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Asymptotic Normality of Non-parametric Estimator for the FGT Poverty Index via Adaptive Kernel

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Abstract. . In this paper, we study the kernel estimator of the measurement class of Foster, Greer and Thorbecke to establish the asymptotic normality of the kernel estimator of the FGT poverty index by the adaptive method for the values of $\alpha = 0$ and $\alpha \geq 1$. We then provide a performance study of this estimator, on simulated data, compared to the estimator from the non-adaptive kernel and the empirical estimator. The study shows that an adaptive kernel estimator is recommended.

Résumé. Dans ce papier, nous étudions l'estimateur à noyau de la classe de mesure de Foster, Greer et Thorbecke afin établir la normalité asymptotique de l'estimateur à noyau de l'indice de pauvreté FGT par la méthode adaptative pour $\alpha = 0$ et $\alpha \geq 1$. A titre d'illustration, nous déterminerons les intervalles de confiance sur des données simulées pour différentes valeurs de z. Par cette étude nous montrons que pour la plupart des valeurs de z, le nouveau estimateur est non seulement plus efficace que les deux autres estimateurs mais génère des intervalles de confiance d'amplitudes plus petites.

Key words: poverty line; poverty aversion; adaptive kernel; Foster, Greer and Thorbecke; uniform convergence; asymptotic normality.
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1. Introduction and definition of the estimator

Let $F(x)$ be the cumulative distribution function (cdf) of the income variable X from a population with continuous density (pdf) $f(x)$ at a point x on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The FGT (Foster, Greer, Thorbecke) Foster *et al.* (1984) class of poverty measures indexed by $\alpha \geq 0$ is defined by

$$P(z,\alpha) = \begin{cases} \int_0^z \left(\frac{z-x}{z}\right)^\alpha f(x)\,dx & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where z is the poverty line,

$$P_{\alpha,H} = \frac{1}{H} \sum_{i \in Q(x)} (\frac{z - x_i}{z})^\alpha$$

for $H = n$ and $H = q$. He showed using the sampling tools that the estimators were respectively unbiased and asymptotically unbiased.
Now let us consider, for an integer $n \geq 1$, a random sample $(X_1, ..., X_n)$ from X, defined on the probability space defined above. The empirical estimator of (1) is given by (See Seidl (1988))

$$\widehat{P}_n(z,\alpha) = \frac{1}{n}\sum_{i=1}^n \left(1 - \frac{X_i}{z}\right)_+^\alpha \quad where \quad x_+ = \max(0,x).$$

This empirical estimator is unbiased consistent and has a limit normal law with zero mean and variance equal to $n^{-1}(P(z,2\alpha) - (P(z,\alpha))^2)$. Therefore, the statistic

$$T = \sqrt{n}\frac{\widehat{P}_n(z,\alpha) - P(z,\alpha)}{\sqrt{(\widehat{P}_n(z,2\alpha) - (\widehat{P}_n(z,\alpha))^2)}}$$

can be used to built confidence interval for the poverty measure (see Dia (2009) and Ciss *et al.*(2016)). So, asymptotic normal estimator with smaller variance will be in general preferable to $\widehat{P}_n(z,\alpha)$.

Seidl (1988) Lo *et al.* (2009) used the empirical process theory and the extreme value theory to study this estimator. This family has been showed to be both a Glivenko-Cantilli and a Donsker one, as a particular case of the family

$$\widehat{P}_n(z,\alpha,g)=\frac{1}{n}\sum_{i=1}^{Q} g\left(1-\frac{X_{i,n}}{z}\right)^{\alpha},$$

where g lies in a suitable family of functions $\mathcal{G}$, and $X_{1,n}\leq ...\leq X_{n,n}$ denote the order statistics associated with $X_1,...,X_n$. Seck (2011) and Seck *et al.* (2009) use some non-weighted poverty measures, viewed as stochastic processes and indexed by real numbers or monotone functions, to follow up the poverty evolution between two periods. But earlier in 2008, Dia (2008) introduced the kernel based estimation of (1) by supposing that the distribution function has a probably density function f. Then replacing f by its Parzen-Rosenblatt kernel estimator, he got the kernel FGT estimator, that he was able to describe in terms of uniform almost sure convergence and uniform squared mean convergence for $\alpha=0$ and $\alpha\geq 1$. The case $\alpha\in]0,1[$ was been introduced in Ciss *et al.*(2015).

Dia (2008) and Ciss *et al.*(2015) consider the classical estimator of the density $f(x)$ (Parzen-Rosenblatt):

$$\hat{f}(x)=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}K\Big(\frac{X_i-x}{h}\Big),$$

where h is a function of n which tends to zero as n tends to infinity and K verifies the following hypotheses:

$$(\mathbf{H_1})\ \sup_{-\infty<x<+\infty}|K(x)|<+\infty,\quad (\mathbf{H_2})\int_{-\infty}^{+\infty}K(x)\,dx=1,\quad (\mathbf{H_3})\ \lim_{x\to\pm\infty}|xK(x)|=0 \qquad (2)$$

and proposed as estimator of FGT poverty index, the following estimator :

$$P_n(z,\alpha)=\frac{1}{n}\sum_{j=1}^{n}\sum_{i=0}^{[z/h]}\Big(\frac{z-ih}{z}\Big)^{\alpha}K\Big(\frac{X_j-ih}{h}\Big)\,dx. \qquad (3)$$

The kernel method requires a prior choice of the smoothing parameter, also called window width, which is then set at any point where the distribution is estimated. This constraint finds its limits when the concentration of the data is particularly heterogeneous in the sample.
To answer this problem, Zakaria *et al.* (2018) consider the adaptive kernel estimator of the density $f(x)$:

$$\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h\lambda_i}K\Big(\frac{X_i - x}{h\lambda_i}\Big).$$

Then they propose as estimator of FGT poverty index, the following estimator :

$$P_n(z,\alpha) = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=0}^{[z/h\lambda_j]}\Big(\frac{z - ih\lambda_j}{z}\Big)^{\alpha}K\Big(\frac{X_j - ih\lambda_j}{h\lambda_j}\Big)\,dx \quad \text{for} \quad \alpha = 0 \quad \text{or} \quad \alpha \geq 1. \tag{4}$$

where λ_i is a parameter that varies according to the local concentration of the data.

An adaptive kernel approach adapts to the sparseness of the data by using a broader kernel over observations located in regions of low density. This is done by varying the bandwidth inversely with the density. As Silverman (1986) puts it, "An obvious practical problem is deciding in the first place whether or not an observation is in a region of low density".

An estimate of the density at the point X_i, denoted $\tilde{f}(X_i)$, measures the concentration of the data around this point: a high value indicates a high concentration while a small value indicates a low concentration. The parameter λ_i can therefore be defined as being inversely proportional to this estimate $\tilde{f}(X_i)$:

$$\lambda_i = \Big[\frac{g}{\tilde{f}(y_i)}\Big]^{\beta},$$

where g is the geometric mean of the $\tilde{f}(X_i)$ and β is the sensitivity parameter, a number satisfying $0 \leq \beta \leq 1$.

The parameter λ_i is even smaller than the density is strong (especially in the center of the distribution) and even greater than the density is low (at the ends of the distribution). the function $\tilde{f}(X_i)$ is called the pilot estimator, which must be calculated in a first step, before the adaptive estimator can be evaluated.

Further, in this paper, we assume that the hypotheses $\mathbf{H_1}, \mathbf{H_2}, \mathbf{H_3}$ hold, K is Riemann integrable, and that f is bounded with support included in $\mathbb{R}_+$.We denote by x_0 the infimum of this support.
The rest of the paper is organized as follows. In section 2, we will state full details of the results. In section 3, as an illustration, we will determine the confidence intervals for simulated data and relevant comments as well as a comparison with results from the classical approach that was used until now. The complete proofs are then given in section 4.

2. Asymptotic Normality

We will need a number of hypotheses and conditions for our theorems. Now additional hypotheses on K are the following:

($\mathbf{H_4}$) K is of bounded variation function $V_{-\infty}^{u}K$ on $\mathbb{R}$ and we denote by $V(\mathbb{R})$ its total variation.
($\mathbf{H_5}$) $\int_{\mathbb{R}}|u|\,|K(u)| < +\infty$.
($\mathbf{H_6}$) There exists a non-increasing function λ such as $\lambda(\frac{u}{h\lambda_i}) = O(h\lambda_i)$ on any bounded interval and for two real numbers x and y

$$|K(x) - K(y)| \leq \lambda\,|x-y| \quad \text{and} \quad \lambda(u) \longrightarrow 0, u \to 0, u \geq 0,$$

($\mathbf{H_7}$) $\frac{|x|}{|h\lambda_i|^{1+\varepsilon}}|K(\frac{x}{h\lambda_i})| \to 0, \quad 0 < \varepsilon < \frac{1}{2}, \quad \text{as} \quad \frac{|x|}{h\lambda_i} \to +\infty.$

Next, these conditions depend of the *pdf* $f(x)$:

$\mathbf{C_1}$: $f(x)$ is uniformly continuous.
$\mathbf{C_2}$: $f(x)$ admits almost everywhere a derivative $f^{'}(x) \in L_1(\mathbb{R})$.
$\mathbf{C_3}$: $f(x)$ satisfies a $C-$Lipschitz condition of order γ, $0 < \gamma \leq 1$.

Under hypothesis $\mathbf{C_1}$ or $\mathbf{C_2}$, the convergence of the above estimator was established in Zakaria *et al.* (2018).
Finally, We consider a family of kernels K_ν, $\nu \in \Gamma \subset \mathbb{R}_+^*$, $\mathbb{R}_+^*$ being the set of strictly positive real numbers, about which we made respectively the same hypotheses $\mathbf{H_1} - \mathbf{H_7}$. We denote by P_n^ν the estimator of $P(z,\alpha)$ when we replace K by K_ν in P_n. We suppose $\int_{\mathbb{R}} K_\nu^2(y)\,dy < 1$ for all $\nu \in \Gamma$ and $\sup_{\nu\in\Gamma}\int_{\mathbb{R}} K_\nu^2(y)\,dy = 1$.
Let $N(0,1)$ be the standard normal distribution function.

Our main resultats are the following.

Theorem 1. Assume that the hypotheses $\mathbf{C_3}, \mathbf{H_6}, \mathbf{H_7}$ hold and $\int_{\mathbb{R}} |y^\gamma K_\nu^2(y)|\,dy < +\infty$ for all $\nu \in \Gamma$.
If $nh^{2\gamma} \to 0$ when $n \to +\infty$, then

$$\frac{P_n^\nu(z,\alpha) - P(z,\alpha)}{\sqrt{Var(P_n^\nu(z,\alpha))}} \to N(0,1)$$

in distribution as $n \to +\infty$, provided

$$(\int_{\mathbb{R}} K_\nu^2(y)\,dy)P(z,2\alpha) - (P(z,\alpha))^2 > 0.$$

Theorem 2. Assume the hypotheses $\mathbf{C_2}, \mathbf{H_5}, \mathbf{H_7}$ hold.
If $nh^{2\gamma} \to 0$ when $n \to +\infty$,
then

$$\frac{P_n^\nu(z,\alpha) - P(z,\alpha)}{\sqrt{Var(P_n^\nu(z,\alpha))}} \to N(0,1)$$

in distribution as $n \to +\infty$, provided

$$(\int_{\mathbb{R}} K_\nu^2(y)\,dy)P(z,2\alpha) - (P(z,\alpha))^2 > 0.$$

We establish these two theorems by proving the two following lemmas which are respectively a generalization in three dimensions of the function K in Lemma 2.9 and in Theorem 2.10 of Zakaria *et al.* (2018).

Lemma 1. Let $0 \le \theta_i \le 1; \quad i = 1,2,3$. Then for all x, y, t pairwise different we have

$$\lim_{n\to+\infty} \sup_{(\theta_1,\theta_2,\theta_3)\in[0,1]\times[0,1]\times[0,1]} \Big((h\lambda_j)^{-3}\int_{-\infty}^{+\infty} |K(\frac{u-x+\theta_1 h\lambda_j}{h\lambda_j})K(\frac{u-y+\theta_2 h\lambda_j}{h\lambda_j}) \times K(\frac{u-t+\theta_3 h\lambda_j}{h\lambda_j})|f(u)\,du\Big) = 0.$$

Lemma 2. Assume that Hypotheses $\mathbf{C_1}$ or $\mathbf{C_2}$ holds.

Then under Hypotheses $\mathbf{H_6}$ and $\mathbf{H_7}$ we have for all $b > 0$,

$$\lim_{n\to+\infty} \sup_{z\in[0,b]} \sum_{0\le i\neq j\neq l\neq i\le[\frac{z}{h\lambda_k}]} \Big(1-\frac{ih\lambda_k}{z}\Big)^\alpha\Big(1-\frac{jh\lambda_k}{z}\Big)^\alpha\Big(1-\frac{lh\lambda_k}{z}\Big)^\alpha \int_{\mathbb{R}} K\Big(\frac{u-ih\lambda_k}{h\lambda_k}\Big)K\Big(\frac{u-jh\lambda_k}{h\lambda_k}\Big) \times K\Big(\frac{u-lh\lambda_k}{h\lambda_k}\Big)f(u)\,du = 0.$$

Remark 1. To construct a confidence interval we proceed as follows:
For $0 < \beta < 1$, let $q_{1-\frac{\beta}{2}}$ be the $\beta-$quantile of the standardized normal distribution. Since $Var(P_n^\nu(z,\alpha)) \ge 0$ for all z and by tTheorems 2.7 and 2.8 in Zakaria *et al.* (2018)

$$\lim_{n\to+\infty} nVar(P_n^\nu(z,\alpha)) = \Big(\int_{\mathbb{R}} K_\nu^2(y)\,dy\Big)P(z,2\alpha) - (P(z,\alpha))^2, \tag{5}$$

we have $\lim_{n\to+\infty} nVar(P_n^\nu(z,\alpha)) = 0$ for all z such that $\Big(\int_{\mathbb{R}} K_\nu^2(y)\,dy\Big)P(z,2\alpha) - (P(z,\alpha))^2 \le 0$. It follows that the asymptotic efficiency $e^{K_\nu}(z,\alpha)$ verifies

$$0 \le e^{K_\nu}(z,\alpha) = \lim_{n\to+\infty} \frac{nVar(P_n^\nu(z,\alpha))}{nVar(\widehat{P}_n(z,\alpha))} \le \int_{\mathbb{R}} K_\nu^2(y)\,dy < 1$$

for conventional kernels in Parzen (1962) p.1068.

We define $100(1-\beta)\%$ the confidence interval CI_ν for $P(z,\alpha)$ in the following form,

$$CI_\nu = P_n^\nu(z,\alpha) \pm q_{1-\frac{\beta}{2}}\{(\int_{\mathbb{R}} K_\nu^2(y)\,dy)P_n^\nu(z,2\alpha) - (P_n^\nu(z,\alpha))^2\}^{\frac{1}{2}}/\sqrt{n} \tag{6}$$

as long as $(\int_{\mathbb{R}} K_\nu^2(y)\,dy) P_n^\nu(z,2\alpha) - (P_n^\nu(z,\alpha))^2 > 0$. Denote this inequality by $(\mathbf{C})$. If it is not verified we increase the size of the sample from n to $n+1$. If for all n the inequality $(\mathbf{C})$ is not satisfied, we vary ν to increase $\int_{\mathbb{R}} K_\nu^2(y)\,dy$. There exists then $\nu \in \Gamma$ and a integer n_0 from which the interval CI_ν is defined. Indeed: let $\nu_k \in \Gamma$ be a sequence such that $\int_{\mathbb{R}} K_{\nu_k}^2(y)\,dy$ converges to 1. Then, since $P(z,2\alpha) - (P(z,\alpha))^2 > 0$ according to the empirical estimator, we have $(\int_{\mathbb{R}} K_{\nu_k}^2(y)\,dy) P(z,2\alpha) - (P(z,\alpha))^2 > 0$ for k large enough greater than or equal to k_0. The inequality $(\mathbf{C})$ is then verified from an integer n_0 and for $\nu = \nu_{k_0}$, under the conditions of Theorems 2.7, 2.8 in Zakaria *et al.* (2018) and the convergence in mean square of $P_n^\nu(z,\alpha)$ to $P(z,\alpha)$ [Theorems 2.7, 2.8 in Zakaria *et al.* (2018)].

We conclude that, the length of the confidence interval CI_ν associated with our estimator is asymptotically lower than that of the empirical estimator, the coefficient being $\{\int_{\mathbb{R}} K_\nu^2(y)\,dy\}^{\frac{1}{2}} < 1$.

Remark 2. Consider the inequality $(\mathbf{C})$. The quantity $P_\nu = \int_{\mathbb{R}} K_\nu^2(y)\,dy$ may be considered as a weight placed in $P(z,2\alpha)$. Greater weight is attached to higher poverty line (It is even heavier than the ratio or proportion $Q_n = \dfrac{P_n^\nu(z,\alpha)}{P_n^\nu(z,2\alpha)}$ is high. So, in order to perform a normality test or to construct a confidence interval of $P(z,\alpha)$, we must calculate $Q_n(z,\nu,\alpha)$ for a kernel K_{ν_0}. If $P_{\nu_0} > Q_n(z,\alpha)$, we determine CI_{ν_0} by the equality (6), otherwise a greater weight kernel K_ν is chosen such that the inequality is verified.

Remark 3. Consider the following family of kernels $K_\nu(x) = \dfrac{K(\nu x)}{\int_{\mathbb{R}} K(\nu x)}, \nu > 0$. We verify that they satisfy the hypotheses $\mathbf{H_1} - \mathbf{H_7}$. Moreover $\int_{\mathbb{R}} K_\nu^2(y)\,dy = \nu \int_{\mathbb{R}} K^2(y)\,dy$. Therefore Γ can be the interval $\left]0, \dfrac{1}{\int_{\mathbb{R}} K^2(y)\,dy}\right[$.

3. Simulations

In this section, we make simulations giving the confidence intervals of a sample of size n of the three estimators that we compared. Our adaptive kernel estimator and the classical one are evaluated by a Gaussian kernel checking assumptions $H_i, i = 1, ..., 6$, taking $h = 1/\sqrt{nlogn}$.

For a Pareto distribution type on $[0,1]$ with parameters $x_0 = 0.02$ and $b = 0.2$, we determine the confidence interval CI1 of $P_n^\lambda(z,1)$, CI2 of $P_n(z,1)$ and CI3 of $\widehat{P}_n(z,1)$ for different values of z.

z	0.1	0.2	0.3	0.4
$\alpha = 1 \quad n = 1000$				
CI1	[0.0583128; 0.073341]	[0.13774014; 0.15763093]	[0.2387292; 0.2525163]	[0.2685651; 0.2747988]
CI2	[0.05986876; 0.07752384]	[0.1359451; 0.1635929]	[0.2180077; 0.2522571]	[0.2451492; 0.2807777]
CI3	[0.1694118; 0.1865882]	[0.2642265; 0.2877735]	[0.3432951; 0.3707049]	[0.3712747; 0.3987253]
$\alpha = 1 \quad n = 10000$				
CI1	[0.0690582; 0.0691582]	[0.1572218; 0.1572392]	[0.241776823; 0.2471099]	[0.27027997; 0.2803871]
CI2	[0.06806406; 0.073594]	[0.1529118; 0.1617329]	[0.241731; 0.2526293]	[0.270199; 0.2815178]
CI3	[0.1830294; 0.1883706]	[0.2827866; 0.2902134]	[0.3718884; 0.3801116]	[0.4000077; 0.4081923]

Table 1. Comparative table of results simulations

A comparative simulations study results shows that for each point z, our estimator provides confidence intervals with smaller amplitudes than those of the other two estimators. Better still, the intervals provided by the other two estimators contain those generated by our estimator.

4. Details of the Proofs

Proof of Lemma 1.It is a generalisation of Lemma 2.9 in Zakaria *et al.* (2018). We assume $\mathbf{C_1}$ holds. Let $\delta > 0$.
Define

$$\begin{aligned} I_n(x,y) &= (h\lambda_j)^{-3} \int_{-\infty}^{+\infty} |K(\frac{u-x+\theta_1 h\lambda_j}{h\lambda_j})K(\frac{u-y+\theta_2 h\lambda_j}{h\lambda_j})K(\frac{u-t+\theta_3 h\lambda_j}{h\lambda_j})| f(u)\,du \\ &= \int_{-\infty}^{+\infty} ((h\lambda_j)^{-1} K(\frac{v}{h\lambda_j}))|((h\lambda_j)^{-1} K(\frac{v+x-\theta_1 h\lambda_j - y + \theta_2 h\lambda_j}{h\lambda_j})) \\ &\times ((h\lambda_j)^{-1} K(\frac{v+x-\theta_1 h\lambda_j - t + \theta_3 h\lambda_j}{h\lambda_j}))| f(x+v-\theta_1 h\lambda_j)\,dv \\ &= \int_{|v|\le\delta} + \int_{|v|>\delta}. \end{aligned}$$

Since f is continuous, it is so bounded on $I = [x-\delta, x+\delta]$. We assume n large enough such that $x + v \pm \theta_1 h\lambda_j \in I$. Then

$$\begin{aligned} \int_{|v|\le\delta} &\le \sup_{|v|\le\delta} f(x+v-\theta_1 h\lambda_j) \int_{-\frac{\delta}{h\lambda_j}\le u\le\frac{\delta}{h\lambda_j}} |K(u)||K(\frac{x-\theta_1 h\lambda_j - y + \theta_2 h\lambda_j}{h\lambda_j} + u) \\ &\times (h\lambda_j)^{-1} K(\frac{x-\theta_1 h\lambda_j - t + \theta_3 h\lambda_j}{h\lambda_j} + u)(h\lambda_j)^{-1}|\,du \qquad (7) \\ &= \sup_{|v|\le\delta} f(x+v-\theta_1 h\lambda_j) \int_{-\infty}^{+\infty} \chi_{-\frac{\delta}{h\lambda_j}\le u\le\frac{\delta}{h\lambda_j}} |K(u)||K(\frac{x-\theta_1 h\lambda_j - y + \theta_2 h\lambda_j}{h\lambda_j} + u)(h\lambda_j)^{-1} \\ &\times K(\frac{x-\theta_1 h\lambda_j - t + \theta_3 h\lambda_j}{h\lambda_j} + u)(h\lambda_j)^{-1}|\,du. \end{aligned}$$

For all u

$$\lim_{n\to+\infty}|K(\frac{x-\theta_1 h\lambda_j - y + \theta_2 h\lambda_j}{h\lambda_j}+u)(h\lambda_j)^{-1}| = 0.$$

Write

$$|K(\frac{x-\theta_1 h\lambda_j - y + \theta_2 h\lambda_j}{h\lambda_j}+u)(h\lambda_j)^{-1}| =$$

$$|(\frac{x-\theta_1 h\lambda_j - y + \theta_2 h\lambda_j}{h\lambda_j}+u)K(\frac{x-\theta_1 h\lambda_j - y + \theta_2 h\lambda_j}{h\lambda_j}+u)||\frac{1}{x-\theta_1 h\lambda_j - y+\theta_2 h\lambda_j + h\lambda_j u}|.$$

We have

$$|\frac{1}{x-\theta_1 h\lambda_j - y+\theta_2 h\lambda_j + h\lambda_j u}| = \frac{1}{|x-y||1-\frac{\theta_1-\theta_2-u}{x-y}h\lambda_j|}.$$

Since $|u| \leq \dfrac{\delta}{h}$ we may choose δ small enough such that for $n \geq n_0$ we have

$$|\frac{\theta_1-\theta_2-u}{x-y}h\lambda_j| \leq \frac{3(h\lambda_j)_{n_0}+\delta}{|x-y|} = \eta_1 < 1.$$

Therefore

$$|\frac{1}{x-\theta_1 h\lambda_j - y+\theta_2 h\lambda_j + h\lambda_j u}| \leq \frac{1}{|x-y|(1-\eta_1)}, \tag{8}$$

since $\mathbf{H_3}$ implies there exists a constant B such that

$$|(\frac{x-\theta_1 h\lambda_j - y + \theta_2 h\lambda_j}{h\lambda_j}+u)K(\frac{x-\theta_1 h\lambda_j - y + \theta_2 h\lambda_j}{h\lambda_j}+u)| \leq B.$$

Then we have

$$\chi_{-\frac{\delta}{h\lambda_j}\leq u\leq \frac{\delta}{h\lambda_j}}(u)|K(\frac{x-\theta_1 h\lambda_j - y + \theta_2 h\lambda_j}{h\lambda_j}+u)(h\lambda_j)^{-1}| \leq \frac{B}{|x-y|(1-\eta_1)}.$$

Similarly, there exists a constant C such that

$$|(\frac{x-\theta_1 h\lambda_j - t + \theta_3 h\lambda_j}{h\lambda_j}+u)K(\frac{x-\theta_1 h\lambda_j - t + \theta_3 h\lambda_j}{h\lambda_j}+u)| \leq C$$

and

$$\chi_{-\frac{\delta}{h\lambda_j}\leq u\leq \frac{\delta}{h\lambda_j}}(u)|K(\frac{x-\theta_1 h\lambda_j - t + \theta_3 h\lambda_j}{h\lambda_j}+u)(h\lambda_j)^{-1}| \leq \frac{C}{|x-t|(1-\eta 2)}.$$

Therefore, if δ is small enough and n sufficiently large we have, for $-\dfrac{\delta}{h\lambda_j} \leq u \leq \dfrac{\delta}{h\lambda_j}$. $|K(u)|$ being integrable, by dominated convergence theorem, the integral in the right-hand side of 7 tends to zero as $n \to +\infty$, uniformly with respect to $(\theta_1, \theta_2, \theta_3)$. Hence

$$\int_{|v|\leq\delta} \to 0 \quad \text{as} \quad n\to+\infty \quad \text{uniformly with respect to}\,(\theta_1,\theta_2,\theta_3).$$

Let $\int_{|v|>\delta}$; write in the form

$$\int_{|v|>\delta} = \int_{|v-\theta_1 h\lambda_j|>\delta} |v(h\lambda_j)^{-1}K(\frac{v}{h\lambda_j})((h\lambda_j)^{-1}K(\frac{v+x-\theta_1 h\lambda_j - y+\theta_2 h\lambda_j}{h\lambda_j}) \times ((h\lambda_j)^{-1}K(\frac{v+x-\theta_1 h\lambda_j - t+\theta_3 h\lambda_j}{h\lambda_j}))\frac{f(x+v-\theta_1 h\lambda_j)}{v}|\,dv.$$

We obtain

$$\int_{|v|>\delta} \leq \frac{2}{\delta}\sup_{|v|>\delta}|\frac{v}{h\lambda_j}K(\frac{v}{h\lambda_j})|\int_{|v|>\delta}((h\lambda_j)^{-1}K(\frac{v+x-\theta_1 h\lambda_j - y+\theta_2 h\lambda_j}{h\lambda_j})) \times ((h\lambda_j)^{-1}K(\frac{v+x-\theta_1 h\lambda_j - t+\theta_3 h\lambda_j}{h\lambda_j}))|f(x+v-\theta_1 h\lambda_j)|\,dv. \quad (9)$$

Making the change of variable

$$v+x-\theta_1 h\lambda_j = u.$$

Then

$$\int_{|v|>\delta} \leq \frac{2}{\delta-\theta_1 h\lambda_j}\sup_{|v|>\delta}|\frac{v}{h\lambda_j}K(\frac{v}{h\lambda_j})|\int_{\mathbb{R}}|(h\lambda_j)^{-1} \times K(\frac{u-y+\theta_2 h\lambda_j}{h\lambda_j})((h\lambda_j)^{-1}K(\frac{u-t+\theta_3 h\lambda_j}{h\lambda_j}))|f(u)\,du$$
$$\leq \int_{\mathbb{R}}|(h\lambda_j)^{-1}K(\frac{u-y+\theta_2 h\lambda_j}{h\lambda_j})((h\lambda_j)^{-1}K(\frac{u-t+\theta_3 h\lambda_j}{h\lambda_j}))|f(u)\,du$$

and this quantity in the right-side tends to zero as $n \to +\infty$ uniformly with respect to (θ_2, θ_3) according to Lemma 2.9 in Zakaria *et al.* (2018) (when we replace K by $|K|$) and $(\mathbf{H_3})$ (valid under assumption $\mathbf{C_1}$ or $\mathbf{C_2}$). Since

$$\sup_{|v|>\delta}|\frac{v}{h\lambda_j}K(\frac{v}{h\lambda_j})| \to 0 \quad \text{as} \quad n \to +\infty,$$

we have

$$|\int_{|v|>\delta}| \to 0 \quad \text{as} \quad n \to +\infty.$$

The proof of Lemma 1 is complete.

Proof of Lemma 2. We Suppose Condition $\mathbf{C_1}$ is verified. Let $\Delta = [0,b]\times[0,b]\times[0,b]$. We can write for all $z \in [0,b]$

$$\sum_{0\leq i\neq j\neq l\neq i\leq[\frac{z}{h\lambda_k}]}\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha}\Big(1-\frac{jh\lambda_k}{z}\Big)^{\alpha}\Big(1-\frac{lh\lambda_k}{z}\Big)^{\alpha} \times \int_{\mathbb{R}}|K(\frac{u-ih\lambda_k}{h\lambda_k})K(\frac{u-jh\lambda_k}{h\lambda_k})K(\frac{u-lh\lambda_k}{h\lambda_k})|f(u)\,du$$
$$\leq \int\int\int_{\{(x,y,t)\in\Delta:|x-y||x-t||t-y|>0\}}\Phi_n(x,y,t)\,dxdydt$$

where

$$\Phi_n(x,y,t) = \frac{1}{(h\lambda_k)^3} \sum_{0\leq i\neq j\neq l\neq i\leq [\frac{z}{h\lambda_k}]} \chi_{\Delta_{h\lambda_k,i}}(x)\chi_{\Delta_{h\lambda_k,j}}(y)\chi_{\Delta_{h\lambda_k,l}}(t) \times \int_{\mathbb{R}} |K(\frac{u-ih\lambda_k}{h\lambda_k})K(\frac{u-jh\lambda_k}{h\lambda_k})K(\frac{u-lh\lambda_k}{h\lambda_k})| f(u)\,du. \tag{10}$$

Let $(x,y,t) \in \Delta_{h\lambda_k,i}\times\Delta_{h\lambda_k,j}\times\Delta_{h\lambda_k,l}$ $\quad i\neq j\neq l\neq i$ with the representations:

$$x = h\lambda_k i + \theta_1 h\lambda_k, \quad y = h\lambda_k j + \theta_2 h\lambda_k \quad t = h\lambda_k l + \theta_3 h\lambda_k \quad 0\leq \theta_i < 1, \quad i = 1,2,3$$

10 becomes

$$\frac{1}{(h\lambda_k)^3}\int_{\mathbb{R}} |K(\frac{u-ih\lambda_k}{h\lambda_k})K(\frac{u-jh\lambda_k}{h\lambda_k})K(\frac{u-lh\lambda_k}{h\lambda_k})| f(u)\,du.$$

Let $\delta = \min(\frac{|x-y|}{2}, \frac{|x-t|}{2}, \frac{|t-y|}{2})$. With the change of variable $v = u - x - \theta_1 h\lambda_k$ we split the integral above in the following form:

$$\frac{1}{(h\lambda_k)^3}\int_{\mathbb{R}} |K(\frac{v}{h\lambda_k})K(\frac{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k})K(\frac{x-\theta_1 h\lambda_k - t + \theta_3 h\lambda_k}{h\lambda_k})| f(u)\,du = \int_{|v|\leq\delta} + \int_{|v|>\delta}.$$

Then, we have

$$\int\int\int_{\{(x,y,t)\in\Delta:|x-y||x-t||t-y|>0\}} \Phi_n(x,y,t)\,dxdydt$$
$$\leq \int\int\int_{\{(x,y,t)\in\Delta:|x-y||x-t||t-y|>0\}} \sum_{0\leq i\neq j\neq l\neq i\leq[\frac{z}{h\lambda_k}]} \chi_{\Delta_{h\lambda_k,i}\times\Delta_{h\lambda_k,j}\times\Delta_{h\lambda_k,l}}(x,y,t)(\int_{|v|\leq\delta} + \int_{|v|>\delta}).$$

The proof is conducted as follows.

First consider

$$\int\int\int_{\{(x,y,t1)\in\Delta:|x-y||x-t||t-y|>0\}} \sum_{0\leq i\neq j\neq l\neq i\leq[\frac{z}{h\lambda_k}]} \chi_{\Delta_{h\lambda_k,i}\times\Delta_{h\lambda_k,j}\times\Delta_{h\lambda_k,l}}(x,y,t)(\int_{|v|\leq\delta}).$$

Let $A = \sup_{x\in\mathbb{R}} f(x)$. The notations being as in the proof of Lemma 2.9 in Zakaria *et al.* (2018) and we suppose $h\lambda_k \leq \frac{b}{4}$ for n large enough with $\delta = \frac{z}{2}$. We have, in accordance with inequality (7)

$$\int_{|v|\leq\delta} \leq A\int_{-\infty}^{+\infty} \chi_{-\frac{\delta}{h\lambda_k}\leq u\leq\frac{\delta}{h\lambda_k}} |K(u)||K(\frac{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k} + u)(h\lambda_k)^{-1}| \times |K(\frac{x-\theta_1 h\lambda_k - t + \theta_3 h\lambda_k}{h\lambda_k} + u)(h\lambda_k)^{-1}|\,du. \tag{11}$$

For all u

$$\lim_{n\to+\infty}|K(\frac{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)(h\lambda_k)^{-1}|=0.$$

Write

$$\begin{aligned}|K(\frac{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)(h\lambda_k)^{-1}| &= |K(\frac{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)\\ &- K(\frac{2b+x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)\\ &+ K(\frac{2b+x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)|(h\lambda_k)^{-1}\\ &\leq (\lambda(\frac{2b}{h\lambda_k})+|K(\frac{2b+x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)|)(h\lambda_k)^{-1}.\end{aligned}$$

Express

$$\begin{aligned}&|K(\frac{2b+x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)|(h\lambda_k)^{-1}\\ &=|\frac{2b+x-y+h\lambda_k u}{h}||K(\frac{2b+x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)|\frac{1}{|2z+x-y+h\lambda_k u|}.\end{aligned}$$

Let $B=\sup_{y\in\mathbb{R}}|y||K(y)|$ and $C=\sup_{y\in\mathbb{R}}|K(y)|$, then, we have

$$|\frac{2b+x-y+h\lambda_k u}{h\lambda_k}||K(\frac{2z+x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)|\leq B+2C.$$

Therefore, we have

$$|K(\frac{2b+x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)|(h\lambda_k)^{-1}\leq\frac{B+2C}{|2b+x-y+h\lambda_k u|}.$$

Hence

$$|K(\frac{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)(h\lambda_k)^{-1}|\leq\lambda(\frac{2b}{h\lambda_k})+\frac{B+2C}{|2b+x-y+h\lambda_k u|}.$$

Similarly, we have

$$|K(\frac{x-\theta_1 h\lambda_k - t+\theta_3 h\lambda_k}{h\lambda_k}+u)(h\lambda_k)^{-1}|\leq\lambda(\frac{2b}{h\lambda_k})+\frac{B+2C}{|2b+x-t+h\lambda_k u|}.$$

We conclude that for $h\lambda_k$ small enough

$$\int_{|v|\leq\delta}\leq\frac{A}{|2b+x-y+h\lambda_k u|}\int_{\mathbb{R}}|K(u)|(B+2C)\,du<\frac{AD}{|2b+x-y+h\lambda_k u|},$$

D being the finite bound of $\int_{\mathbb{R}}|K(u)|(B+2C)\,du.$

We have

$$\int_{|v|\leq\delta}\leq\frac{AD}{|2b+x-y+h\lambda_k u|}+O(h\lambda_k).$$

Since $-\delta \leq h\lambda_k u \leq \delta$ we have

$$\frac{b}{4} \leq |2b + x - y + h\lambda_k u|, \text{and } \frac{b}{4} \leq |2b + x - t + h\lambda_k u|.$$

Hence

$$\int_{|v|\leq\delta} \leq \frac{16AD}{b^2} + O(h\lambda_k).$$

This inequality is true for all

$$(x, y, t) \in \{(x, y, t) \in \Delta : |x-y||x-t||t-y| > 0\}.$$

For all u we have according to the proof of the Lemma 2.3 in Zakaria *et al.* (2018)

$$\sup_{(\theta_1,\theta_2,\theta_3)\in[0,1]^3} \int_{-\infty}^{+\infty} \chi_{-\frac{\delta}{h\lambda_k}\leq u\leq\frac{\delta}{h\lambda_k}}(u)|K(u)||K(\frac{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k} + u)(h\lambda_k)^{-1}| \times |K(\frac{x-\theta_1 h\lambda_k - t + \theta_3 h\lambda_k}{h\lambda_k} + u)(h\lambda_k)^{-1}|\, du$$

tends to zero as $n \to +\infty$ except the complement in Δ of

$$\{(x, y, t) \in \Delta, x \neq y \neq t \neq x\}$$

which is a $dxdydt$-null set. Therefore by Lebesgue-dominated convergence theorem

$$\lim_{n\to+\infty} \int\int\int_\Delta \sum_{0\leq i\neq j\neq l\neq i\leq[\frac{z}{h\lambda_k}]} \chi_{\Delta_{h\lambda_k,i}\times\Delta_{h\lambda_k,j}\times\Delta_{h\lambda_k,l}}(x, y, t)(\int_{|v|\leq\delta})\, dxdydt = \int\int\int_\Delta \lim_{n\to+\infty}(\int_{|v|\leq\delta})\, dxdydt = 0. \tag{12}$$

Consider finally

$$\int\int\int_{\{(x,y,t)\in\Delta:|x-y||x-t||t-y|>0\}} \sum_{0\leq i\neq j\neq l\neq i\leq[\frac{z}{h\lambda_k}]} \chi_{\Delta_{h\lambda_k,i}\times\Delta_{h\lambda_k,j}\times\Delta_{h\lambda_k,l}}(x, y, t) \int_{|v|>\delta}. \tag{13}$$

We use expression (3) in the second part of Lemma 2.9 in Zakaria *et al.* (2018) by analogous reasoning, we obtain

$$\int_{|v|>\delta} \leq \frac{1}{\delta} \sup_{|v|>\delta} |\frac{v}{h\lambda_k}||K(\frac{v}{h\lambda_k})| \int_{|v|>\delta} |(h\lambda_k)^{-1}K(\frac{v + x - \theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k}) \times (h\lambda_k)^{-1}K(\frac{v + x - \theta_1 h\lambda_k - t + \theta_3 h\lambda_k}{h\lambda_k})|f(x + v - \theta_1 h\lambda_k)\, dv. \tag{14}$$

Making a change of variable $u = x + v - \theta_1 h$, the integral of the right-hand side of (15) does not exceed

$$\int_{\mathbb{R}} |(h\lambda_k)^{-1}K(\frac{u - y + \theta_2 h\lambda_k}{h\lambda_k})(h\lambda_k)^{-1}K(\frac{u - t + \theta_3 h\lambda_k}{h\lambda_k}|f(u)\, du.$$

Let $\delta \geq (h\lambda_k)^{\varepsilon}, 0 < \varepsilon < \frac{1}{2}$. We have

$$\int_{|v|>\delta} \leq \sup_{|v|>(h\lambda_k)^{\varepsilon}} \frac{|v|}{(h\lambda_k)^{1+\varepsilon}} |K(\frac{v}{h\lambda_k})| \sup_{(\theta_2,\theta_3)\in[0,1]^2} \int_{\mathbb{R}} |(h\lambda_k)^{-1} K(\frac{u-y+\theta_2 h\lambda_k}{h\lambda_k})(h\lambda_k)^{-1} \times K(\frac{u-t+\theta_3 h\lambda_k}{h\lambda_k}) |f(u)\, du. \quad (15)$$

When writing

$$\int_{\mathbb{R}} |(h\lambda_k)^{-1} K(\frac{u-y+\theta_2 h\lambda_k}{h\lambda_k})(h\lambda_k)^{-1} K(\frac{u-t+\theta_3 h\lambda_k}{h\lambda_k}) |f(u)\, du \leq \int_{|v|\leq\bar{\delta}} + \int_{|v|>\bar{\delta}},$$

with the change of variable $v = u - t + \theta_3 h\lambda_k$ and $\bar{\delta} = \frac{|t-y|}{2}$, according to the proof of Theorem 2.10 in Zakaria *et al.* (2018) we have that, for $\bar{\delta} > (h\lambda_k)^{\varepsilon}$

$$|\int_{|v|\leq\bar{\delta}} \leq \frac{4AD}{b} + O(h\lambda_k), \qquad y \neq t$$

and

$$|\int_{|v|>\bar{\delta}} | \leq \sup_{|v|>h^{\varepsilon}} \frac{|v|}{(h\lambda_k)^{1+\varepsilon}} |K(\frac{v}{h\lambda_k})| \int_{\mathbb{R}} |(h\lambda_k)^{-1} K(\frac{u}{h\lambda_k})| f(u+y-\theta_2 h\lambda_k)\, du.$$

Since under hypothesis $\mathbf{C_1}$ or $\mathbf{C_2}$

$$\int_{\mathbb{R}} |(h\lambda_k)^{-1} K(\frac{v}{h\lambda_k})| f(u+y-\theta_2 h\lambda_k)\, du$$

is bounded, we deduce that the right-hand side of 15 is bounded except on set $\bar{\delta} = 0$. Hence

$$\int_{|v|>\delta}$$

is bounded except for $\{(x,y,t) \in \Delta : \delta = 0\}$ which is a $(dxdydt)$-null measure.
Δ being bounded, hypothesis $\mathbf{H_7}$ implies that the integral (13) tends to zero as $n \to +\infty$.
Consequently

$$\lim_{n\to+\infty} \int\int_{\Delta}\int_{\mathbb{R}} \to 0, \quad n \to +\infty$$

since Δ is bounded. The proof of the lemma is then complete.

Proof of Theorem 1. For sake of simplicity in the notations, K stands for K_ν and P_n for P_n^{ν}. It is sufficient to prove: on the one hand,

1. $\sqrt{n}(\mathbb{E}(P_n(z,\alpha) - P(z,\alpha)) \to 0 \quad$ as $\quad n \to +\infty$
   and on the other hand,

2. $\dfrac{P_n(z,\alpha)-\mathbb{E}(P_n(z,\alpha))}{\sqrt{\mathbb{V}(P_n(z,\alpha))}} \to N(0,1)$ as $n \to +\infty$.

Let x_0 be the infimum of the support of f. Let's firstly, observe that $F(z)$ is bounded. Let $\bar{\Delta}_{h\lambda_k,i} = \Delta_{h\lambda_k,i} \cap [0,z]$; and χ_B the indicator function of B. Let $z \in [0,b]$. We have

$$\mathbb{E}(P_n(z,\alpha)) = \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \left(1-\frac{ih\lambda_k}{z}\right)^{\alpha} \int_{\mathbb{R}} K(u) f(uh\lambda_k - ih\lambda_k)\,du$$

which can be written in the following form

$$\begin{aligned}
&\int_0^z \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \chi_{\bar{\Delta}_{h\lambda_k,i}}(x)\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha} \int_{-\infty}^{+\infty} K(u) f(uh\lambda_k - ih\lambda_k)\,du dx \\
&+ (h\lambda_k([\frac{z}{h\lambda_k}]+1)-z)\Big(1-\frac{h\lambda_k[\frac{z}{h\lambda_k}]}{z}\Big)^{\alpha} \int_{-\infty}^{+\infty} K(u) f(uh\lambda_k + \frac{h\lambda_k[\frac{z}{h\lambda_k}]}{z})\,du.
\end{aligned} \tag{16}$$

We have

$$\begin{aligned}
&\sup_{z\in\mathbb{R}} |(h\lambda_k([\frac{z}{h\lambda_k}]+1)-z)\Big(1-\frac{h\lambda_k[\frac{z}{h\lambda_k}]}{z}\Big)^{\alpha} \int_{-\infty}^{+\infty} K(u) f(uh\lambda_k + \frac{h\lambda_k[\frac{z}{h\lambda_k}]}{z})\,du| \\
&\qquad \le 2h\lambda_k \sup_{x\in\mathbb{R}} f(x) \int_{-\infty}^{+\infty} |K(u)|\,du.
\end{aligned} \tag{17}$$

Since we have $|h\lambda_k([\frac{z}{h\lambda_k}]+1)-z| \le 2h\lambda_k$. Because of Assumption $\mathbf{H_2}$, we can write

$$P(z,\alpha) = \int_0^z \left(1-\frac{x}{z}\right)^{\alpha} K(u)\,du\ f(x)\,dx. \tag{18}$$

Let $x \in \bar{\Delta}_{h\lambda_k,i}$. By considering the terms (16) and (18), we get

$$\begin{aligned}
&|\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha} \int_{-\infty}^{+\infty} K(u) f(uh\lambda_k - ih\lambda_k)\,du - \Big(1-\frac{x}{z}\Big)^{\alpha} K(u)\,du f(x)| \\
&\qquad = |\int_{-\infty}^{+\infty} \Big[\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha} f(uh\lambda_k - ih\lambda_k) - \Big(1-\frac{x}{z}\Big)^{\alpha} f(x)\Big] K(u)\,du| \\
&\qquad \le \int_{-\infty}^{+\infty} |\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha} - \Big(1-\frac{x}{z}\Big)^{\alpha} |f(x)| K(u)|\,du \\
&\qquad + \int_{-\infty}^{+\infty} |\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha} || f(uh\lambda_k - ih\lambda_k) - f(x)||K(u)|\,du.
\end{aligned} \tag{19}$$

Let $x \in \Delta_{h,i}$, we have, by the Lipschitz condition applied to the function,

$$g(x) = \Big(1-\frac{x}{z}\Big)^{\alpha},$$

$$\left|\left(1-\frac{ih}{z}\right)^{\alpha}-\left(1-\frac{x}{z}\right)^{\alpha}\right| \leq \frac{2\alpha h}{z} \quad \text{because} \quad \alpha = 0 \quad \text{or} \quad \alpha \geq 1.$$

Therefore, denoting by $I_1^i(x)$ the first integral of the right hand-side of (19) and

$$I_1(x) = \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \chi_{\bar{\Delta}_{h\lambda_k,i}}(x) I_1^i(x).$$

We have

$$\int_0^z I_1(x)\,dx \leq \frac{2\alpha h\lambda_k}{z}\int_0^z \left(\int_{-\infty}^{+\infty} f(x)|K(u)|\,du\right)dx = 2\alpha h\lambda_k\left(\int_{-\infty}^{+\infty}|K(u)|\,du\right)\frac{F(z)}{z}. \tag{20}$$

Denoting by $I_2^i(x)$ the second integral of the right hand-side of (19) and

$$I_2(x) = \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \chi_{\bar{\Delta}_{h\lambda_k,i}}(x) I_2^i(x).$$

Applying the lipschitz condition to the points ih and z. We have

$$I_2^i(x) \leq \frac{2\alpha h\lambda_k}{z}\left(\int_{-\infty}^{+\infty}|f(uh\lambda_k - ih\lambda_k) - f(ih\lambda_k)||K(u)|\,du + \int_{-\infty}^{+\infty}|f(ih\lambda_k) - f(x)||K(u)|\,du\right) \tag{21}$$

$$\begin{aligned}\int_0^z I_2(x)\,dx &\leq \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \chi_{\Delta_{h\lambda_k,i}}(x)\left(\int_{-\infty}^{+\infty}|f(uh\lambda_k + x - \theta_1 h\lambda_k) - f(x)||K(u)|\,du\right)\\ &\leq C(h\lambda_k)^{\gamma} z\int_0^z\left(\int_{\mathbb{R}}(|u|^{\gamma}+1)|K(u)|\,dudx.\right.\end{aligned} \tag{22}$$

Since

$$\int_0^z I_1(x)\,dx \leq \frac{2\alpha h\lambda_k}{z}\int_0^z \left(\int_{-\infty}^{+\infty} f(x)|K(u)|\,du\right)dx = 2\alpha h\lambda_k\left(\int_{-\infty}^{+\infty}|K(u)|\,du\right)\frac{F(z)}{z},$$

we get, together with the right hand-side of (17)

$$\begin{aligned}\sqrt{n}(\mathbb{E}(P_n(z,\alpha) - P(z,\alpha)) \leq h^{\gamma}\sqrt{n}\Big\{&Cz\int_0^z\left(\int_{\mathbb{R}}(|u|^{\gamma}+1)|K(u)|\,du\right)dx\\ &+ (h\lambda_k)^{(1-\gamma)}\sup_{x\in\mathbb{R}} f(x)\int_{-\infty}^{+\infty}|K(u)|\,du\\ &+ 2(h\lambda_k)^{(1-\gamma)}\alpha\left(\int_{-\infty}^{+\infty}|K(u)|\,du\right)F(z)\Big\}.\end{aligned}$$

The integrals in the braces exist. Hence the first part is proved.

For the second part, we define

$$U_i = \frac{1}{n}\sum_{0\leq l\leq[\frac{z}{h\lambda_k}]}\left(1-\frac{lh\lambda_k}{z}\right)^{\alpha} K\left(\frac{X_i - lh\lambda_k}{h\lambda_k}\right),$$

$$i = 1...n, \quad \mu_i = \mathbb{E}(U_i) \text{ and } \beta_i = \mathbb{E}(|U_i - \mu_i|^3).$$

Let $B_n = (\sum_{i=1}^{n} \beta_i)^{\frac{1}{3}}$. We shall obtain the statement 2) if, by Liapounov's theorem, we prove

$$\frac{B_n}{\sqrt{\mathbb{V}ar(P_n(z,\alpha))}} \to 0 \quad \text{as}\, n \to +\infty.$$

Consider $n^3\mathbb{E}(U_i^3)$. We have

$$\begin{aligned}
n^3\mathbb{E}(U_j^3) &= \mathbb{E}\Big[\big(\sum_{0\le l\le[\frac{z}{h\lambda_k}]}(1-\frac{lh\lambda_k}{z})^{\alpha}K(\frac{X_j - lh\lambda_k}{h\lambda_k})\big)^3\Big] \\
&= \mathbb{E}\Big[\Big\{\sum_{0\le l\le[\frac{z}{h\lambda_k}]}(1-\frac{lh\lambda_k}{z})^{3\alpha}K^3(\frac{X_j - lh\lambda_k}{h\lambda_k}) \\
&+ \sum_{0\le l\ne i\le[\frac{z}{h\lambda_k}]}(1-\frac{lh\lambda_k}{z})^{2\alpha}K^2(\frac{X_j - lh\lambda_k}{h\lambda_k})(1-\frac{ih\lambda_k}{z})^{\alpha}K(\frac{X_j - jh\lambda_k}{h\lambda_k}) \\
&+ \sum_{0\le l\ne i\ne j\ne l\le[\frac{z}{h\lambda_k}]}(1-\frac{lh\lambda_k}{z})^{\alpha}K(\frac{X_j - lh\lambda_k}{h\lambda_k})(1-\frac{ih\lambda_k}{z})^{\alpha} \\
&\times K(\frac{X_j - ih\lambda_k}{h\lambda_k})(1-\frac{jh\lambda_k}{z})^{\alpha}K(\frac{Xj - jh\lambda_k}{h\lambda_k})\Big\}\Big] \\
&\le \sum_{0\le i\le[\frac{z}{h\lambda_k}]}\int_{\mathbb{R}}(|K^3(\frac{u-ih\lambda_k}{h\lambda_k})|f(u)\,du \\
&+ \sum_{0\le l\ne i\le[\frac{z}{h\lambda_k}]}\int_{\mathbb{R}}K^2(\frac{u-lh\lambda_k}{h\lambda_k})|K(\frac{u-jh\lambda_k}{h\lambda_k})|f(u)\,du \\
&+ \sum_{0\le l\ne i\ne j\ne l\le[\frac{z}{h\lambda_k}]}\int_{\mathbb{R}}|K(\frac{u-lh\lambda_k}{h\lambda_k})K(\frac{u-ih\lambda_k}{h\lambda_k})K(\frac{u-jh\lambda_k}{h\lambda_k})|f(u)\,du.
\end{aligned}$$

The first term of the right hand-side of this inequality tends to $(\int_{\mathbb{R}}|K^3(y)|\,dy)P(z,2\alpha)$ as $n \to +\infty$.

The second term of the right hand-side of this inequality tends to zero as $n \to +\infty$ because of Theorem 2.10 Zakaria *et al.* (2018) in the unidimensional case.

The latter two terms of the right hand-side of this inequality tend to zero as $n \to +\infty$ because of the Fubini's theorem , the Theorem 2.10 and Corollary 2.5 Zakaria *et al.* (2018) in the one dimensional case.

Therefore, their limits exist. Hence the Liapounov's condition is satisfied. We have

$$n^3\mathbb{E}(|U_i - \mu_i|)^3 < +\infty \quad \text{and} \quad B_n = \frac{(\sum_{i=1}^n n^3\mathbb{E}(|U_i - \mu_i|))^{1/3}}{n}.$$

Let cst be a constant such that $n^3\mathbb{E}(|U_i - \mu_i|))^3 \leq cst$ therefore

$$\begin{aligned}\frac{B_n}{\sqrt{\mathbb{V}ar(P_n(z,\alpha))}} &= \frac{(\sum_{i=1}^n n^3\mathbb{E}(|U_i - \mu_i|))^{1/3}}{n\sqrt{\mathbb{V}ar(P_n(z,\alpha))}} \\ &= \frac{n^{1/3}cst}{n\sqrt{\mathbb{V}ar(P_n(z,\alpha))}}.\end{aligned}$$

On the other hand, we have

$$\mathbb{V}ar(P_n(z,\alpha)) = O(\frac{1}{n}) > 0,$$

.
Therefore

$$\frac{B_n}{\sqrt{\mathbb{V}ar(P_n(z,\alpha))}} = O(\frac{n^{1/3}}{nO(\frac{1}{n})}) \cong (\frac{n^{1/3}}{n^{\frac{1}{2}}})$$

which tends to zero as $n \to +\infty$. This completes the proof.

Proof of Theorem 2. We have

$$\sqrt{n}\Big(\mathbb{E}(P_n^{\nu}(z,\alpha)) - P(z,\alpha)\Big) \to 0 \quad \text{as}\, n \to +\infty$$

by Lemma 2.4 Zakaria *et al.* (2018) one dimensional case. The second part remains unchanged. The proof is complete.

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