Afrika Statistika

Afrika Statistika
Vol. 15 (1), 2020, pages 2263 - 2273.
DOI: http://dx.doi.org/10.16929/as/2020.2263.157

ISSN 2316-090X

Some characterizations of Pearson's Two-Unequal Step Random Walk

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Received February 14, 20; Accepted March 30, 2020

Abstract. More than a century ago, Pearson (1905) proposed the following: *A man starts from a point 0 and walks ℓ yards in a straight line, then he turns any angle whatever and walks ℓ yards in a straight line. He repeats this process n times. I require the probability that after n steps he is at a distance r and $r + dr$ from the starting point 0.* In this paper some characterizations of the the random walk of unequal step size are given.

Résumé (French) Il y'a plus de cent ans, Pearson (1905) a ecrit : *Une personne fait un trajet de ℓ yards sur une ligne droite commençant par le point zéro. Ensuite il tourne d'un certain angle et fait aussi un trajet de ℓ yards. Il répète cela n fois. Je cherche la probabilité que la position de lq personne soit à un distance de zéro entre r et $r + dr$.* Dans ce papier, nous donnons quelques caracterisations de la marche aléatoire avec des pas de déplacement inégaux.

Key words: random walk; Pearson; characterization of laws; truncated moment; order statistics; upper record values and times
AMS 2010 Mathematics Subject Classification Objects : 60E05.

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Résumé. .

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1. Introduction

A uniform random walk is a walk in the plane that starts at a point 0 and consists of n step of length ℓ each taken uniformly random direction. Pearson (1905) posed this problem. Let X be the distance traveled in n steps. Kluyver (1906) gave the probability density function (*pdf*) $p_n(x, 1..., 1)$ of X when step is of length 1. The *pdf* $p_n(x, 1, ..., 1)$ is given as

$$p_n(x, 1, ..., 1) - \int_0^{\infty} xtJ_0(xt)(J_0(t))^n dt, \ 0 \leq x \leq n, \tag{1}$$

where $J_0(t)$ is the Bessel function of the first kind and zeroth order. For $n = 2$, the solution of Equation (1) is

$$p_2(x) = \frac{2}{\pi}(4 - x^2)^{-\frac{1}{2}}, \ 0 \leq x \leq 2. \tag{2}$$

Serino and Redner (2010) discussed various properties of Pearson random walk with shrinking steps. In this paper we will consider some basic properties and characterizations of the Pearson two step random walk with different step sizes.

2. Main results

Suppose one goes in a straight line a distance and randomly goes another b distance from it. Let x the distance at the end of second step. Then

$$x^2 = a^2 + b^2 - 2ab\cos\theta, \tag{3}$$

where θ is the angle between two steps. Thus

$$cos\theta = \frac{a^2 + b^2 - x^2}{2ab}. \tag{4}$$

Differentiating both sides and on assimilations, we obtain

$$\frac{d\theta}{dx} = \frac{x}{ab}\frac{1}{\sqrt{1 - (\frac{a^2+b^2-x^2}{2ab})}}$$

The *pdf* $f_2(x, a, b)$ is given by

$$f_2(x, a, b) = \frac{x}{\pi ab}\frac{1}{\sqrt{1 - (\frac{a^2+b^2-x^2}{2\lambda})}}, \ |a - b| \leq x \leq a + b. \tag{5}$$

$\lambda\backslash p$		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.25	x_p	0.7661	0,8112	0.8777	0.9549	1.0308	1.1032	1.1646	1.2112	1.2402
0.5	x_p	0.5468	0.6641	0.8138	0.9700	1.1180	1.2486	1.3556	1.4349	1.4836
0.75	x_p	0.3687	0.5907	0.8251	1.0483	1.2500	1.4234	1.5634	1.6661	1.7289
1.00	x_p	0.3129	0.6180	0.9080	1.1756	1.4142	1.6180	1.7820	1.9021	1.9754

Table 1. The percentage points (X_P) of the two-step random walk

For simplicity , we assume $a = 1$ and $b = \lambda \leq 1$. We have

$$f_2(x, 1, \lambda) = \frac{x}{\pi\lambda} \frac{1}{\sqrt{1 - (\frac{1+\lambda^2-x^2}{2\lambda})}}, 1 - \lambda \leq x \leq 1 + \lambda.$$

The cumulative distribution function (*cdf*) $F_2(x, 1, \lambda)$ of X is given by

$$\begin{aligned} F_2(x, 1, \lambda) &= \int_{1-\lambda}^{x} \frac{u}{\pi\lambda} \frac{1}{\sqrt{1 - (\frac{1+\lambda^2-u^2}{2\lambda})}} \, du \\ &= \frac{1}{2} - \frac{1}{\pi} \arcsin(\frac{1 + \lambda^2 - x^2}{2\lambda}) \end{aligned} \quad (6)$$

$$\begin{aligned} \mathbb{E}(X) &= \int_{1-\lambda}^{1+\lambda} \frac{x^2}{\pi\lambda} \frac{1}{\sqrt{1 - (\frac{1+\lambda^2-x^2}{2\lambda})}} \, dx \\ &= \int_{1-\lambda}^{1+\lambda} \frac{4\lambda}{\pi} \frac{x^2}{\sqrt{((1 + \lambda)^2 - x^2)(x^2 - (1 - \lambda)^2)}} dx \\ &= \frac{4\lambda}{\pi}(1 + \lambda)F(p, q), \end{aligned}$$

where $p = \pi/2$,

$$q = \frac{2\sqrt{\lambda}}{1 + \lambda}$$

and $F(p, q)$ is the elliptical integral of the first order:

$$\begin{aligned} F(\pi/2, q) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin\theta}} d\theta \\ = \frac{\pi}{2}(1 + (\frac{1}{2})^2 q^2 + (\frac{1.3}{2.4})^2 q^4 + ... \end{aligned}$$

The percentage points are given in Table 1.

The conditional first moment $\mathbb{E}(X/(X \leq x))$ is as

$$\begin{aligned}\mathbb{E}(X/(X \le x)) &= \int_{1-\lambda}^{x} \frac{u^2}{\pi\lambda} \frac{1}{\sqrt{1-(\frac{1+\lambda^2-u^2}{2\lambda})}} du \\ &= \int_{1-\lambda}^{x} \frac{4\lambda}{\pi} \frac{u^2}{\sqrt{((1+\lambda)^2-u^2)(u^2-(1-\lambda)^2)}} du \\ &= \frac{4\lambda}{\pi}(1+\lambda)F(\chi,q) - \frac{64\lambda^3}{\pi x}\sqrt{(1+\lambda)^2-x^2},\end{aligned}$$

where

$$\chi = \arcsin \frac{1+\lambda}{2x}\sqrt{\frac{x^2-(1-\lambda)^2}{\lambda}}$$

and F is the elliptical integral of first kind $p(x)$, say.

We present here two characterizations in terms of the conditional moments. For that we need the following two lemmas and the following assumption.

Assumption (2.1). Suppose the random variable X is absolutely continuous with cumulative distribution function (cdf) $F(x)$ and probability density function (*pdf*) f(x) .

We denote

$$\gamma = \sup\{x|F(x) > 0\} \; and \; \delta = \inf\{x|F(x) < 1\}$$

and we further assume that $\mathbb{E}(X)$ exists.

Lemma 1. *Under the Assumption (2.1), if*

$$\mathbb{E}(X/(X \le x)) = g(x)\tau(x),$$

where $\tau(x) = \frac{f(x)}{F(x)}$ *and* $g(x)$ *is a continuous differentiable function of* x *with the condition that*

$$\int_{\gamma}^{x} \frac{u-g'(u)}{g(u)} du$$

is finite for all x *such that* $\gamma \le x \le \delta$*, then*

$$f(x) = ce^{\int \frac{x-g'(x)}{g(x)} dx},$$

where c is determined by the condition

$$\int_{\gamma}^{\delta} f(x)dx = 1.$$

Proof of lemma 2. See Ahsanullah (2017).

Lemma 2. *Under the Assumption **(2.1)**, if*

$$\mathbb{E}(X/(X \leq x)) = h(x)r(x),$$

where

$$r(x) = \frac{f(x)}{1 - F(x)}$$

and $h(x)$ is a continuous differentiable function of x with the condition that

$$\int_{\gamma}^{x} \frac{u + h'(u))}{h(u)} du$$

is finite for all x such that $\gamma < x < \delta$, then

$$f(x) = ce^{-\int \frac{x+h'(x)}{h(x)} dx},$$

where c is determined by the condition

$$\int_{\gamma}^{\delta} f(x)dx = 1.$$

Proof of lemma 2. Also, see Ahsanullah (2017).

We are going to organize the presentations of our characterizations into three parts (a), (b) and (c).

(a) Characterization by Truncated Moment. The following two theorems are based on truncated first moment.

Theorem 1. *Suppose that the random variable X is an absolutely continuous random variable with* cdf $F(x)$ *with* $\gamma = 1-\lambda$ *and* $\delta = 1+\lambda$ *and* $\mathbb{E}(X)$ *exists. Then* $\mathbb{E}(X/(X \le x)) = g(x)\tau(x)$, *where*

$$g(x) = \frac{\pi\lambda p(x)}{x}\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})} \tag{7}$$

and $\tau(x) = \frac{f(x)}{F(x)}$, *if and only if*

$$f(x) = \frac{x}{\pi\lambda}\frac{1}{\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}}$$

Proof of Theorem 1. Suppose

$$f(x) = \frac{x}{\pi\lambda}\frac{1}{\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}}, 1-\lambda \le x \le 1+\lambda,$$

then

$$\begin{aligned} g(x)f(x) &= \int_{1-\lambda}^{x}\frac{u^2}{\pi\lambda}\frac{1}{\sqrt{1-(\frac{1+\lambda^2-u^2}{2\lambda})}}du \\ &= p(x). \end{aligned}$$

Thus

$$g(x) = \frac{\pi\lambda p(x)}{x}\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}. \tag{8}$$

Suppose that

$$g(x) = \frac{\pi\lambda p(x)}{x}\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}.$$

Then

$$\begin{aligned} g'(x) &= x - \frac{\pi\lambda p(x)}{x}\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}(\frac{1}{x}-\frac{x}{x^2-\lambda^2+2\lambda-1}) \\ &= x - g(x)(\frac{1}{x}-\frac{x}{x^2-\lambda^2+2\lambda-1}). \end{aligned}$$

Thus

$$\frac{x - g\prime(x)}{g(x)} = \frac{1}{x} - \frac{x}{x^2 - \lambda^2 + 2\lambda - 1}.$$

By Lemma 1, we will have

$$\frac{f'(x)}{f(x)} = \frac{1}{x} - \frac{x}{x^2 - \lambda^2 + 2\lambda - 1}. \tag{9}$$

Integrating both sides of (9) with respect to x, we obtain on simplification, we obtain

$$f(x) = c\frac{x}{\sqrt{1 - (\frac{1+\lambda^2-x^2}{2\lambda})}},$$

where c is a constant. Using the condition

$$\int_{1-\lambda}^{1+\lambda} f(x)dx = 1,$$

we obtain

$$f(x) = \frac{x}{\pi\lambda}\frac{1}{\sqrt{1 - (\frac{1+\lambda^2-x^2}{2\lambda})}}, 1 - \lambda \le x \le 1 + \lambda. \blacksquare$$

Theorem 2. *Suppose that the random variable X is an absolutely continuous random variable with cdf with $F(x)$ with $\gamma = 1 - \lambda$ and $\delta = 1 + \lambda$ and $\mathbb{E}(X)$ exists. Then $\mathbb{E}(X/(X \le x)) = h(x)r(x)$, where*

$$h(x) = \frac{\pi\lambda(\mathbb{E}(X) - p(x))}{x}\sqrt{1 - (\frac{1 + \lambda^2 - x^2}{2\lambda})}$$

and $r(x) = \frac{f(x)}{1-F(x)}$, if and only if, for $1 - \lambda \le x \le 1 + \lambda$,

$$f(x) = \frac{x}{\pi\lambda}\frac{1}{\sqrt{1 - (\frac{1+\lambda^2-x^2}{2\lambda})}}. \tag{10}$$

Proof of Theorem 2. If

$$f(x) = \frac{x}{\pi\lambda}\frac{1}{\sqrt{1 - (\frac{1+\lambda^2-x^2}{2\lambda})}}, 1 - \lambda \le x \le 1 + \lambda, \tag{11}$$

then, for $1 - \lambda \le x \le 1 + \lambda$,

$$
\begin{aligned}
f(x)h(x) &= \int_x^{1+\lambda} = \frac{u^2}{\pi\lambda}\frac{1}{\sqrt{1-(\frac{1+\lambda^2-u^2}{2\lambda})}}du \\
&= \mathbb{E}(X) - \int_0^{1+\lambda} = \frac{u^2}{\pi\lambda}\frac{1}{\sqrt{1-(\frac{1+\lambda^2-u^2}{2\lambda})}}du \\
&= \mathbb{E}(X) - p(x).
\end{aligned}
$$

Thus

$$
h(x) = \frac{\pi\lambda(\mathbb{E}(X)-p(x))}{x}\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}. \tag{12}
$$

Suppose $h(x)$ is as given in (12). Then we obtain on differentiation with respect to x. We obtain

$$
\begin{aligned}
h'(x) &= -x - \frac{\pi\lambda(\mathbb{E}(X)-p(x))}{x}\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}(\frac{1}{x} - \frac{x}{x^2-\lambda^2+2\lambda-1}) \\
&= -x - h(x)(\frac{1}{x} - \frac{x}{x^2-\lambda^2+2\lambda-1}).
\end{aligned}
$$

Thus

$$
-\frac{x+h'(x)}{h(x)} = \frac{1}{x} - \frac{x}{x^2-\lambda^2+2\lambda-1}.
$$

Thus by Lemma 2,

$$
\frac{f'(x)}{f(x)} = \frac{1}{x} - \frac{x}{x^2-\lambda^2+2\lambda-1}. \tag{13}
$$

On integrating both sides of 13 with respect to x, we obtain on simplification,

$$
, f(x) = c\frac{x}{\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}}
$$

where c is a constant. Using the condition

$$
\int_{1-\lambda}^{1+\lambda} f(x)dx = 1,
$$

we obtain

$$f(x) = \frac{x}{\pi\lambda}\frac{1}{\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}}, 1-\lambda \le x \le 1+\lambda.\blacksquare$$

(b) Characterizations by Order Statistics. Here, we will provide two characterizations of distribution of the random variable X based on order statistics, for which we first recall the following well-known results.
Let X_1,X$_2$, $\cdots$, X_n be n ($n \ge 1$) independent copies of the random variable X having absolutely continuous distribution function $F(x)$ and *pdf* $f(x)$. Suppose that $X_{1,n} \le X_{2,n} \le ... \le X_{n,n}$ are the corresponding order statistics. It is known that $X_{j,n}|X_{k,n}$=x , for 1$\le$ $k < j \le n$, is distributed as $(j-k)$-th order statistics from independent observations from the random variable V having the *pdf* $f_{v|x}$ where $f_{v|x}(w|x) = \frac{f(v)}{F(x)}$, $0 \le v < x$ (See Ahsanullah *et al.* (2013), Chapter 5 or Arnold *et al.* (205), Chapter 2, among others).

Let

$$S_{k-1} = \frac{(X_{1,n} + X_{2.n} + ... + X_{k-1,n})}{k-1},$$

and

$$T_{k,n} = \frac{1}{n-k}(X_{k+1} + X_{k+2,n} + \cdots + X_{n,n}).$$

In the following two theorems, we will provide the characterizations of the random variable X based on order statistics.

Theorem 3. *Suppose the random variable X satisfies the Assumption **(2.1)** with $\gamma = 1-\lambda$ and $\delta = 1+\lambda$, then $\mathbb{E}S_{k-1}|X_{k,n} = x) = g(x)\tau(x)$, where*

$$\tau(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = \frac{\pi\lambda p(x)}{x}\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}$$

if and only if

$$\frac{x}{\pi\lambda}\frac{1}{\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}}, 1-\lambda \le x \le 1+\lambda.$$

Proof of Theorem 3. It is known from Ahsanullah *et al.* (2013) and David and Nagaraj (2003) that

$$\mathbb{E}(S_{k-1}|X_{k,n} = x) = \mathbb{E}(X|X \leq x).$$

Therefore, the result follows from Theorem 1. □//

Theorem 4. *Suppose the random variable X satisfies the Assumption **(2.1)** with $\gamma = 0$ and $\delta = 1$, then $\mathbb{E}(T_{k,n}|X_{k,n} = x) = h(x)\tau(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and and*

$$h(x) = \frac{\pi\lambda(\mathbb{E}(X) - p(x))}{x}\sqrt{1 - (\frac{1+\lambda^2 - x^2}{2\lambda})}$$

if and only if

$$f(x) = \frac{x}{\pi\lambda}\frac{1}{\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}}, 1 - \lambda \leq x \leq 1 + \lambda.$$

Proof of Theorem 4. It is known from Ahsanullah *et al.* (2013) and David and Nagaraj (2003) that

$$\mathbb{E}(T_{k,n}|X_{k,n} = x) = \mathbb{E}(X|X \geq x).$$

Therefore, the result follows from Theorem 2. □

(c) Characterization by Upper Record Values. Here, we will provide the characterizations based on upper record values, for which we first recall the following definitions. Suppose that X_1, X_2, $\cdots$, is a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and *pdf* $f(x)$. Let $Y_n = \max(X_1, X_2, ..., X_n)$ for $n \geq 1$. X_1 is the first upper record value, by convention. For $j \geq 2$, we say that X_j is an upper record value of $\{X_n, n \geq 1\}$ if and only if $X_j = Y_j$. If X_j is an upper record, it is strong whenever $X_s < X_j$ for all $1 \leq s \leq j-1$. The indices at which the upper records occur are given by the record times $U(n)$, $n \geq 1$, defined by : $U(1) = 1$ and for any $n \geq 2$,

$$U(n+1) = \min\{j,\ j > U(n),\ X_j > X_{U(n)}\}.$$

We will denote the n-th upper record value as $X(n) = X_{U(n)}, \neq 1$. For details, see for example Ahsanullah (1995), Lo and Ahsnaullah (2019), among others. In the following theorem, we will provide the characterization of the random variable based on upper record values.

Theorem 5. *Suppose the random variable X satisfies the Assumption* **(2.1)** *with* $\gamma = 1-\lambda$ *and* $\delta = 1+\lambda$, *then* $\mathbb{E}(X(n+1)/(X(n)=x)) = h(x)r(x)$, *where* $r(x) = \frac{f(x)}{1-F(x)}$ *and*

$$h(x) = \frac{\pi\lambda(E(X)-p(x))}{x}\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}$$

if and only if, for $1-\lambda \leq x \leq 1$,

$$f(x) = \frac{x}{\pi\lambda}\frac{1}{\sqrt{1-(\frac{1+\lambda^2-x^2}{2\lambda})}}.$$

Proof of Theorem 5. It is known from Ahsanullah (1995) and Nevzorov (2001) that

$$\mathbb{E}(X(n+1)/(X(n)=x)) = \mathbb{E}(X/(X \geq x)).$$

The result follows from Theorem 2. □

3. Conclusion

Several basic characterizations of the Pearson's two-step random walk are given. We sincerely believe that the paper will be quite useful for researchers and practitioners in the field of probability, statistics and other applied sciences.

Acknowledgment

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