\lambda+1}}d\sigma, \tag{25}$$

$$\pi^*_{(\beta T)} = \Omega_T\, \beta^{j_2-1}\int_{\sigma}\frac{\int_{\theta}\theta^{\alpha-1}e^{\theta(T_{02}-1)}exp\left\{-\sigma\left(\frac{W_3+W_4}{\theta}+1\right)\right\}d\theta}{\sigma^{-j_1-j_2-\lambda+1}}d\sigma \tag{26}$$

and

$$\pi^*_{(\sigma T)} = \Omega_T\, \sigma^{j_1+j_2+\lambda-1}e^{-\sigma}\int_{\theta}\frac{\int_{\beta}\beta^{j_2-1}e^{\theta T_{02}}exp\left\{-\sigma\left(\frac{W_3+W_4}{\theta}\right)\right\}d\beta}{\theta^{1-\alpha}\,e^{\theta}}d\theta; \tag{27}$$

where $\Omega_T = \left(\int_{\sigma}\frac{\left\{\int_{\theta}\left\{\int_{\beta}\beta^{j_2-1}e^{\theta(T_{02}-1)}exp\left\{-\sigma\left(\frac{W_3+W_4}{\theta}+1\right)\right\}d\beta\right\}\theta^{\alpha-1}d\theta\right\}}{\sigma^{-j_1-j_2-\lambda+1}}d\sigma\right)^{-1}$.

Further, the Bayes estimator corresponding to the parameter $\Theta\,(=\theta,\sigma,\beta)$ under ILLF is obtained again by simplifying given equality

$$\int_{\Theta}\left\{\frac{1}{\Theta}exp\left(-a\frac{\hat{\Theta}_{lT}}{\Theta}\right)\right\}\pi^*_{\Theta T}d\Theta = e^{a}\int_{\Theta}\frac{1}{\Theta}\pi^*_{\Theta T}d\Theta\;;\Theta\,(=\theta,\sigma,\beta)\,. \tag{28}$$

5.3. Type-II (T-II) Censoring

Usual Type-II (T-II) or item failure censoring scheme is a particular case of PT-II censoring by substituting $r_1 = r_2 = ... = r_{m_1-1} = 0 \Rightarrow R_{1i} = n_1 - m_1$ and $r_1 = r_2 = ... = r_{m_2-1} = 0 \Rightarrow R_{2i} = n_2 - m_2$. Therefore, the joint probability function under T-II censoring on above scenario is given as

$$L_I \propto \beta^{m_2}\sigma^{m_1+m_2}e^{\theta T_{01}}exp\left(-\frac{\sigma}{\theta}\left(W_5+W_6\right)\right). \tag{29}$$

On similar lines, the joint and marginal posterior densities are obtained as

$$\pi^*_{(\theta,\sigma,\beta)} = \Omega_I\sigma^{m_1+m_2+\lambda-1}\theta^{\alpha-1}\beta^{m_2-1}e^{\theta(T_{01}-1)}exp\left\{-\sigma\left(\frac{W_5+W_6}{\theta}+1\right)\right\},$$

$$\pi^*_{(\theta I)} = \Omega_I\, \theta^{\alpha-1}e^{-\theta}\int_{\sigma}\frac{\int_{\beta}\beta^{m_2-1}e^{\theta T_{01}}exp\left\{-\sigma\left(\frac{W_5+W_6}{\theta}+1\right)\right\}d\beta}{\sigma^{-m_1-m_2-\lambda+1}}d\sigma, \tag{30}$$

$$\pi^*_{(\beta I)} = \Omega_I\, \beta^{m_2-1}\int_{\sigma}\frac{\int_{\theta}\theta^{\alpha-1}e^{\theta(T_{01}-1)}exp\left\{-\sigma\left(\frac{W_5+W_6}{\theta}+1\right)\right\}d\theta}{\sigma^{-m_1-m_2-\lambda+1}}d\sigma \tag{31}$$

and

$$\pi^*_{(\sigma I)} = \Omega_I\sigma^{m_1+m_2+\lambda-1}e^{-\sigma}\int_{\theta}\frac{\int_{\beta}\beta^{m_2-1}e^{\theta T_{01}}exp\left\{-\sigma\left(\frac{W_5+W_6}{\theta}\right)\right\}d\beta}{\theta^{1-\alpha}e^{\theta}}d\theta; \tag{32}$$

where $\Omega_1 = \left(\int_\sigma \frac{\int_\theta \left\{ \int_\beta \beta^{m_2-1} e^{\theta(T_{01}-1)} exp\left\{ -\sigma \left(\frac{W_5+W_6}{\theta}+1 \right) \right\} d\beta \right\} \theta^{\alpha-1} d\theta}{\sigma^{-m_1-m_2-\lambda+1}} d\sigma \right)^{-1}$, $W_5 = \sum_{i=1}^{m_1} \left(e^{\theta x_{1i}} - 1 \right) + (n_1 - m_1) \left(e^{\theta x_{1m_1}} - 1 \right)$ and $W_6 = \sum_{i=1}^{m_2} \left(e^{\theta \beta x_{2i}} - 1 \right) + (n_2 - m_2) \left(e^{\theta \beta x_{2m_2}} - 1 \right)$.

Similarly, the Bayes estimator under ILLF corresponding to the parameter $\Theta\,(=\theta, \sigma, \beta)$ is obtained by solving Eq. (33)

$$\int_\Theta \left\{ \frac{1}{\Theta} exp \left(-a \frac{\hat{\Theta}_{lI}}{\Theta} \right) \right\} \pi^*_{\Theta I} d\Theta = e^a \int_\Theta \frac{1}{\Theta} \pi^*_{\Theta I} d\Theta \,; \Theta\,(=\theta, \sigma, \beta)\,. \tag{33}$$

6. Numerical Analysis

We studied the properties of the Bayes risks under ILLF obtained on above scenario, by a simulation study with the help of M-H algorithm. The M-H algorithm has been deliberated by Hastings (1970) and Metropolis *et al.* (1953) for simulating samples from a given posterior distribution by use of an arbitrary proposal distribution. The major objective of the present paper is to study the properties of the Bayes risk on above considered scenario and presents a comparison between Bayes risks on different censoring schemes. The Bayes risks obtained under ILLF by using T-IPH, PT-II, T-IP and T-II censoring schemes for different pre-considered parametric values by using Monte-Carlo simulations on 10,000 replications.

A total of $30(=n)$ test units was assumed in this numerical study. Two censoring stages $(05, 10)$, along with four different progressive censoring patterns $(1, 3, 0, 4, 2)$, $(2, 1, 4, 2, 1)$, $(1, 0, 1, 0, 1, 0, 1, 0, 0, 1)$ and $(0, 1, 1, 0, 1, 1, 0, 0, 0, 1)$ for both groups has been assumed. The numerical values for the underlying parameters θ, σ and acceleration factor β assumed in the study are $\theta = \sigma = 0.40(0.10)2.50 = \beta$. Based on Bayes risks under the above scenario on ILLF, we are trying to find out the values of the parameters θ, σ and β for which the magnitude of Bayes risks minimizes numerically. The numerical values of the shape parameter of ILLF were assumed here as $a\,(= 0.20(0.05)0.90)$. When other parametric values considered to be fixed, under T-IPH censoring criterion, it has been observed that, the Bayes risks for the parameter θ under ILLF minimized at $\theta = 1.30, \beta = 1.40, \sigma = 1.60$ and $a = 0.60$. Hence, the numerical finding is presented here only for $\theta(= 0.40, 1.30, 2.50)$, $\sigma(= 0.40, 1.60, 2.50)$, $\beta(= 0.40, 1.40, 2.50)$ and $a(= 0.20, 0.60, 0.90)$.

Case 1: Type-I Progressive Hybrid Censoring

Tables (1)-(3) presents the Bayes risks under T-IPH censored data on CS-PALT for selected parametric values. For simplicity in numerical findings and avoiding numerous tables, time intervals for both groups was assumed equal $(t_1 = 04, 07 = t_2)$. The censored sample size is assumed in three groups $(m_1, m_2 = (5, 5), (5, 10), (10, 10))$ with assumed censoring patterns.

Table 1. Bayes Risk on Simulated Data for Parameter θ

T-IPH Censoring				$t_1 = 04 = t_2$			$t_1 = 07 = t_2$		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	0.6364	0.6199	0.7162	0.7524	0.7397	0.8481
		5	10	0.7723	0.7021	0.8499	0.9125	0.8986	1.0225
		10	10	1.0105	0.8929	1.0271	1.1529	1.1234	1.2023
	1.30, 1.60	5	5	0.4403	0.4252	0.4542	0.5204	0.5085	0.5351
		5	10	0.4967	0.4661	0.5104	0.5862	0.5739	0.6059
		10	10	0.6067	0.5489	0.5789	0.6917	0.6701	0.6842
	2.50, 2.50	5	5	0.9219	0.9036	1.1327	1.0907	1.0768	1.3482
		5	10	1.2046	1.0577	1.4197	1.4239	1.4078	1.7303
		10	10	1.6886	1.4532	1.8289	1.9272	1.8846	2.1188
1.40	0.40, 0.40	5	5	0.5468	0.5311	0.6151	0.6465	0.6342	0.7281
		5	10	0.6634	0.6016	0.7296	0.7835	0.7702	0.8775
		10	10	0.8676	0.7653	0.8816	0.9896	0.9629	1.0316
	1.30, 1.60	5	5	0.3788	0.3641	0.3906	0.4475	0.4359	0.4597
		5	10	0.4272	0.3995	0.4388	0.5041	0.4919	0.5208
		10	10	0.5215	0.4706	0.4976	0.5944	0.5745	0.5877
	2.50, 2.50	5	5	0.7915	0.7741	0.9721	0.9361	0.9228	1.1566
		5	10	1.0336	0.9064	1.2179	1.2217	1.2064	1.4841
		10	10	1.4485	1.2454	1.5685	1.6529	1.6151	1.8169
2.50	0.40, 0.40	5	5	0.7915	0.7741	0.9721	0.9361	0.9228	1.1566
		5	10	1.0337	0.9064	1.2179	1.2217	1.2064	1.4842
		10	10	1.4485	1.2454	1.5686	1.6529	1.6151	1.8168
	1.30, 1.60	5	5	0.5468	0.5309	0.6151	0.6464	0.6341	0.7281
		5	10	0.6633	0.6015	0.7296	0.7835	0.7702	0.8775
		10	10	0.8676	0.7654	0.8816	0.9895	0.9629	1.0316
	2.50, 2.50	5	5	1.1478	1.1281	1.5394	1.3583	1.3434	1.8407
		5	10	1.6141	1.3661	2.0375	1.9087	1.8904	2.5152
		10	10	2.4238	2.0276	2.7976	2.7666	2.7098	3.2057

The numerical findings show that, the magnitude of Bayes risks first decreases as the numerical values of the parameters increases from initial assumed point to a particular point and touches its minimum. Then the magnitude of Bayes risks increases as the parametric values further increases. Table (1) presents the Bayes risks under ILLF for the parameter θ. From the table it was observed that the minimum Bayes risks exists for $a = 0.60, \theta = 1.30, \sigma = 1.60$ and $\beta = 1.40$ respectively when other parametric values assumed to be fixed. It is further observed that the smaller censored samples produce smaller risks. Similarly, the smaller Bayes risks magnitude was noted for small t. Table (2)-(3) presents the Bayes risks for parameters σ and β respectively. All properties have discussed above have been seen similar. It was further observed that the minimum Bayes risks obtained at $\beta = 1.40$ for the underlying parameter β. For others values of β, the Bayes risks of the parameter θ has been seen smaller.

Case 2: Progressive Type-II Censoring

Tables (6)-(9) shows the Bayes risks under the PT-II censoring for the parameters θ, σ and β respectively. All properties have been discussed above for T-IPH censoring criteria was seen similarly for PT-II censoring criteria. The remarkable point is that, the T-IPH censoring scheme produced smaller Bayes risks as compared with Bayes risks obtained under PT-II censored data for all considered parametric values except for $\beta = 1.40$.

Case 3: Type-I Progressive Censoring

Tables (10)-(12) shows the Bayes risks under the T-IP censoring for the parameters θ, σ and β respectively. For T-IP censoring, $m_1 = 12 = m_2$ was assumed with censoring pattern $(0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0)$. All properties have been seen similar as discussed above for T-IPH censoring criteria. The magnitude of Bayes risks was observed higher as compared to the PT-II censoring.

Case 4: Type-II Censoring

Tables (13)-(15) presents the Bayes risks obtained under ILLF on usual T-II censoring for the parameters θ, σ and β respectively. For T-II censoring, the censored sample size was taken as $m_1 = 5, 5, 10$ and $m_2 = 5, 10, 10$. All properties have been seen similar as discussed above for other censoring criterion. Further, the magnitude of Bayes risks was observed higher as compared to the T-IP censoring.

7. Numerical Illustration on Real Data

For numerical illustration of the proposed method, a real life data about cancer survival times in years provided by Bekker *et al.* (2000) has been analyzed for practical applicability of the results. A total of 30 data set has been assumed for the analysis with all numerical assumptions used as discussed in the previous section. The validity of the model is checked by computing the Kolmogorov-Smirnov (K-S) distance test between the empirical distribution function and the fitted distribution function when the parameters are obtained by the method of maximum likelihood estimation. The resultant K-S distance test value (D = 0.0674) with corresponding p-value (p=0.492). This result indicates that the Gompertz distribution provides a good fit to the given data set of 30 units.

The Bayes risks for the underlying parameters are presented in the Tables (4)-(6) under T-IPH censored data, Tables (7)-(9) for PT-II Censoring, Tables (10)-(12) for T-IP Censoring and Tables (13)-(15) for T-II censored data under above considered scenario. All the properties have been seen similar as discussed above. One remarkable point is that, the risk magnitude has been seen higher as compared to simulated data when other parametric values considered to be fixed. Except for $\beta = 1.40$ here the magnitude of the Bayes risks was observed smaller as compared to simulated data when other parametric values considered to be fixed.

Table 2. Bayes Risk on Simulated Data for Parameter σ

T-IPH Censoring				$t_1 = 04 = t_2$			$t_1 = 07 = t_2$		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	0.8796	0.8449	1.0768	1.0388	1.0115	1.2787
		5	10	1.1458	0.9931	1.3471	1.3521	1.3217	1.6392
		10	10	1.6011	1.3645	1.7323	1.8256	1.7693	2.0027
	1.30, 1.60	5	5	0.6121	0.5791	0.6866	0.7223	0.6961	0.8104
		5	10	0.7409	0.6596	0.8131	0.8728	0.8448	0.9758
		10	10	0.9661	0.8395	0.9811	1.1003	1.0562	1.1443
	2.50, 2.50	5	5	1.2692	1.2321	1.6971	1.5006	1.4716	2.0268
		5	10	1.7808	1.4954	2.2433	2.1033	2.0696	2.7663
		10	10	2.6678	2.2193	3.0761	3.0435	2.9665	3.5215
1.40	0.40, 0.40	5	5	0.4499	0.4251	0.5038	0.5311	0.5111	0.5945
		5	10	0.5439	0.4843	0.5961	0.6408	0.6201	0.7152
		10	10	0.7083	0.6161	0.7185	0.8066	0.7752	0.8384
	1.30, 1.60	5	5	0.3152	0.2912	0.3236	0.3713	0.3519	0.3791
		5	10	0.3543	0.3221	0.3626	0.4164	0.3967	0.4289
		10	10	0.4304	0.3797	0.4102	0.4895	0.4635	0.4821
	2.50, 2.50	5	5	0.6464	0.6202	0.7903	0.7634	0.7428	0.9384
		5	10	0.8411	0.7289	0.9879	0.9925	0.9702	1.2021
		10	10	1.1745	1.0015	1.2699	1.3391	1.2986	1.4687
2.50	0.40, 0.40	5	5	1.0913	1.0554	1.4579	1.2896	1.2615	1.7402
		5	10	1.5297	1.2817	1.9259	1.8064	1.7738	2.3744
		10	10	2.2991	1.9023	2.6397	2.6121	2.5424	3.0211
	1.30, 1.60	5	5	0.7573	0.7234	0.9262	0.8942	0.8674	1.0991
		5	10	0.9857	0.8511	1.1579	1.1627	1.1331	1.4082
		10	10	1.3765	1.1698	1.4884	1.5687	1.5168	1.7199
	2.50, 2.50	5	5	1.5774	1.5384	2.3029	1.8657	1.8355	2.7634
		5	10	2.3822	1.9311	3.2149	2.8153	2.7785	4.0162
		10	10	3.8241	3.0958	4.6995	4.3636	4.2643	5.3226

Table 3. Bayes Risk on Simulated Data for Parameter β

T-IPH Censoring				$t_1 = 04 = t_2$			$t_1 = 07 = t_2$		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	1.1331	1.0777	1.5095	1.3375	1.2934	1.7985
		5	10	1.5848	1.3134	1.9915	1.8688	1.8178	2.4518
		10	10	2.3666	1.9494	2.7257	2.6973	2.6054	3.1139
	1.30, 1.60	5	5	0.7915	0.7384	0.9658	0.9332	0.8906	1.1432
		5	10	1.0285	0.873	1.2058	1.2102	1.1626	1.4635
		10	10	1.4322	1.2001	1.5481	1.6302	1.5562	1.7832
	2.50, 2.50	5	5	1.6306	1.5719	2.3739	1.9269	1.8808	2.8454
		5	10	2.4572	1.9775	3.3099	2.9009	2.8456	4.1308
		10	10	3.9358	3.1698	4.8323	4.4892	4.3669	5.4682
1.40	0.40, 0.40	5	5	0.3484	0.3181	0.3879	0.4102	0.3858	0.4562
		5	10	0.4192	0.3649	0.4574	0.4925	0.4674	0.5473
		10	10	0.5432	0.4645	0.5498	0.6176	0.5844	0.6395
	1.30, 1.60	5	5	0.2472	0.2176	0.2527	0.2903	0.2663	0.2943
		5	10	0.2768	0.2432	0.2822	0.3241	0.2997	0.3325
		10	10	0.3346	0.2869	0.3184	0.3794	0.3503	0.3719
	2.50, 2.50	5	5	0.4959	0.4647	0.6032	0.5847	0.5597	0.7143
		5	10	0.6423	0.5486	0.7517	0.7565	0.7303	0.9133
		10	10	0.8933	0.7537	0.9638	1.0172	0.9773	1.1127
2.50	0.40, 0.40	5	5	1.4034	1.3463	2.0404	1.6575	1.6123	2.4442
		5	10	2.1122	1.6951	2.8429	2.4928	2.4392	3.5473
		10	10	3.3803	2.7173	4.1483	3.8544	3.7428	4.6923
	1.30, 1.60	5	5	0.9771	0.9225	1.2997	1.1528	1.1093	1.5473
		5	10	1.3648	1.1262	1.7133	1.6083	1.5587	2.1079
		10	10	2.0363	1.6717	2.3434	2.3192	2.2338	2.6754
	2.50, 2.50	5	5	2.0241	1.9631	3.2179	2.3931	2.3452	3.8762
		5	10	3.2836	2.5531	4.7397	3.8793	3.8194	5.9928
		10	10	5.6371	4.4211	7.3776	6.4315	6.2763	8.2604

Table 4. Bayes Risk on Real Data for Parameter θ

T-IPH Censoring				$t_1 = 04 = t_2$			$t_1 = 07 = t_2$		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	0.9382	0.9027	1.1488	1.1082	1.0797	1.3643
		5	10	1.2223	1.0599	1.4373	1.4426	1.4108	1.7497
		10	10	1.7084	1.4563	1.8484	1.9485	1.8885	2.1371
	1.30, 1.60	5	5	0.6525	0.6183	0.7322	0.7702	0.7429	0.8644
		5	10	0.7902	0.7041	0.8673	0.9316	0.9017	1.0409
		10	10	1.0305	0.8966	1.0466	1.1737	1.1274	1.2207
	2.50, 2.50	5	5	1.3541	1.3153	1.8110	1.6014	1.5708	2.1629
		5	10	1.9001	1.5963	2.3939	2.2444	2.2092	2.9522
		10	10	2.8469	2.3690	3.2828	3.2486	3.1665	3.7583
1.40	0.40, 0.40	5	5	0.5079	0.4813	0.5690	0.5995	0.5783	0.6716
		5	10	0.6141	0.5480	0.6733	0.7238	0.7017	0.8082
		10	10	0.8002	0.6972	0.8119	0.9115	0.8773	0.9477
	1.30, 1.60	5	5	0.3553	0.3297	0.3650	0.4187	0.3981	0.4278
		5	10	0.3995	0.3644	0.4091	0.4698	0.4488	0.4841
		10	10	0.4857	0.4295	0.4630	0.5525	0.5244	0.5444
	2.50, 2.50	5	5	0.7302	0.7021	0.8932	0.8626	0.8405	1.0609
		5	10	0.9505	0.8249	1.1169	1.1219	1.0986	1.3593
		10	10	1.3280	1.1334	1.4360	1.5142	1.4697	1.6612
2.50	0.40, 0.40	5	5	1.1641	1.1266	1.5555	1.3758	1.3464	1.8569
		5	10	1.6321	1.3682	2.0551	1.9274	1.8934	2.5339
		10	10	2.4438	2.0305	2.8171	2.7875	2.7138	3.2241
	1.30, 1.60	5	5	0.8077	0.7723	0.9880	0.9538	0.9258	1.1726
		5	10	1.0514	0.9084	1.2353	1.2403	1.2095	1.5026
		10	10	1.4685	1.2487	1.5881	1.6737	1.6190	1.8352
	2.50, 2.50	5	5	1.6832	1.6423	2.4576	1.9909	1.9592	2.9493
		5	10	2.5421	2.0612	3.4312	3.0045	2.9658	4.2865
		10	10	4.0812	3.3047	5.0158	4.6572	4.5519	5.6810

Table 5. Bayes Risk on Real Data for Parameter σ

T-IPH Censoring				$t_1 = 04 = t_2$			$t_1 = 07 = t_2$		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	1.0489	0.9958	1.3966	1.2377	1.1955	1.6637
		5	10	1.4664	1.2144	1.8421	1.7287	1.6802	2.2675
		10	10	2.1889	1.8023	2.5207	2.4947	2.4081	2.8794
	1.30, 1.60	5	5	0.7331	0.6821	0.8942	0.8643	0.8234	1.0581
		5	10	0.9521	0.8069	1.1165	1.1202	1.0747	1.3543
		10	10	1.3253	1.1094	1.4324	1.5085	1.4385	1.6496
	2.50, 2.50	5	5	1.5085	1.4525	2.1952	1.7825	1.7383	2.6312
		5	10	2.2726	1.8276	3.0606	2.6826	2.6298	3.8193
		10	10	3.6393	2.9297	4.4677	4.1507	4.0363	5.0553
1.40	0.40, 0.40	5	5	0.3422	0.3117	0.3808	0.4028	0.3782	0.4476
		5	10	0.4116	0.3577	0.4496	0.4834	0.4582	0.5371
		10	10	0.5331	0.4553	0.5396	0.6059	0.5728	0.6273
	1.30, 1.60	5	5	0.2431	0.2131	0.2482	0.2853	0.2614	0.2889
		5	10	0.2721	0.2384	0.2772	0.3183	0.2938	0.3265
		10	10	0.3286	0.2813	0.3127	0.3726	0.3435	0.3651
	2.50, 2.50	5	5	0.4867	0.4553	0.5916	0.5739	0.5487	0.7006
		5	10	0.6302	0.5377	0.7373	0.7422	0.7158	0.8955
		10	10	0.8761	0.7388	0.9454	0.9977	0.9579	1.0911
2.50	0.40, 0.40	5	5	1.2986	1.2442	1.8873	1.5336	1.4905	2.2605
		5	10	1.9539	1.5666	2.6293	2.3057	2.2543	3.2798
		10	10	3.1256	2.5114	3.8354	3.5639	3.4593	4.3384
	1.30, 1.60	5	5	0.9045	0.8524	1.2027	1.0671	1.0255	1.4314
		5	10	1.2632	1.0408	1.5849	1.4882	1.4408	1.9496
		10	10	1.8835	1.5451	2.1674	2.1454	2.0648	2.4743
	2.50, 2.50	5	5	1.8721	1.8139	2.9754	2.2133	2.1676	3.5837
		5	10	3.0363	2.3596	4.3818	3.5867	3.5352	5.5465
		10	10	5.2115	4.0859	6.8198	5.9456	5.8006	7.6354

Table 6. Bayes Risk on Real Data for Parameter β

T-IPH Censoring				$t_1 = 04 = t_2$			$t_1 = 07 = t_2$		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	1.1646	1.0961	1.6871	1.3736	1.3187	2.0168
		5	10	1.7477	1.3853	2.3468	2.0593	1.9934	2.9232
		10	10	2.7883	2.2208	3.4182	3.1768	3.0588	3.8593
	1.30, 1.60	5	5	0.8168	0.7506	1.0831	0.9621	0.9087	1.2856
		5	10	1.1382	0.9212	1.4253	1.3377	1.2757	1.7492
		10	10	1.6921	1.3679	1.9461	1.9248	1.8281	2.2143
	2.50, 2.50	5	5	1.6713	1.5993	2.6474	1.9735	1.9166	3.1854
		5	10	2.7035	2.0855	3.8934	3.1901	3.1195	4.9175
		10	10	4.6292	3.6098	6.0516	5.2789	5.1254	6.7694
1.40	0.40, 0.40	5	5	0.2322	0.2009	0.2563	0.2725	0.2473	0.2996
		5	10	0.2775	0.2331	0.3008	0.3245	0.2986	0.3583
		10	10	0.3568	0.2968	0.3642	0.4045	0.3735	0.4164
	1.30, 1.60	5	5	0.1684	0.1371	0.1706	0.1964	0.1712	0.1969
		5	10	0.1871	0.1558	0.1896	0.2176	0.1922	0.2223
		10	10	0.2243	0.1841	0.2131	0.2533	0.2248	0.2465
	2.50, 2.50	5	5	0.3259	0.2939	0.3931	0.3833	0.3574	0.4636
		5	10	0.4192	0.3497	0.4877	0.4922	0.4655	0.5906
		10	10	0.5791	0.4804	0.6228	0.6582	0.6229	0.7168
2.50	0.40, 0.40	5	5	1.4396	1.3693	2.2768	1.6994	1.6433	2.7373
		5	10	2.3254	1.7875	3.3455	2.7433	2.6741	4.2241
		10	10	3.9775	3.0947	5.1967	4.5342	4.3932	5.8103
	1.30, 1.60	5	5	1.0058	0.9379	1.4541	1.1855	1.1313	1.7365
		5	10	1.5067	1.1878	2.0241	1.7739	1.7096	2.5147
		10	10	2.4005	1.9046	2.9404	2.7332	2.6232	3.3174
	2.50, 2.50	5	5	2.0716	1.9972	3.5853	2.4481	2.3893	4.3352
		5	10	3.6088	2.6915	5.5707	4.2623	4.1863	7.1292
		10	10	6.6251	5.0341	9.2336	7.5575	7.3654	10.225

Table 7. Bayes Risk on Progressive Censoring for Parameter θ

PT-II Censoring				Simulated Data			Real Data		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	0.9497	0.9132	1.1628	1.0986	1.0701	1.3525
		5	10	1.2372	1.0731	1.4549	1.4301	1.3984	1.7337
		10	10	1.7294	1.4743	1.8711	1.9309	1.8719	2.1182
	1.30, 1.60	5	5	0.6604	0.6259	0.7411	0.7635	0.7365	0.8569
		5	10	0.7997	0.7128	0.8779	0.9228	0.8939	1.0319
		10	10	1.0431	0.9071	1.0593	1.1635	1.1175	1.2101
	2.50, 2.50	5	5	1.3707	1.3316	1.8332	1.5872	1.5569	2.1439
		5	10	1.9234	1.6161	2.4233	2.2248	2.1899	2.9264
		10	10	2.8818	2.3982	3.3232	3.2196	3.1386	3.7254
1.40	0.40, 0.40	5	5	0.4854	0.4595	0.5436	0.5612	0.5407	0.6284
		5	10	0.5868	0.5232	0.6432	0.6774	0.6563	0.7561
		10	10	0.7644	0.6657	0.7756	0.8528	0.8202	0.8865
	1.30, 1.60	5	5	0.3397	0.3147	0.3489	0.3922	0.3723	0.4005
		5	10	0.3822	0.3479	0.3912	0.4399	0.4197	0.4532
		10	10	0.4642	0.4101	0.4425	0.5171	0.4903	0.5096
	2.50, 2.50	5	5	0.6976	0.6703	0.8532	0.8071	0.7858	0.9924
		5	10	0.9079	0.7876	1.0667	1.0494	1.0264	1.2712
		10	10	1.2683	1.0821	1.3714	1.4162	1.3739	1.5534
2.50	0.40, 0.40	5	5	1.1784	1.1406	1.5746	1.3639	1.3346	1.8408
		5	10	1.6522	1.3853	2.0803	1.9105	1.8768	2.5116
		10	10	2.4738	2.0556	2.8518	2.7631	2.6945	3.1958
	1.30, 1.60	5	5	0.8176	0.7819	1.0001	0.9456	0.9177	1.1623
		5	10	1.0642	0.9196	1.2504	1.2297	1.1988	1.4893
		10	10	1.4865	1.2642	1.6076	1.6591	1.6049	1.8191
	2.50, 2.50	5	5	1.7039	1.6626	2.4878	1.9735	1.9422	2.9234
		5	10	2.5733	2.0867	3.4735	2.9783	2.9398	4.2491
		10	10	4.1314	3.3454	5.0775	4.6164	4.5119	5.6313

Table 8. Bayes Risk on Progressive Censoring for Parameter σ

PT-II Censoring				Simulated Data			Real Data		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	0.9212	0.8732	1.2257	1.1965	1.1554	1.6082
		5	10	1.2873	1.0646	1.6164	1.6709	1.6237	2.1913
		10	10	1.9208	1.5801	2.2114	2.4112	2.3272	2.7828
	1.30, 1.60	5	5	0.6442	0.5981	0.7853	0.8353	0.7956	1.0229
		5	10	0.8364	0.7077	0.9798	1.0828	1.0383	1.3089
		10	10	1.1636	0.9729	1.2573	1.4581	1.3902	1.5943
	2.50, 2.50	5	5	1.3241	1.2736	1.9261	1.7228	1.6799	2.5429
		5	10	1.9942	1.6026	2.6849	2.5925	2.5411	3.6908
		10	10	3.1924	2.5689	3.9186	4.0112	3.9091	4.8853
1.40	0.40, 0.40	5	5	0.2848	0.2576	0.3166	0.3682	0.3449	0.4088
		5	10	0.3423	0.2963	0.3732	0.4415	0.4176	0.4905
		10	10	0.4428	0.3767	0.4483	0.5532	0.5221	0.5726
	1.30, 1.60	5	5	0.2029	0.1761	0.2074	0.2612	0.2381	0.2641
		5	10	0.2269	0.1973	0.2312	0.2911	0.2678	0.2984
		10	10	0.2737	0.2329	0.2605	0.3406	0.3133	0.3338
	2.50, 2.50	5	5	0.4044	0.3762	0.4909	0.5241	0.5002	0.6393
		5	10	0.5231	0.4448	0.6114	0.6773	0.6523	0.8171
		10	10	0.7265	0.6112	0.7836	0.9102	0.8731	0.9952
2.50	0.40, 0.40	5	5	1.1401	1.0908	1.6561	1.4823	1.4402	2.1846
		5	10	1.7144	1.3737	2.3065	2.2286	2.1783	3.1695
		10	10	2.7421	2.2024	3.3642	3.4441	3.3429	4.1926
	1.30, 1.60	5	5	0.7946	0.7473	1.0558	1.0317	0.9912	1.3835
		5	10	1.1089	0.9128	1.3908	1.4385	1.3921	1.8842
		10	10	1.6532	1.3552	1.9017	2.0734	1.9953	2.3913
	2.50, 2.50	5	5	1.6428	1.5903	2.6101	2.1393	2.0945	3.4633
		5	10	2.6636	2.0691	3.8432	3.4663	3.4111	5.3537
		10	10	4.5708	3.5827	5.9812	5.7457	5.6051	7.3789

Table 9. Bayes Risk on Progressive Censoring for Parameter β

PT-II Censoring				Simulated Data			Real Data		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	1.6124	1.5221	2.3383	1.8829	1.8111	2.7674
		5	10	2.4221	1.9228	3.2542	2.8252	2.7383	4.0124
		10	10	3.8665	3.0825	4.7416	4.3602	4.2022	5.2989
	1.30, 1.60	5	5	1.1298	1.0424	1.4997	1.3174	1.2478	1.7627
		5	10	1.5758	1.2784	1.9744	1.8336	1.7522	2.3994
		10	10	2.3446	1.8983	2.6976	2.6402	2.5108	3.0386
	2.50, 2.50	5	5	2.3156	2.2204	3.6716	2.7073	2.6328	4.3724
		5	10	3.7489	2.8941	5.4013	4.3788	4.2855	6.7528
		10	10	6.4223	5.0109	8.3976	7.2488	7.0417	9.2973
1.40	0.40, 0.40	5	5	0.3007	0.2633	0.3325	0.3495	0.3195	0.3852
		5	10	0.3599	0.3047	0.3906	0.4171	0.3864	0.4611
		10	10	0.4634	0.3881	0.4679	0.5205	0.4831	0.5365
	1.30, 1.60	5	5	0.2166	0.1798	0.2202	0.2509	0.2212	0.2522
		5	10	0.2414	0.2035	0.2452	0.2784	0.2484	0.2844
		10	10	0.2901	0.2403	0.2756	0.3245	0.2905	0.3164
	2.50, 2.50	5	5	0.4233	0.3851	0.5113	0.4931	0.4625	0.5977
		5	10	0.5453	0.4574	0.6352	0.6342	0.6027	0.7621
		10	10	0.7544	0.6284	0.8122	0.8491	0.8063	0.9257
2.50	0.40, 0.40	5	5	1.9945	1.9016	3.1571	2.3304	2.2572	3.7573
		5	10	3.2242	2.4811	4.6408	3.7645	3.6734	5.8009
		10	10	5.5175	4.2958	7.2106	6.2255	6.0357	7.9796
	1.30, 1.60	5	5	1.3922	1.3026	2.0148	1.6245	1.5537	2.3823
		5	10	2.0873	1.6486	2.8004	2.4331	2.3482	3.4511
		10	10	3.3281	2.6437	4.0782	3.7508	3.6031	4.5542
	2.50, 2.50	5	5	2.8717	2.7732	4.9734	3.3594	3.2824	5.9527
		5	10	5.0059	3.7361	7.7302	5.8517	5.7515	9.7917
		10	10	9.1933	6.9883	12.811	10.379	10.115	14.035

Table 10. Bayes Risk on Type-I Progressive Censoring for Parameter θ

T-IP Censoring				Simulated Data			Real Data		
$n = 30, m_1 = 12 = m_2$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	t_1	t_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	4	4	1.0105	0.8929	1.0271	1.7084	1.4563	1.8484
		7	10	1.1643	1.1345	1.2142	1.9672	1.9072	2.1582
		10	10	2.0225	1.7885	2.0557	3.1501	3.0698	3.2844
	1.30, 1.60	4	4	0.6067	0.5489	0.5789	1.0305	0.8962	1.0466
		7	10	0.6985	0.6767	0.6914	1.1853	1.1385	1.2328
		10	10	1.2191	1.1041	1.1637	1.8947	1.8359	1.8744
	2.50, 2.50	4	4	1.6886	1.4532	1.8289	2.8469	2.369	3.2828
		7	10	1.9462	1.9032	2.1397	3.2801	3.1978	3.7954
		10	10	3.3717	2.9032	3.6509	5.2581	5.1423	5.7797
1.40	0.40, 0.40	4	4	0.5468	0.5313	0.6151	0.5079	0.4813	0.5692
		7	10	0.6529	0.6405	0.7353	0.6054	0.5842	0.6782
		10	10	1.5835	1.3981	1.6089	2.0866	2.0308	2.1746
	1.30, 1.60	4	4	0.3788	0.3641	0.3906	0.3553	0.3297	0.3652
		7	10	0.4519	0.4402	0.4642	0.4228	0.4022	0.4321
		10	10	0.9565	0.8643	0.9132	1.2578	1.2165	1.2442
	2.50, 2.50	4	4	0.7915	0.7741	0.9723	0.7302	0.7021	0.8932
		7	10	0.9453	0.9319	1.1683	0.8711	0.8488	1.0714
		10	10	2.6357	2.2678	2.8532	3.4772	3.3977	3.8209
2.50	0.40, 0.40	4	4	0.7915	0.7741	0.9721	1.1641	1.1266	1.5555
		7	10	0.9453	0.9319	1.1683	1.3894	1.3597	1.8752
		10	10	2.6357	2.2678	2.8532	3.4772	3.3977	3.8208
	1.30, 1.60	4	4	0.5468	0.5309	0.6151	0.8077	0.7723	0.9883
		7	10	0.6528	0.6404	0.7352	1.3894	1.3597	1.8752
		10	10	1.5835	1.3983	1.6089	2.0863	2.0308	2.1746
	2.50, 2.50	4	4	1.1478	1.1281	1.5394	1.6832	1.6423	2.4576
		7	10	1.3717	1.3567	1.8589	2.0106	1.9786	2.9784
		10	10	4.4023	3.6846	5.0792	5.8121	5.6933	6.7327

Table 11. Bayes Risk on Type-I Progressive Censoring for Parameter β

T-IP Censoring				Simulated Data			Real Data		
$n = 30, m_1 = 12 = m_2$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	t_1	t_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	4	4	2.3666	1.9494	2.7257	2.7883	2.2208	3.4182
		7	10	2.5561	2.4689	2.5507	3.0103	2.8985	3.6571
		10	10	2.8892	2.5541	2.6364	3.8108	3.7137	3.9734
	1.30, 1.60	4	4	1.4322	1.2001	1.5484	1.6921	1.3679	1.9461
		7	10	1.5448	1.4747	1.6896	1.8239	1.7322	2.0983
		10	10	1.7395	1.5747	1.6653	2.2912	2.2199	2.2666
	2.50, 2.50	4	4	3.9358	3.1698	4.8323	4.6292	3.6098	6.0516
		7	10	4.2541	4.1381	5.1817	5.0023	4.8568	6.4147
		10	10	4.8198	4.1491	5.2193	6.3628	6.2222	6.9943
1.40	0.40, 0.40	4	4	0.3484	0.3181	0.3879	0.2322	0.2009	0.2563
		7	10	0.3887	0.3656	0.4321	0.2582	0.2341	0.2839
		10	10	2.2609	1.9956	2.2971	2.5235	2.4559	2.6309
	1.30, 1.60	4	4	0.2472	0.2176	0.2527	0.1683	0.1371	0.1706
		7	10	0.2751	0.2523	0.2789	0.1861	0.1622	0.1866
		10	10	1.3636	1.2315	1.3015	1.5203	1.4701	1.5036
	2.50, 2.50	4	4	0.4959	0.4647	0.6031	0.3259	0.2939	0.3931
		7	10	0.5541	0.5304	0.6769	0.3632	0.3387	0.4393
		10	10	3.7665	3.2401	4.0776	4.2068	4.1107	4.6228
2.50	0.40, 0.40	4	4	1.4034	1.3463	2.0404	1.4396	1.3693	2.2768
		7	10	1.5706	1.5278	2.3161	1.6104	1.5572	2.5939
		10	10	3.7665	3.2401	4.0782	4.2068	4.1107	4.6228
	1.30, 1.60	4	4	0.9771	0.9225	1.2997	1.0058	0.9379	1.4541
		7	10	1.0924	1.0512	1.4662	1.6104	1.5572	2.5939
		10	10	2.2609	1.9957	2.2971	2.5232	2.4559	2.6321
	2.50, 2.50	4	4	2.0241	1.9631	3.2179	2.0716	1.9972	3.5851
		7	10	2.2677	2.2223	3.6731	2.3198	2.2641	4.1078
		10	10	6.2942	5.2674	7.2633	7.0333	6.8893	8.1481

Table 12. Bayes Risk on Type-I Progressive Censoring for Parameter σ

T-IP Censoring				Simulated Data			Real Data		
$n = 30, m_1 = 12 = m_2$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	t_1	t_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	4	4	1.6011	1.3645	1.7323	2.1889	1.8021	2.5207
		7	10	1.6102	1.5509	1.7554	2.1967	2.1108	2.5239
		10	10	2.2049	1.9496	2.2409	2.9074	2.8333	3.0312
	1.30, 1.60	4	4	0.9616	0.8395	0.9811	1.3253	1.1094	1.4324
		7	10	0.9745	0.9258	1.0031	1.3323	1.2609	1.4459
		10	10	1.3285	1.2031	1.2681	1.7491	1.6949	1.7304
	2.50, 2.50	4	4	2.6678	2.2193	3.0762	3.6393	2.9297	4.4377
		7	10	2.6678	2.6003	3.0867	3.6433	3.5377	4.4412
		10	10	3.6763	3.1653	3.9809	4.8523	4.7453	5.3338
1.40	0.40, 0.40	4	4	0.4499	0.4251	0.5038	0.3422	0.3117	0.3808
		7	10	0.4654	0.4484	0.5211	0.3531	0.3315	0.3923
		10	10	1.7261	1.5238	1.7537	1.9262	1.8747	2.0073
	1.30, 1.60	4	4	0.3152	0.2912	0.3236	0.2431	0.2131	0.2482
		7	10	0.3255	0.3085	0.3322	0.2501	0.2288	0.2532
		10	10	1.0422	0.9415	0.9948	1.1616	1.1233	1.1489
	2.50, 2.50	4	4	0.6464	0.6202	0.7903	0.4867	0.4553	0.5916
		7	10	0.6692	0.6511	0.8225	0.5033	0.4812	0.6141
		10	10	2.8737	2.4724	3.1109	3.2093	3.1361	3.5264
2.50	0.40, 0.40	4	4	1.0913	1.0554	1.4579	1.2986	1.244	1.8873
		7	10	1.1304	1.1058	1.5254	1.3443	1.3065	1.9814
		10	10	2.8737	2.4724	3.1112	3.2093	3.1365	3.5263
	1.30, 1.60	4	4	0.7573	0.7234	0.9262	0.9045	0.8524	1.2027
		7	10	0.7838	0.7603	0.9634	1.3443	1.3065	1.9814
		10	10	1.7262	1.5243	1.7537	1.9259	1.8747	2.0073
	2.50, 2.50	4	4	1.5774	1.5384	2.3029	1.8721	1.8139	2.9754
		7	10	1.6354	1.6089	2.4222	1.9403	1.9009	3.1413
		10	10	4.8004	4.0175	5.5388	5.3635	5.2537	6.2131

Table 13. Bayes Risk on Type-II Censoring for Parameter θ

T-II Censoring				Simulated Data			Real Data		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	2.3609	2.2161	3.7277	2.7551	2.6405	4.4291
		5	10	3.8088	2.9008	5.4735	4.4415	4.2953	6.8315
		10	10	6.5034	5.0231	8.4936	7.3327	7.0575	9.3836
	1.30, 1.60	5	5	1.6572	1.5175	2.3945	1.9313	1.8199	2.8248
		5	10	2.4821	1.9291	3.3252	2.8868	2.7488	4.0896
		10	10	3.9482	3.0942	4.8369	4.4451	4.2178	5.3854
	2.50, 2.50	5	5	3.3844	3.2336	5.8475	3.9561	3.8375	6.9919
		5	10	5.8893	4.3654	9.0786	6.8774	6.7211	11.499
		10	10	10.793	8.1642	15.032	12.184	11.827	16.454
1.40	0.40, 0.40	5	5	0.5935	0.5087	0.6541	0.6892	0.6209	0.7561
		5	10	0.7083	0.5915	0.7668	0.8197	0.7502	0.9038
		10	10	0.9092	0.7532	0.9171	1.0204	0.9379	1.0493
	1.30, 1.60	5	5	0.4306	0.3471	0.4366	0.4981	0.4305	0.4986
		5	10	0.4789	0.3955	0.4848	0.5511	0.4829	0.5616
		10	10	0.5735	0.4673	0.5447	0.6409	0.5651	0.6231
	2.50, 2.50	5	5	0.8308	0.7447	1.0003	0.9669	0.8978	1.1675
		5	10	1.0673	0.8872	1.2405	1.2401	1.1691	1.4865
		10	10	1.4727	1.2189	1.5833	1.6566	1.5637	1.8029
2.50	0.40, 0.40	5	5	2.9167	2.7691	5.0295	3.4071	3.2904	6.0097
		5	10	5.0666	3.7426	7.8018	5.9142	5.7614	9.8703
		10	10	9.2752	6.9993	12.910	10.444	10.137	14.155
	1.30, 1.60	5	5	2.0391	1.8965	3.2134	2.3787	2.2655	3.8141
		5	10	3.2841	2.4873	4.7121	3.8266	3.6836	5.8774
		10	10	5.5994	4.3082	7.3064	6.3096	6.0516	8.0663
	2.50, 2.50	5	5	4.1946	4.0389	7.9173	4.9061	4.7839	9.5153
		5	10	7.8601	5.6351	12.986	9.1867	9.0196	16.657
		10	10	15.465	11.351	22.971	17.437	16.992	24.849

Table 14. Bayes Risk on Type-II Censoring for Parameter σ

T-II Censoring				Simulated Data			Real Data		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	1.3694	1.2864	1.9828	1.7359	1.6668	2.5498
		5	10	2.0542	1.6265	2.7578	2.6028	2.5196	3.6949
		10	10	3.2765	2.6075	4.0159	4.0161	3.8662	4.8788
	1.30, 1.60	5	5	0.9612	0.8816	1.2741	1.2156	1.1483	1.6254
		5	10	1.3389	1.0817	1.6757	1.6917	1.6124	2.2114
		10	10	1.9896	1.6062	2.2879	2.4335	2.3107	2.7996
	2.50, 2.50	5	5	1.9638	1.8771	3.1102	2.4943	2.4224	4.0259
		5	10	3.1767	2.4478	4.5737	4.0328	3.9423	6.2158
		10	10	5.4378	4.2386	7.1079	6.6728	6.4781	8.5575
1.40	0.40, 0.40	5	5	0.3314	0.2886	0.3661	0.4261	0.3899	0.4698
		5	10	0.3963	0.3343	0.4298	0.5083	0.4714	0.5624
		10	10	0.5098	0.4255	0.5146	0.6344	0.5895	0.6542
	1.30, 1.60	5	5	0.2393	0.1969	0.2429	0.3056	0.2697	0.3071
		5	10	0.2666	0.2232	0.2702	0.3392	0.3033	0.3464
		10	10	0.3197	0.2637	0.3038	0.3954	0.3547	0.3858
	2.50, 2.50	5	5	0.4658	0.4219	0.5626	0.6014	0.5645	0.7289
		5	10	0.5996	0.5015	0.6979	0.7735	0.7353	0.9296
		10	10	0.8288	0.6891	0.8922	1.0358	0.9841	1.1294
2.50	0.40, 0.40	5	5	1.6925	1.6076	2.6754	2.1478	2.0772	3.4601
		5	10	2.7326	2.0984	3.9306	3.4677	3.3796	5.3394
		10	10	4.6722	3.6337	6.1037	5.7311	5.5532	7.3451
	1.30, 1.60	5	5	1.1829	1.1013	1.7094	1.4986	1.4299	2.1951
		5	10	1.7715	1.3948	2.3743	2.2425	2.1606	3.1786
		10	10	2.8214	2.2363	3.4551	3.4552	3.3155	4.1938
	2.50, 2.50	5	5	2.4339	2.3442	4.2109	3.0944	3.0198	5.4794
		5	10	4.2392	3.1597	6.5423	5.3869	5.2913	9.0115
		10	10	7.7805	5.9098	10.833	9.5527	9.3094	12.913

Table 15. Bayes Risk on Type-II Censoring for Parameter β

T-II Censoring				Simulated Data			Real Data		
$n = 30$				$\leftarrow a \rightarrow$			$\leftarrow a \rightarrow$		
β	θ, σ	m_1	m_2	0.20	0.60	0.90	0.20	0.60	0.90
0.40	0.40, 0.40	5	5	1.4117	1.3457	1.8817	1.5949	1.5438	2.1461
		5	10	1.9752	1.6394	2.4833	2.2295	2.1703	2.9255
		10	10	2.9512	2.4331	3.3996	3.2186	3.1105	3.7161
	1.30, 1.60	5	5	0.9851	0.9219	1.2028	1.1121	1.0632	1.3634
		5	10	1.2808	1.0895	1.5024	1.4429	1.3881	1.7458
		10	10	1.7848	1.4977	1.9294	1.9443	1.8579	2.1274
	2.50, 2.50	5	5	2.0325	1.9627	2.9606	2.2988	2.2453	3.3958
		5	10	3.0642	2.4682	4.1291	3.4617	3.3976	4.9307
		10	10	4.9099	3.9565	6.0295	5.3584	5.2139	6.5277
1.40	0.40, 0.40	5	5	0.5636	0.5222	0.6289	0.6362	0.6034	0.7089
		5	10	0.6793	0.5972	0.7425	0.7651	0.7314	0.8516
		10	10	0.8825	0.7596	0.8938	0.9607	0.9145	0.9961
	1.30, 1.60	5	5	0.3978	0.3571	0.4072	0.4483	0.4161	0.4555
		5	10	0.4462	0.3974	0.4554	0.5011	0.4687	0.5149
		10	10	0.5403	0.4688	0.5146	0.5874	0.5477	0.5771
	2.50, 2.50	5	5	0.8052	0.7622	0.9813	0.9096	0.8761	1.1137
		5	10	1.0449	0.8979	1.2246	1.1789	1.1433	1.4245
		10	10	1.4557	1.2337	1.5719	1.5873	1.5302	1.7379
2.50	0.40, 0.40	5	5	1.7489	1.6811	2.5444	1.9772	1.9249	2.9172
		5	10	2.6339	2.1156	3.5462	2.9744	2.9122	4.2334
		10	10	4.2164	3.3916	5.1757	4.6003	4.4689	5.6014
	1.30, 1.60	5	5	1.2169	1.1521	1.6197	1.3745	1.3243	1.8456
		5	10	1.7005	1.4052	2.1356	1.9188	1.8608	2.5147
		10	10	2.5385	2.0862	2.9223	2.7673	2.6672	3.1928
	2.50, 2.50	5	5	2.5239	2.4508	4.0142	2.8555	2.8009	4.6267
		5	10	4.0959	3.1868	5.9141	4.6301	4.5605	7.1545
		10	10	7.0337	5.5185	9.2067	7.6776	7.4942	9.8621

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