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An introduction to a general records theory both for dependent and high dimensions

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Abstract. The probabilistic investigation on record values and record times of a sequence of random variables defined on the same probability space has received much attention from 1952 to now. A great deal of such theory focused on *iid* or independent real-valued random variables. There exists a few results for real-valued dependent random variables. Some papers deal also with multivariate random variables. But a large theory regarding vectors and dependent data has yet to be done. In preparation of that, the probability laws of records are investigated here, without any assumption on the dependence structure. The results are extended sequences with values in partially ordered spaces whose order is compatible with measurability. The general characterizations are checked in known cases mostly for *iid* sequences. The frame is ready for undertaking a vast study of records theory in high dimensions and for types of dependence.
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Résumé. L'investigation probabiliste des records d'une suite de variables aleatoires a reçu une grande attention depuis 1952 jusqu'à nos jours. Une grande partie de cette théorie a concerné les suites de variables aléatoires indépendantes, identiquement distribuées ou non, à valeurs réelles. Il existe quelques résultats pour les variables aléatoires dépendantes à valeurs réelles. Certains articles traitent également de variables aléatoires multivariées. Cependant, une théorie majeure relative aux données dépendantes ou multivariées est encore à faire. En préparation de cela, les lois de probabilité des records sont étudiées ici, sans aucune hypothèse sur la structure de dépendance des variables. Les résultats sont étendus aux suite de variables à valeurs dans un espace partiellement ordonné sur lequel l'ordre est compatible avec la mesurabilité. Les caractérisations générales sont testées sur les résultats connus portant principalemententent sur les suites *iid*. Le cadre est prêt pour passer à la théorie en haute dimension et pour differents types de dépendance.

1. Introduction

The theory of records both dealing with record values and record times for a sequence of random variables $(X_n)_{n\geq 1}$, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and taking their values in some measurable space E endowed with a partial order $(\leq)$ is a relatively recent sub-discipline of Probability Theory and Statistics. That theory goes back to Chandler (1952) and back to Feller (1966) who applied it to gambling problems.

By now, the theory has had extraordinary developments in a great variety of directions including characterization of distributions or of stochastic processes, statistical estimations. A few number of attempts beyond the scheme of independent and identically distributed (*iid*) sequences are available.

A stochastic process view has led to the extremal process (see Dwass (1964)) which has helped to solve many problems in Extreme Value Theory (see Resnick (1987)).

Especially, in the *iid* case, that theory is tremendously developed in a number of papers and more than a dozen of books (as quoted by Ahsanullah (2015)) has been reported. To cite a few, we have the following books : Ahsanullah (2008), Ahsanullah (1988), Ahsanullah (1995), Ahsanullah (2004), Ahsanullah (2006), Arnold *et al.* (1998), Gulatis and Padgett (2003), Nevzorev (2001), Resnick (1987) (partially).

Although several departures from the real-valued sequences frame, from the *iid* case and even from the independent assumption, it seems that there does not exist a complete review on the results of such generalizations, including the ones related to partial order relations, for examples in finite dimensional spaces like $\mathbb{R}^d$, $d > 1$.

The aim of this paper is two-fold. First, we wish to propose a general frame for the theory of records values and record times in an arbitrary partially or totally ordered space, including random fields and to find out the finite-dimensional probability laws for the records values and times. Such a general frame would be an appropriate place to summarize available generalizations and to be the basis to get new extensions.

It is expected that this presentations and the general formulas therein will quick off new trends of innovative research works from the readers.

2. General setting

2.1. Basic definitions about records on $\mathbb{R}$

We are going to introduce all the needed definitions on a sequence of real numbers $x = (x_n)_{n\geq 1}$.

(A) - Strong record times and strong upper records.

Let us define them by induction. The first record time, in general, is set to one, and we write $u(1) = 1$ and the first record is defined by

$$x^{(1)} = x_{u(1)} =: x_1.$$

Next we search

$$u(2) = \begin{cases} \inf\{j > u(1), x_j > x_{u(1)}\} \equiv \inf A_2 & \text{if } A_2 \neq \emptyset \\ +\infty & \text{otherwise} \end{cases}.$$

If $u(2) < +\infty$, we call $u(2)$ the second strong record time and the strong record value is given by

$$x^{(2)} = x_{u(2)}.$$

Given that the n-th strong record time exists, we may define

$$u(n+1) = \begin{cases} \inf\{j > u(n), x_j > x_{u(n)}\} \equiv A_n & \text{if } A_n \neq \emptyset \\ =+\infty & \text{otherwise} \end{cases}.$$

And, as previously, $u(n+1)$ is the $(n+1)$th strong record time if $u(n+1) < +\infty$, and

$$x^{(n+1)} = x_{u(n+1)}$$

is the $(n+1) - th$ strong record value.

Either me way proceed indefinitely and the sequence of record times $(u(n))_{n\geq 1}$ is unbounded or we stop the first time we have $u(n) = +\infty$ and in such a case, the sequence x has only $(n-1)$ record values where n is necessarily greater than one.

(B) - Weak records times and weak upper record values.

We may define weak versions of record times and values by allowing repetitions of the record values. The first weak record time set to one, and we write $u^{(w)}(1) = 1$ and the weak first record value is defined by

$$x^{(1,w)} = x_{u^{(w)}(1)} =: x_1.$$

Next we define

$$u(2) = \begin{cases} \inf\{j > u^{(w)}(1), x_j \geq x_{u(1)}\} \equiv \inf A_2 & \text{if } A_{n-1} \neq \emptyset \\ +\infty & \text{otherwise} \end{cases}.$$

If $u^{(w)}(2) < +\infty$, we call $u^{(w)}(2)$ the second weak record time and the weak record value is given by

$$x^{(2,w)} = x_{u^{(w)}(2)}.$$

Given that the n-th weak record time exists, we may define the

$$u^{(w)}(n+1) = \begin{cases} \inf\{j > u^{(w)}(n), x_j \geq x_{u(n)}\} = \inf A_n & \text{if } A_n \neq \emptyset \\ +\infty & \text{otherwise} \end{cases}.$$

Next we have, as previously, that if $u^{(w)}(n+1)$ is the $(n+1)-th$ weak record time if $u^{(w)}(n+1) < +\infty$, we may define

$$x^{(n+1,w)} = x_{u^{(w)}(n+1)}$$

is the $(n+1)$-th weak strong record value.

C - Lower records.

Generally we set the first record time to one for any kind of record value. The strong lower record times $(\ell(n))_{n\geq 1}$ and lower record values $(x_{\ell(n)})_{n\geq 1}$ are similarly defined by setting $\ell(1) = 1$ and $x_{(1)} = x_{\ell(1)} = 1$ and next by induction, if $\ell(n)$ is finite, by

$$\ell(n+1) = \begin{cases} \inf\{j > \ell(n), x_j < x_{\ell(n)}\} \equiv A_n & \text{if } A_2 \neq \emptyset \\ +\infty & \text{otherwise} \end{cases}.,$$

and $\ell(n+1)$ is the $(n+1)$-th strong lower record time if it is finite and the $(n+1)-th$ is given by

$$x_{(n+1)} = x_{\ell(n+1)}.$$

As well, the weak lower record times $(\ell^{(w)}(n))_{n\geq 1}$ and lower record value $(x_{\ell^{(w)}(n)})_{n\geq 1}$ are similarly defined by setting $\ell^{(w)}(1) = 1$ and $y_{(n,w)} = x_{\ell^{(w)}(1)} = 1$ and next by induction, if $\ell^{(w)}(n)$ is finite, by

$$\ell^{(w)}(n+1) = \begin{cases} +\infty & if \quad \{j > \ell^{(w)}(n), x_j \leq x_{\ell^{(w)}(n)}\} = \emptyset \\ \inf\{j > \ell^{(w)}(n), x_j \leq x_{\ell^{(w)}(n)}\} & \text{otherwise.} \end{cases},$$

and $\ell^{(w)}(n+1)$ is the $(n+1)$-th weak lower record time if it is finite and the $(n+1)$-th lower record value is given by

$$x_{(n+1,w)} = x_{\ell^{(w)}(n+1)}.$$

2.2. Random records and general formulas

We are going to move from statistic records to random ones.

(I) - Introduction.

To make short, we call *record values* by *records* simply and *records*, without any further precision, are *strong upper records*.

In this section, we consider now a sequence of random real random variables

$$X_1, \ X_2, \ldots$$

defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We then define the random records associated to this sequence. The records times and records become random variables defined by capital letters as follows

$$\begin{aligned} &U(n), \ L(n), \ U^{(w)}(n), \ L^{(w)}(n), \\ &X^{(n)} = X_{U(n)}, \ Y^{(n)} = X_{L(n)}, \\ &X^{(n,w)} = X_{U^{(w)}(n)}, \\ &X_{(n,w)} = X_{L^{(w)}(n)} \end{aligned}$$

We are going to give a series of general facts on the laws of the record values and record times. We may focus only of the upper records, since we are able to derive the results on lower records from those on upper records by using the transform

$$(X_1, \ X_2, \ldots) \ \mapsto \ (-X_1, \ -X_2, \ldots)$$

or, if the X_i's are *a.s.* positive, by using the transform

$$(X_1, \ X_2, \ldots) \ \mapsto \ (1/X_1, \ 1/X_2, \ldots).$$

Hence, we are going to study mainly the strong records in this paper.

First of all, we are going to see that the records times and the record values are Markovian.

(II) - Markovian properties of the records and the record times.

We have the following fact concerning the strong record times.

Proposition 1. *The sequence $(U(n))_{n\geq 1}$ of strong record times is a Markovian chain with transition probabilities*

$$p_{t,n}(k,j) = \begin{cases} \mathbb{P}\left(X_j > X_k, \max_{k<h<j} X_h \leq X_k\right) & \text{if} \quad j>k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us assume that $n \geq 2$. Let $j \geq n+1$, $1 = k_1 < k_2 < \ldots < k_n$. Conditionally on $(U(1) = k_1, \ \ldots, \ U(n-1) = k_{n-1}, \ U(n) = k_n)$, the event $(U(n+1) = j)$ depends only on the observations X_h, $k_n < h \leq j$ and reduces to

$$\left(X_{k_n+1} \leq X_{k_n}, X_{k_n+2} \leq X_{k_n}, \ldots X_{j-1} \leq X_{k_n}, X_j > X_{k_n}\right),$$

Thus we have

$$\begin{aligned} &\mathbb{P}(U(n+1) = j)/(U(1) = k_1, \ \ldots, \ U(n-1) = k_{n-1}, \ U(n) = k_n) \\ &= \mathbb{P}\left(X_{k_n+1} \leq X_{k_n}, X_{k_n+2} \leq X_{k_n}, \ldots X_{j-1} \leq X_{k_n}, X_j > X_{k_n}\right), \end{aligned}$$

which proves that behavior of $U(n)$ depends only on the most recent past k_n, and by the way, provides the probability transition.

Let us move to the record values. We have

Proposition 2. *The sequence $(X^{(n)})_{n\geq 1}$ of strong records is a Markovian chain with transition probability*

$$p_{r,n}(x, A) = \sum_{k=n}^{+\infty} \mathbb{P}\left((X_{min(j>k, \ X_j > x_n)} \in A)/(X^{(n)} = x)\right).$$

where x is a real number and A a Borel set of $\mathbb{R}$.

Proof. Let $n \geq 2$. Let $x_1 < x_2 < \ldots < x_n < x$. Conditionally on the intersection $B = (X^{(1)} = x_1, \ \ldots, \ X^{(n-1)} = x_{n-1}, \ X^{(n)} = x_n)$, the record $X^{(n+1)}$ is defined on $(U(n) = k)$ with $k \geq n$ by

$$X^{(n+1)} = X_{min(j>k, \ X_j > x_n)}$$

so that

$$\mathbb{P}((X^{(n+1)} \in A) \cap (U(n) = k)/B) = \mathbb{P}\left((X_{min(j>k,\ X_j>x_n)}) \in A) \cap (U(n) = k)/(X^{(n)} = x_n)\right).$$

We get

$$\mathbb{P}((X^{(n+1)} \in A)/B) = \sum_{k=n}^{+\infty} \mathbb{P}((X_{min(j>k,\ X_j>x_n)}) \in A) \cap (U(n) = k)/(X^{(n)} = x_n)),$$

which concludes the proof.

Now let us face the general probability laws of the sequences of record values $(X^{(n)})_{n\geq 1}$, of record times $(U(n))_{n\geq 1}$, of inter-record times $(\Delta(n))_{n\geq 1} = (\Delta_n)_{n\geq 1}$, of the number of record values in a sample $(N(n)+_{n\geq 1}$.

In each case, we give a general probability law regardless to the dependence between the X_j's. Next, we adapt the results to the situation where the X_j's are independent.

Finally, we give detailed results for the *iid* case. In the last case, we apply the previous results but compare our outcomes with formulas with those available in the literature, in particular in books of Ahsanullah.

2.3. General Joint Cumulative Distribution Functions

We are going to provide the most general expression, so that we will be able to refine it if we know more about the structure of the dependence of the finite-distributions.

Theorem 1. *For each $n \geq 1$, we have :*

(a) The joint cdf of the vector of records $X = (X^{(1)}, X^{(2)}, \cdots, X^{(n)})^T$ is given, for any $y = (y_1, ..., y_n) \in \mathbb{R}^n$,

$$\mathbb{P}(X^{(1)} \leq y_1, \cdots, X^{(n)} \leq y_n) \qquad (FD1)$$
$$= \int \mathbb{P}\left(\bigcap_{j=1}^{n} \left(\max_{z_{j-1}+1\leq h\leq z_j} X_h \leq y_j^*\right)\right) d\mathbb{P}_{(U(1),\cdots,U(n))}(z_1, \cdots, z_n),$$

where $y_i^* = \wedge_{j=i}^{n} y_j = min(y_i, \cdots, y_n)$, $i \in \{1, \cdots, n\}$.

(b) For k-tuple $(n_1, \cdots, n_k = n)$, $1 \leq k \leq n$ with $n_0 = 0 < 1 \leq n_1 < \cdots < n_k$, for any $y = (y_1, ..., y_k) \in \mathbb{R}^k$, we have

$$\mathbb{P}(X^{(n_1)} \leq y_1, \cdots, X^{(n_k)} \leq y_k = n) \qquad (FD2)$$
$$= \int \mathbb{P}\left(\bigcap_{j=1}^{k} \left(\max_{z_{j-1}+1 \leq h \leq z_j} X_h \leq y_j^*\right)\right) \, d\mathbb{P}_{(U(n_1), \cdots, U(n_k))}(z_1, \cdots, z_k.$$

Proof. Let us prove Formula (b). Let us consider an increasing sequence $(x_j)_{j \geq 1}$. For $1 \leq h \leq \ell \leq n$, we define by $A(h, \ell, y)$ the assertion:

$$\left(\text{All observations } x_j,\ h \leq j \leq \ell \text{ is or are less or equal to } y\right).$$

Let us see simple cases. It is not hard to see that for $1 \leq n_1 < n_2 < \cdots$. We easily see that

$$(x_1 \leq x, x_{n_1} \leq y) = (A(1,1,x) \ and \ A(1,n_1,y)) = (A(1,1,min(x,y)) \ and \ A(2,n_1,y))$$

and

$$(x_{n_1} \leq x, x_{n_2} \leq y) = (A(1,n_1,x) \ and \ A(1,n_2,y)) = (A(1,n_1,min(x,y))$$
$$and \ A(n_1+1,n_2,y))$$

If we understand the two previous examples, we can see that

$$(x_{n_1} \leq x_1, x_{n_2} \leq x_2, \cdots, x_{n_k} \leq x_k) \qquad (1)$$
$$= \Big((A(1,n_1,min(x_1,\cdots,x_k) \ and \ A(n_1+1,n_2,min(x_2,\cdots,x_k)$$
$$and \ A(n_2+1,n_3,min(x_3,\cdots,x_k),$$
$$\cdots,$$
$$A(n_{k-1}+1,n_k,x_k)\Big)$$

By Applying this simple rule, we have that the event $(Y^{(n_1)} \leq x_1, Y^{(n_2)} \leq x_2, \cdots, Y^{(n_2)} \leq x_k)$ given the event

$$(U(n_1) = z_1, \cdots, U(n_k) = z_k)$$

(with $U(0) = 0$ and $z_0 = 0$) is equivalent to that all the observations in the bloc of observations from X_{z_i+1} to X_{z_i} are less or equal to $y_i^* = \wedge_{j=i}^{n} y_j$ for $1 \leq i \leq n$. We get that $\mathbb{P}(X^{(n_1)} \leq y_1, \cdots, X^{(n_k)} \leq y_k = n)$ is equal to

$$\int \mathbb{P}\left(\bigcap_{i=1}^{k}\left(\max_{z_{j-1}+1<h<z_j} X_h \leq y_j^*\right)\right) \ d\mathbb{P}_{(U(n_1),\cdots,U(n_k))}(z_1,\cdots,z_k),$$

which is Formula (FD2) of which (FD2) is a particular form.

We have the following corollary.

Corollary 1. *We have:*

(a) If the random variables X_j's are independent, then for k-tuple $(n_1, \cdots, n_k = n)$, $1 \leq k \leq n$ with $n_0 = 0 < 1 \leq n_1 < \cdots < n_k$, for any $y = (y_1, ..., y_k) \in \mathbb{R}^k$, we have

$$\mathbb{P}(X^{(n_1)} \leq y_1, \cdots, X^{(n_k)} \leq y_k = n) \qquad (FDI1)$$
$$= \int \prod_{j=1}^{k} \prod_{z_{j-1}+1\leq h\leq z_j} F_{X_h}(y_j^*) \ \ d\mathbb{P}_{(U(n_1),\cdots,U(n_k))}(z_1,\cdots,z_k.$$

(b) If the random variables X_j's are iid with common cdf F, then for a k-tuple $(n_1, \cdots, n_k = n)$, $1 \leq k \leq n$ with $n_0 = 0 < 1 \leq n_1 < \cdots < n_k$, for any $y = (y_1, ..., y_k) \in \mathbb{R}^k$, we have

$$\mathbb{P}(X^{(n_1)} \leq y_1, \cdots, X^{(n_k)} \leq y_k = n) \qquad (FDI2)$$
$$= \int \prod_{j=1}^{k} F^{z_j - z_{j-1}}(y_j^*) \ \ d\mathbb{P}_{(U(n_1),\cdots,U(n_k))}(z_1,\cdots,z_k. \ \Diamond$$

2.4. Finiteness or Infiniteness of the total number of records

Let us begin by a general law.

Proposition 3. *For each $k \geq 1$, set*

$$X_k^\star = \sup_{h>k} X_h.$$

and denote

$$D_- = \{(x,y) \in \mathbb{R}^2, \ x \leq y\}.$$

We have for any $n \geq 2$

$$\mathbb{P}(U(n+1) = +\infty) = \sum_{k\geq n} \mathbb{P}_{(X_k^\star, X_k)}(D_-)\mathbb{P}(U(n) = k).$$

Proof. Conditioning on $(U(n)=k)$, $(U(n+1)=+\infty)$ means that all the X_h, $h>k$, are less than X_k. The proof is ended by the remark

$$\mathbb{P}(\max_{h>k} X_h \leq X_k) = \mathbb{P}(X_k^\star \leq X_k) = \mathbb{P}((X_k^\star, X_k) \in D_-) = \mathbb{P}_{(X_k^\star, X_k)}(D_-).\blacksquare$$

Let us give an application of Proposition 3 in the independent case.

Proposition 4. *Suppose that $X_1, X_2, \ldots$ are independent random variables with respective cumulative distribution functions (cdf) F_j, $j \geq 1$. Then, whenever $U(n)$ is finite, we have*

$$\mathbb{P}(U(n+1)=+\infty) = \sum_{k\geq n}\left(\int_{\mathbb{R}}\left(\prod_{j>k} F_j(x)\right) d\mathbb{P}_{X_k}(x)\right)\mathbb{P}(U(n)=k).$$

Proof. Here $X_k^\star$ and X_k are independent and we have

$$\mathbb{P}(U(n+1)=+\infty) = \sum_{k\geq n} \mathbb{P}_{X_k^\star} \otimes \mathbb{P}_{X_k}(D_-)\mathbb{P}(U(n)=k).$$

Let us use Fubini's theorem to have

$$\begin{aligned}
&\mathbb{P}_{X_k^\star} \otimes \mathbb{P}_{X_k}(D_-)\\
&= \int_{\mathbb{R}} d\mathbb{P}_{X_k}(x) \int_{\mathbb{R}} 1_{D_-}(x,y) d\mathbb{P}_{X_k^\star}\\
&= \int_{\mathbb{R}} \mathbb{P}(X_k^\star \leq x) d\mathbb{P}_{X_k}(x)\\
&= \int_{\mathbb{R}} \left(\prod_{j>k} F_j(x)\right) d\mathbb{P}_{X_k}(x).
\end{aligned}$$

We get the announced result by combining the above lines.

Now let us see what happens if the sequence is stationary, that is $F_j = F$ for all $j \geq 1$. Define the lower and the upper endpoints (lep and uep) of F by

$$uep(F) = \inf\{x \in \mathbb{R},\ F(x) > 1\} \text{ } and \text{ } uep(F) = \sup\{x \in \mathbb{R},\ F(x) < 1\}$$

We have

$$\int_{\mathbb{R}} \left(\prod_{j>k} F_j(x)\right) d\mathbb{P}_{X_k}(x) = \int_{-\infty}^{uep(F)} F(x)^{+\infty} dF(x).$$

But $F(x)^{+\infty} = 0$ unless $x = uep(F)$. This gives

$$\int_{\mathbb{R}}\left(\prod_{j>k} F_j(x)\right) d\mathbb{P}_{X_k}(x) = \int_{-\infty}^{uep(F)} 1_{\{uep(F)\}} dF(x) = \mathbb{P}(X = uep(F)).$$

We conclude that

$$\mathbb{P}(U(n+1) = +\infty) = \sum_{k \geq n} \mathbb{P}(X = uep(F))\mathbb{P}(U(n) = k) = \mathbb{P}(X = uep(F))$$

which leads to the simple result:

Proposition 5. *Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables with common cdf F and let $uep(F)$ denote the upper endpoint of F. Then*

$$\mathbb{P}(U(n+1) = +\infty) = \mathbb{P}(X = uep(F)).$$

As a consequence, the sequence of record values (and of record times) is finite if and only if $uep(F)$ is finite and is an atom of F, that is $\mathbb{P}_X(uep(F)) > 0$.

Consequences. The number of time records *a.s.* is infinite in the following cases.

(1) $uep(F) = +\infty$.

(2) $uep(F) < +\infty$ but $\mathbb{P}(X = uep(F)) = 0$. Example : $X \sim \mathcal{U}(0, 1)$.

The number of time records may be finite in the following cases.

(1) X is discrete and takes a finite number of points.

(2) X is discrete, takes an infinite number of values such the strict values set $\mathcal{V}_X$ of X has a maximum value. We mean by strict values set, the set of points taken by X with a non-zero probability.

2.5. Probability law of the sequence of increments of the record times

We make the convention that $U(0) = 0$. Let $n \geq 2$. If $U(n)$ is finite, we define $\Delta_n = U(n) - U(n-1)$. We have :

Proposition 6. *If $U(n)$ is finite, then the joint probability law of*

$$(\Delta_1, \cdots, \Delta_n)$$

is given by

$$= \int_{(x_1<x_2<\cdots<x_n)} \mathbb{P}\left(\left(\bigcap_{1\leq j\leq n-1} \max_{1+\overline{k}_j\leq h\leq \overline{k}_{j+1}-1} X_h \leq x_j\right)\right) \, d\mathbb{P}_{(X_{\overline{k}_1},\cdots,X_{\overline{k}_n})}(x_1,\cdots,x_n),$$

with $k_1=1$, $k_j\geq 1$ *for* $j\in\{2,\cdots,n\}$ *and* $\overline{k}_j=k_1+\cdots+k_j$ *for* $1\leq j\leq n$*.*

Proof. It is clear that $\Delta_1=k_1$ is possible only for $k_1=1$. So in the sequel, we fix $k_1=1$. Let us find the probability law of $(\Delta_2,\ldots,\Delta_n)$ through its discrete probability density

$$(\Delta_2=k_2,...,\Delta_n=k_n).$$

for $k_j\geq 1$, $2\leq j\leq n$, $k_1=1$. Let us define

$$C_j=\left(\max_{1+\overline{k}_j\leq h\leq \overline{k}_{j+1}-1} X_h \leq X_{\overline{k}_j}, X_{\overline{k}_{j+1}}>X_{\overline{k}_j}\right),\ j=1,\cdots,n-1$$

and

$$D_j(t)=\left(\max_{1+\overline{k}_j\leq h\leq \overline{k}_{j+1}-1} X_h \leq t\right),\ t\in\mathbb{R},\ j=1,\cdots,n-1$$

we have

$$(\Delta_2=k_2,\cdots,\Delta_n=k_n)=\bigcap_{1\leq j\leq n} C_j. \tag{2}$$

So, by conditioning by

$$Z=(X_{\overline{k}_1},\cdots,X_{\overline{k}_n})=(x_1,\cdots,x_n),$$

we get

$$\mathbb{P}(\Delta_2=k_2,...,\Delta_n=k_n)$$
$$= \int_{(x_1<x_2<\cdots<x_n)} \mathbb{P}\left(\bigcap_{1\leq j\leq n-1} D_j(x_j)\right) \, d\mathbb{P}_{(X_{\overline{k}_1},\cdots,X_{\overline{k}_n})}(x_1,\cdots,x_n).\blacksquare$$

We have the following corollary in the independent and in the *iid* cases.

Proposition 7. *Let* $k_1 = 1$, $k_j \geq 1$ *for* $j \in \{2, \cdots, n\}$ *and* $\overline{k}_j = k_1 + \cdots + k_j$ *for* $1 \leq j \leq n$.

Suppose that $U(n)$ *is finite.*

(a) If the X_j*'s are independent, we have*

$$\begin{aligned}&\mathbb{P}(\Delta_2 = k_2, ..., \Delta_n = k_n)\\&= \int_{(x_1<x_2<\cdots<x_n)} \prod_{1\leq j\leq n-1} \left(\prod_{1+\overline{k}_j \leq h \leq \overline{k}_{j+1}-1} F_{X_h}(x_j) \right) d \otimes_{1\leq j\leq n} \mathbb{P}_{(X_{\overline{k}_j}}(x_j).\end{aligned}$$

(b) (See Ahnsanullah (2001), page 32) If the X_j*'s are iid with common cdf* F*, we have for* $\overline{k}_j^* = k_2 + \cdots + k_j$,

$$\mathbb{P}(\Delta_2 = k_2, ..., \Delta_n = k_n) = \left(\left(\overline{k}_n^* + 1\right) \prod_{2\leq j\leq n} \overline{k}_j^* \right)^{-1} \prod_{2\leq j\leq n} 1_{k_j\geq 0}.$$

Before we give the proof, let us introduce the following lemma.

Lemma 1. *Let us define, for an arbitrary cdf F of measure of Lebesgue-Stieljes $\mathbb{L}$ and $k_j \in \mathbb{N} \setminus \{0\}$,*

$$\gamma(F, n, k_1, k_2, ..., k_n) \tag{3}$$

$$= \int_{(lep(F) \leq x_1 < x_2 < \cdots < x_n) \leq uep(F)} \prod_{1 \leq j \leq n} F(x_j)^{k_j - 1} \, d \bigotimes_{1 \leq j \leq n} \mathbb{L}^{\otimes n}(x_j). \tag{4}$$

Then for

$$\gamma(F, n, k_1, k_2, ..., k_n) = \left(\prod \overline{k}_j\right)^{-1}.$$

Proof of Proposition Lemma 1. Let us proceed by induction. For $n = 1$ and $k \geq 1$, we clearly have

$$\begin{aligned}
\gamma(F, n, k_1, k_2, ..., k_n) &= \int_{(lep(F) \leq x \leq uep(F)} F(x)^{k-1} \, d\mathbb{L}(x) \\
&= \int_{(lep(F) \leq x \leq uep(F)} F(x)^{k-1} \, dF(x) \\
&= \left[F(x)^k / k\right]_{lep(F)}^{uep(F)} = k^{-1}.
\end{aligned}$$

For $n = 2$, $k \geq 1$ and $\ell \geq 1$, we have

$$\begin{aligned}
\gamma(F, 2, k, \ell) &= \int_{(lep(F) \leq x \leq y \leq uep(F)} F(x)^{k-1} F(x)^{k-1} F(y)^{\ell - 1} \, d\mathbb{L}(x) d\mathbb{L}(y) \\
&= \int_{(lep(F) \leq x \leq uep(F)} F(x)^{k-1} d\mathbb{L}(x) \left(\int_{(x \leq y \leq uep(F)} F(y)^{\ell - 1} d\mathbb{L}(y) \right) \\
&= \int_{(lep(F) \leq x \leq uep(F)} F(x)^{k-1} d\mathbb{L}(x) \left[F(y)^{\ell} / \ell \right]_x^{uep(F)} \, d\mathbb{L}(x) \\
&= \frac{1}{\ell} \int_{(lep(F) \leq x \leq uep(F)} F(x)^{k-1} \left[1 - F(x)^{\ell} \right] \, d\mathbb{L}(x) \\
&= \frac{1}{\ell} \int_{(lep(F) \leq x \leq uep(F)} F(x)^{k + \ell - 1} d\mathbb{L}(x) \\
&+ \frac{1}{\ell} \int_{(lep(F) \leq x \leq uep(F)} F(x)^{k} d\mathbb{L}(x) \\
&= \frac{1}{\ell} (\gamma(F, 1, k) - \gamma(F, 1, k + \ell, k + 1)) \\
&= \frac{1}{\ell} \left(\frac{1}{k} - \frac{1}{k + \ell} \right) \\
&= (k(k + \ell)^{-1}.
\end{aligned}$$

So, Formula (1) holds for $n \in \{1,2\}$. Now, suppose that it holds for $n \geq 1$. We have

$$\begin{aligned}
&\gamma(F, n+1, k_1, \cdots, k_{n+1}) \\
&= \int_{(lep(F) \leq x_1 < x_2 < \cdots < x_{n+1}) \leq uep(F)} \prod_{1 \leq j \leq n+1} F(x_j)^{k_j - 1} \prod_{1 \leq j \leq n+1} d\mathbb{L}(x_j) \\
&= \int_{(lep(F) \leq x_1 < x_2 < \cdots < x_n \leq uep(F)} \prod_{1 \leq j \leq n} F(x_j)^{k_j - 1} \prod_{1 \leq j \leq n} d\mathbb{L}(x_j) \\
&\times \left(\int_{(x_n \leq x_{n+1} \leq uep(F)} F(x_{n+1})^{k_{n+1}-1} d\mathbb{L}(x_{n+1}) \right) \\
&= \frac{1}{k_{n+1}} \int_{(lep(F) \leq x_1 < x_2 < \cdots < x_n \leq uep(F)} \prod_{1 \leq j \leq n} F(x_j)^{k_j - 1} \left(1 - F(x_n)^{k_{n+1}} \right) \prod_{1 \leq j \leq n} d\mathbb{L}(x_j) \\
&= \frac{1}{k_{n+1}} \left(\gamma(F, n, k_1, \cdots, k_n) - \gamma(F, n+1, k_1, \cdots, k_n + k_{n+1}) \right) \\
&= \frac{1}{k_{n+1}} \left(\prod_{1 \leq j \leq n} \overline{k}_j \right)^{-1} \left(\frac{1}{\overline{k}_n} - \frac{1}{\overline{k}_n + k_{n+1}} \right) \\
&= \left(\prod_{1 \leq j \leq n} \overline{k}_j \right)^{-1}.
\end{aligned}$$

The proof is complete of Lemma 1. ■

Proof of Proposition of 7. The first is a translation of the result in Proposition 6 when the X_j's are independent. If the X_j are *iid*, we get, for $\overline{k}_j^* = k_2 + \cdots + k_j$, $j \geq 2$,

$$\begin{aligned}
A_n &\equiv \mathbb{P}(\Delta_2 = k_2, ..., \Delta_n = k_n) \\
&= \int_{(x_1 < x_2 < \cdots < x_n)} \prod_{1 \leq j \leq n-1} F(x_j)^{k_{j+1}-1} \, d\mathbb{P}_X^{\otimes n}(x_j).
\end{aligned}$$

Now, by Lemma 1, by applying the Fubini Theorem where we inetgrate on x_n, we have

$$\begin{aligned}
A_n &= \int_{(x_1 < x_2 < \cdots < x_n)} \prod_{1 \leq j \leq n-1} F(x_j)^{k_{j+1}-1} \bigotimes_{1 \leq j \leq n} d\mathbb{P}_X^{\otimes n}(x_j) \\
&= \int_{(x_1 < x_2 < \cdots < x_{n-1})} \prod_{1 \leq j \leq n-1} F(x_j)^{k_{j+1}-1} (1 - F(x_{n-1})) \, d\mathbb{P}_X^{\otimes (n-1)}(x_j) \\
&= \gamma(F, n, k_2, ..., k_n) - \gamma(F, n, k_2, ..., k_n + 1)
\end{aligned}$$

$$= \left(\overline{k}_n^* \prod_{2\leq j\leq n-1} \overline{k}_j^*\right)^{-1} - \left(\left(\overline{k}_n^*+1\right) \prod_{2\leq j\leq n-1} \overline{k}_j^*\right)^{-1}$$
$$= \left(\left(\overline{k}_n^*+1\right) \prod_{2\leq j\leq n} \overline{k}_j^*\right)^{-1}.$$

■.

Applications. For $n = 2$, we have

$$\mathbb{P}(\Delta_2 = k) = \frac{1}{k(k+1)}, \tag{5}$$

for $n = 2$, we get

$$\mathbb{P}\left(\Delta_2 = k, \Delta_3 = \ell\right) = \frac{1}{k(k+\ell)(k+\ell+1)}. \tag{6}$$

2.6. Probability Law of the sequence of the arrival times

Since we know the probability law of the sequence of the inter-record times, we may get the records times by the general formula, with $\ell_1 = 1 < \ell_2 < \cdots < \ell_n$,

$$\mathbb{P}(U(2) = \ell_2, ..., U(n) = \ell_n) = \mathbb{P}(\Delta_2 = \ell_2 - \ell_1, ..., \Delta_n = \ell_n - \ell_{n-1}) \tag{7}$$

Now, we may derive the law of $(U(2), \cdots, U(n))$, $n \leq 2$. Since we have for $\ell_2 < \ell_3 < \cdots < \ell_n$),

$$\mathbb{P}(U(2) = \ell_2, ..., U(n) = \ell_n) = \mathbb{P}(\Delta_2 = \ell_2 - 1, ..., \Delta_n = \ell_n - \ell_{n-1}) \tag{8}$$
$$\tag{9}$$

By applying Point (b) in proposition 7, we get

Theorem 2. *(See Ahnsanullah (2001), page 33) If the X_j's are real-valued iid random variables, the joint probability law of the record times $(U(2), \cdots, U(n))$, for $n \leq 2$, is given by*

$$\mathbb{P}(U2 = \ell_2, ..., Un = \ell_n) = \ell_n^{-1} \prod_{2\leq j\leq n} (\ell_j - 1)^{-1} 1_{(2\leq \ell_2 < \cdots < \ell_n)}. \tag{10}$$

3. Probability laws of strong records from iid sequences

Here, we are going to give the probability laws of the sequence of record values for independent and identically distributed random variables with common probability law $\mathbb{P}_X$. First, we give the joint cumulative distribution. Secondly, we treat the case where $\mathbb{P}_X$ is absolutely continuous with respect to the Lebesgue measure and finally, the case where $\mathbb{P}_X$ is a discrete probability.

Let us suppose that X, X_1, X_2, ... is a sequence of independent and identically distributed real-valued random variables, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with a common *cdf* F and common *pdf* f.

3.1. General Joint Cumulative Distribution Functions

Before we treat records, let us focus on the sequences of the maxima : $M_n = \max_{1\leq j\leq n}$, $n \geq 1$. We have

Theorem 3. *For each $n \geq 1$, we have :*

(a) The joint cdf of the vector of records $(X^{(1)}, X^{(2)}, \cdots, X^{(n)})^T$ is given, for any $y = (y_1, ..., y_n) \in \mathbb{R}^n$,

$$\mathbb{P}(M_1 \leq y_1, \cdots, M_n \leq y_n) = \prod_{i=j}^{n} F\left(\bigwedge_{j=i}^{n} y_j\right). \quad (PEX1)$$

where $y_i^* = \wedge_{j=i}^{n} y_j = min(y_i, \cdots, y_n)$, $1 \leq i \leq n$.

(b) For a k-tuple $(n_1, \cdots, n_k = n)$, $1 \leq k \leq n$ with $n_0 = 0 < 1 \leq n_1 < \cdots < n_k$, for any $y = (y_1, ..., y_k) \in \mathbb{R}^k$, we have

$$\mathbb{P}(M_{(n_1)} \leq y_1, \cdots, M_{(n_k)} \leq y_k = n) = \prod_{j=1}^{k} F^{(n_j - n_{j-1})}\left(\bigwedge_{i=j}^{k} y_i\right). \quad (PEX2)$$

Proof. We repeat the proof of Theorem 1 to get (a) by applying the principle described in (1) (page 2026). The formula in (b) represents a marginal law of *dcf* in (a). It is get by taking $x - j = +\infty$ in (a) for $j \notin \{n_1, \cdots, n_k\}$. $\square$.

To link this with record values, we notice that $(X^{(1)}, \cdots, X^{(n)}) = (M_{U(1)}, \cdots, M_{U(n)})$, for $n \geq 1$. We find a gain the joint *cdf* for records values as given in Theorem 1 in the *iid* case, that is given $(U(1) = n_1, \cdots, U(n) = n_k)$, we have

$$(X^{(1)}, \cdots, X^{(n)}) = (M_{n_1}, \cdots, M_{n_k})$$

and applying (b) in Theorem 3 allows us get the conditional law. So we have :

Theorem 4. *Let $n \geq 1$. Define*

$$\Gamma_n = \{(\ell_1, \cdots, \ell_n) \in (\mathbb{N} \setminus \{0\})^n, \quad \ell_1 = 1 < \ell_e < \cdots < \ell_n\}.$$

(a) The joint cdf of the vector of records $(X^{(1)}, X^{(2)}, \cdots, X^{(n)})^T$ is given, for any $y = (y_1, ..., y_n) \in \mathbb{R}^n$,

$$\mathbb{P}((X^{(1)} \leq y^1, \cdots, (X^{(n)} \leq y_n)$$
$$= \sum_{(\ell_1,\cdots,\ell_n)\in\in\Gamma_n} \prod_{j=1}^{k} F^{(n_j - n_{j-1})}\left(\bigwedge_{i=j}^{k} y_i\right) \mathbb{P}(U(1) = \ell_1, \cdots, U(n) = \ell_n).$$

3.2. Probability laws of strong records from absolutely continuous random variable

We are going to give the finite distributional probability laws of the sequence of strong records, the individual marginal distributions and different marginal laws involved two or more two individual margins.

But before we begin, we wish to explain a general method which will be systematically used. All the computations below which are related on n-th record $X^{(n)}$, $n \geq 2$, are based on conditioning on the past record $X^{(n-1)}$. For this, the reader is supposed to know the general formulas below. Let A be a measurable set of $(\Omega, \mathcal{A})$, S and T be two real valued variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If T has an absolute probability density function f_T with respect to the Lebesgue measure and supported by $\mathcal{V}_X$, we have

$$\mathbb{E}(S) = \int_{\mathcal{V}_X} \mathbb{E}(S|(T = t)) f_T(t) dt \tag{11}$$

and

$$\mathbb{P}(A) = \int_{\mathcal{V}_X} \mathbb{P}(A|(T = t)) f_T(t) dt. \tag{12}$$

If T is discrete with values set $\mathcal{V}_X = \{x_j, \ j \in J\}$, $J \subset \mathbb{N}$, we have

$$\mathbb{E}(S) = \sum_{j\mathcal{V}_X} \mathbb{E}(S|(T = x_j)) \mathbb{P}(T = x_j) \tag{13}$$

and

$$\mathbb{P}(A) = \sum_{j\mathcal{V}_X} \mathbb{P}(A|(T = x_j)) \mathbb{P}(T = x_j). \tag{14}$$

For advanced readers, we may use the counting measure ν on the discrete set $\mathcal{V}_X$ and the *pdf* $f_T(x) = \mathbb{P}(T = x), \ x \in \mathcal{V}_X$, with respect to the counting measure to unify both formulas in

$$\mathbb{E}(S) = \int_{\mathcal{V}_X} \mathbb{E}(S|(T=t)) f_T(t) d\mu(t) \tag{15}$$

$$\mathbb{P}(A) = \int_{\mathcal{V}_X} \mathbb{P}(A|(T=t)) f_T(t) d\mu(t). \tag{16}$$

where μ is the Lebesgue measure ($\mu = \lambda$) if T has an *pdf* f_T with respect to the Lebesgue measure and supported by $\mathcal{V}_X$, is the counting measure ($\mu = \nu$) on the discrete values set $\mathcal{V}_T$ if T takes at most countable values.

In order to be in a better position for handling the conditioning on the immediate past $X^{(n-1)}$, we introduce the random variable

$$\begin{aligned} & N(n-n, n) \\ = & \text{ Number of observations with time strictly between } U(n-1) \text{ and } U(n). \end{aligned}$$

In the sequel, f, $\mathcal{V}_X$ and F denote the *pdf* of X and its the support and its *cdf*, respectively, in the case of stationary sequence. We also denote

$$R(x) = -\log(1-F(x)) \ \ and \ \ r(x) = \frac{f(x)}{1-F(x)}, \ x \in \mathcal{V}_X, \ F(x) < 1.$$

where $r(.)$ is the hazard function of X, with obvious relation :

$$r(x)/dx = dR(x), \ x \in \mathcal{V}_X, \ F(x) < 1.$$

Let us give at once the joint absolutely probability law from which marginals probability will follow.

Theorem 5. *(a) The joint probability law of* $(X^{(1)}, X^{(2)}, \cdots, X^{(n)})^T$, $n \geq 1$, *is absolutely continuous to the Lebesgue mesure* $\lambda^{\otimes n}$ *on* $\mathbb{R}^n$ *with joint pdf*

$$f^{(1,2,\cdots,n)}(y) = \left(\prod_{i=1}^{n-1} r(y_j)\right) f(y_n), 1_{((y_1<y_2<\cdots<y_n)\cap \mathcal{V}_X^n)}(y). \quad (ADR1)$$

(b) More over for any k*-tuple* $(n_1, \cdots, n_k)$, $1 \leq k \leq n$ *with* $1 < n_1 < \cdots < n_k$, *the joint pdf with respect to* $\lambda^{\otimes k}$ *with support* $\mathcal{V}_X^k$ *is given by*

$$\begin{aligned} & f^{(n_1,n_2,\ldots,n_k)}(y_1, \cdots, y_k) \quad (ADR2) \\ = & \frac{R(y_1)}{\Gamma(n_1)} \prod_{j=2}^{k} \frac{(R(y_j) - R(y_{j-1}))^{n_j - n_{j-1} - 1}}{\Gamma(n_j - n_{j-1})} f(y_k) \\ & \times 1_{((y_1<\cdots<y_k)\cap \mathcal{V}_X^k)}(y_1, \cdots, y_k). \end{aligned}$$

c) Each $X^{(n)}$, $n \geq 1$, is absolutely continuous with pdf

$$f^{(n)}(x) = \frac{R(x)^{n-1}}{\Gamma(n)} f(x) 1_{\mathcal{V}_X}(x). \quad (ADR3)$$

Remark. The strict inequality in the domain $(y_1 < \cdots < y_k)$ is due the the fact for an absolutely continuous random vector, the event of the equality of some components is a $null$-event. We may also say that ii comes from the fact that the y_j's are strong records. Actually in the case where the *iid* sequence is associated to an absolutely *pdf*, there are string records only.

Proofs. Let us organize the proofs into points.

Direct Proof of (c). This part might be proved as a consequence of (b). But we need to learn how to use the Markov property in a simple case. So, we give a direct proof. Let us begin by the two first cases $n = 1$ and $n = 2$ and and next, we proceed by induction. For $n = 1$, we have that $X^{(1)} = X_1$ and the *pdf* f of X_1 is given by Point (c). For $n = 2$, we may use the conditioning formula (16) to have

$$\mathbb{P}(X^{(2)} \in [y - dy/2, y + dy/2])$$
$$= \int_{-\infty}^{y} \mathbb{P}(X^{(2)} \in [y - dy/2, y + dy/2] | X_1 = x) f(x) dx$$
$$= \int_{-\infty}^{y} \left(\sum_{j=0}^{-\infty} \mathbb{P}((X^{(2)} \in [y - dx/2, y + dy/2],\ N(1, 2) = j)) | (X_1 = x)) \right.$$
$$\left. f(x) \right) dx.$$

The following reasoning will be quoted in the sequel as :

(ARG) The event

$$((X^{(2)} \in [y - dx/2, y + dy/2],\ N(1, 2) = j)) | (X_1 = x))$$

means exactly that X_1 is given and we have j independent random variables X_h less that x and the following random variables (in the enumeration) is in $[y - dx/2, y + dy/2]$. The bounds of the integral come from the fact that the second record in $[y - dy/2, y + dy/2]$ should be greater that the first x and the fact that no point falls in $[y - dy/2, y + dy/2]$ except the second record based on the continuity assumption. So, the probability of this event is still

$$F(x)^j f(y) dy.$$

Then we have, for $0 < F(x) < 1$

$$
\begin{aligned}
&\mathbb{P}(X^{(2)} \in [y - dx/2, y + dy/2])\\
&= \int_{-\infty}^{y} \sum_{j=0}^{-\infty} F(x)^j f(y) f(x) dx dy\\
&= f(y)dy \int_{-\infty}^{y} \frac{f(x)}{1-F(x)} dx\\
&= f(y)dy \int_{-\infty}^{y} r(x)dx = f(y)dy \int x^{+\infty} dR(x)\\
&= R(y)f(y)dy.
\end{aligned}
$$

We get the *pdf* of $X^{(2)}$ by letting $dy \rightarrow 0$ and find again Formula $(PLSRC)$ for $n = 2$. Now for the general, we proceed by induction by assuming that $(PLSRC)$ holds for $n \geq 1$. Let us find the *pdf* for $X^{(n+1)}$. We repeat the same method used for $n = 2$ to get

$$
\begin{aligned}
&\mathbb{P}(X^{(n+1)} \in [y - dy/2, y + dy/2])\\
&= \int_{-\infty}^{y} \mathbb{P}(X^{(n+1)} \in [y - dy/2, y + dy/2] | (X^{(n)} = x)) f^{(n)}(x) dx\\
&= \int_{-\infty}^{y} \left(\sum_{j=0}^{-\infty} \mathbb{P}((X^{(n+1)} \in [y - dy/2, y + dy/2],\ N(n+1, n) = j)) | (X^{(n)} = x)) \right.\\
&\left. \times f^{(n)}(x) \right) dx.
\end{aligned}
$$

By using the argument **(ARG)** (page 2038), we see that the probability of the event $((X^{(n+1)} \in [y - dx/2, y + dy/2],\ N(n+1,n) = j))|(X^{(n)} = x))$ is

$$F(x)^j f(y)dy.$$

Next, we have

$$\begin{aligned}
&\mathbb{P}(X^{(n+1)} \in [y - dy/2, y + dy/2]) \\
&= \int_x^{+\infty} \sum_{j=0}^{-\infty} F(x)^j f(y) f^{(n)}(x) dx dy \\
&= f(y)dy \int_x^{+\infty} \frac{f(x)}{1 - F(x)} f^{(n)}(x) dx \\
&= f(y)dy \int_x^{+\infty} \frac{r(x)R(x)^{n-1}}{\Gamma(n)} dx = f(y)dy \int_x^{+\infty} \frac{d(R(x)^n)}{n\Gamma(n)} \\
&= \frac{R(x)^n}{\Gamma(n+1)} f(y)dy.
\end{aligned}$$

We get the *pdf* of $X^{(n+1)}$ by letting $dy \to 0$ and find again Formula (c) for $n+1$. The proof by induction is finished.□

Next we have :

Proof of (a) : The finite distributional probability law.

Let us begin by proving there result for $n = 2$. We have, for x fixed such that $0 \leq F(x) < 1$ and $y > x$,

$$\begin{aligned}
&\mathbb{P}(X^{(1)} \in [x - dx/2, x + dx/2], X^{(2)} \in [y - dy/2, y + dy/2]) \\
&= \mathbb{P}((X^{(2)} \in [y - dy/2, y + dy/2])|(X^{(1)} \in [x - dx/2, x + dx/2])) \\
&\times \mathbb{P}(X^{(1)} \in [x - dx/2, x + dx/2]) \\
&= \mathbb{P}((X^{(2)} \in [y - dy/2, y + dy/2])|(X^{(1)} \in [x - dx/2, x + dx/2])) f(x)dx.
\end{aligned}$$

Since f is the *pdf* of $X^{(1)}$. Using the argument **(ARG)** (page 2038) above, the set

$$(X^{(2)} \in [x - dx/2, x + dy/2])|(X^{(1)} \in [x - dx/2, x + dy/2]),$$

when decomposed over the events $N(1,2) = j)$, leads to

$$\begin{aligned}
&\mathbb{P}(X^{(1)} \in [x - dx/2, x + dx/2], X^{(2)} \in [x - dy/2, x + dy/2]) \\
&= \int_{\mathcal{V}_X} \sum_{j=0}^{+\infty} \mathbb{P}\Big((X^{(2)} \in [x - dy/2, x + dy/2], N(1,2) = j) | (X^{(1)} \in [x - dx/2, x + dx/2])\Big) \\
&\times f(x)dx \\
&= \sum_{j=0}^{+\infty} F(x - dx/2)^j f(y) dy f(x) dx \\
&= \frac{f(x)}{1 - F(x - dx/2)} dxdy.
\end{aligned}$$

As dx and dy go to zero, we exploit the continuity of F at x, do have

$$f^{(1,2)} = r(x)f(y)1_{(x<y)\cap \mathcal{V}_X^2},$$

which proves (b) for $n = 2$. Suppose it is true for $n \geq 2$. Let us prove it for $n+1$. Let us fix $x_1 < \ldots x_n < x_{n+1}$, all of them in $\mathcal{V}_X$ and let us write, for short, $[x_i - dx_i/2, x_i + dx_i/2] = x_i \pm dx_i/2, \ i = 1, \ldots, n$. We have

$$\begin{aligned}
&\mathbb{P}(X^{(1)} \in x_1 \pm dx_1/2, \ldots, X^{(n)} \in x_n \pm dx_n/2, X^{(n+1)} \in x_{n+1} \pm dx_{n+1}/2) \\
&= \mathbb{P}((X^{(n+1)} \in x_{n+1} \pm dx_{n+1}/2) | (X^{(1)} \in x_1 \pm dx_1/2, \ldots, X^{(n)} \in x_n \pm dx_n/2) \\
&\times \mathbb{P}((X^{(1)} \in x_i \pm dx_i/2, \ldots, X^{(n)} \in x_n \pm dx_n/2) \\
&= P((X^{(n+1)} \in x_{n+1} \pm dx_{n+1}/2) | (X^{(1)} \in x_1 \pm dx_1/2, \ldots, X^{(n)} \in x_n \pm dx_n/2)) \\
&\times f^{(1,\ldots,n)}(x_1 \cdots, x_n) dx_1 \ldots dx_n (1 + o(1)))
\end{aligned}$$

By Markov property of the records, we have

$$\begin{aligned}
&\mathbb{P}((X^{(1)} \in x_i \pm dx_i/2, \ldots, X^{(n)} \in x_n \pm dx_n/2) \\
&= ((X^{(n)} \in x_{n+1} \pm dx_{n+1}/2) | (X^{(n)} \in x_n \pm dx_n/2)).
\end{aligned}$$

By using again the argument **(ARG)** (page 2038), we obtain

$$\begin{aligned}
&\mathbb{P}(X^{(1)} \in x_1 \pm dx_1/2, \ldots, X^{(n)} \in x_n \pm dx_n/2, X^{(n+1)} \in x_{n+1} \pm dx_{n+1}/2) \\
&= \sum_{j=0}^{+\infty} \mathbb{P}((X^{(n+1)} \in x_{n+1} \pm dx_{n+1}/2, N(n, n+1) = j) | (X^{(n)} \in x_n \pm dx_n/2)) \\
&\times \ f^{(1,\ldots,n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n (1 + o(1)).
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{+\infty} \mathbb{P}((X^{(n+1)} \in x_{n+1} \pm dx_{n+1}/2, N(n, n+1) = j) | (X^{(n)} \in x_n \pm dx_n/2)) \\
&\times \ f^{(1,\ldots,n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n \\
&= \frac{1}{1 - F(x_n - dx_n/2)} f(x_{n+1}) f^{(1,\ldots,n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n dx_{n+1}
\end{aligned}
$$

The induction hypothesis gives

$$
\begin{aligned}
&\mathbb{P}(X^{(1)} \in x_1 \pm dx_1/2, \ldots, X^{(n)} \in x_n \pm dx_n/2, X^{(n+1)} \in x_{n+1} \pm dx_{n+1}/2) \\
&= r(x_1) \ldots r(x_{n-1}) \frac{f(x_n)}{1 - F(x_n - dx_n/2)} f(x_{n+1}) dx_1 dx_{n+1}.
\end{aligned}
$$

We get (a) for $n+1$ by letting all the dx_i go to zero for $i = 1, ..., n$.■

The combination of Points (a) and (c) allows to have for $y \in \mathcal{V}_X$,

$$
\int_{+\infty < x_1 < \ldots < x_{n-1} < y} \left(\prod_{j=1}^{n-1} r(x_i) \right) dx_1 \ldots dx_{n-1} = \frac{R(y)^{n-1}}{\Gamma(n)}. \tag{17}
$$

Indeed, the marginal *pdf* $f^{(n)}$ of $X^{(n)}$ is obtained from the joint *pdf* $f^{(1,\ldots,n)}$ by

$$
\begin{aligned}
f^{(n)}(y) &= \int_{(x_1 < \ldots < x_{n-1} < y)} \left(\prod_{j=1}^{(n-1)} r(x_i) \right) f(y) dx_1 \ldots \ dx_n \\
&= f(y) \int_{(+\infty < x_1 < \ldots < x_{n-1} < y)} \left(\prod_{j=1}^{n-1} r(x_i) \right) dx_1 \ldots \ dx_n.
\end{aligned}
$$

From Point (c), we make an identification and get Formula (17). We may replace $-\infty$ by $z \in \mathcal{V}_X, z < y$ and expect to have

$$
\int_{(z < x_1 < \ldots < x_{n-1} < y)} \left(\prod_{j=1}^{n-1} r(x_i) \right) = \frac{(R(y) - R(z))^{n-1}}{\Gamma(n)}. \tag{18}
$$

This is proved in the Appendix Part 2. □

Proof of (b) : Distribution of the sub-vector of $(X^{(1)}, X^{(2)}, \cdots, X^{(n)})^t$**,** $n \geq 1$.

From Point (a), may find the *pdf* of $(X^{(n_1)}, X^{(n_2)}, \ldots X^{(n_k)})$, denoted $f^{(n_1,n_2,\ldots,n_k)}$ for $1 \leq n_1 < n_2 < \ldots < n_k$. Indeed, we have to integrate the joint **pdf** $f^{(1,2,\ldots,n_k)}$ with respect to dx_i,

$$i \in \{1, \ldots, n_k\} \setminus \{n_1, \ldots, n_k\},$$

that is, for $x_{n_1} < x_{n_2} < \ldots < x_{n_{k-1}} < x_{n_k}$,

$$\begin{aligned}
&f^{(n_1,n_2,\ldots,n_k)}(x_{n_1}, x_{n_2}, \ldots, x_{n_{k-1}}, x_{n_k}) \\
&= \int_{x_1<\ldots<x_{n_k}} f^{(1,2,\ldots,n_k)}(x_1, \ldots, x_{n_k}) dx_1 \ldots dx_{n_1-1}\ dx_{n_1+1} \\
&\ldots dx_{n_2-1} \ldots dx_{n_{k-1}+1} \ldots dx_{n_k-1} \\
&= r(x_{n_1}) \int_{x_1<\ldots<x_{n_1-1}<x_{n_1}} \prod_{j=1}^{n_1-1} r(x_j)\ dx_1 < \ldots < dx_{n_1-1} \\
&r(x_{n_2}) \int_{x_{n_1}<x_{n_1+1}<\ldots<x_{n_2-1}<x_{n_2}} \prod_{j=n_1+1}^{n_2-1} r(x_j)\ dx_{n_1+1} < \ldots < dx_{n_2-1} \\
&r(x_{n_3}) \int_{x_{n_2}<x_{n_2+1}<\ldots<x_{n_3-1}<x_{n_3}} \prod_{j=n_2+1}^{n_3-1} r(x_j)\ dx_{n_2+1} < \ldots < dx_{n_3-1} \\
&\ldots \\
&1 \times \int_{x_{n_{k-1}}<x_{n_{k-1}+1}\ldots<x_{n_{k-1}-1}<x_{n_k}} \prod_{j=n_{k-1}+1}^{n_{k/}-1} r(x_j)\ dx_{n_{k-1}+1} \ldots < dx_{n_{k-1}-1}\ \times f(x_{n_k}).
\end{aligned}$$

By applying Formulas (17) and (17), we get, for $x_{n_0} = lep(F)$,

$$\begin{aligned}
&f^{(n_1,n_2,\ldots,n_k)}(x_{n_1}, x_{n_2}, \ldots, x_{n_{k-1}}, x_{n_k}) \\
&= f(x_{n_k}) \prod_{j=1}^{k-1} r(x_{n_j}) \prod_{j=1}^{k} \frac{(R(x_{n_j}) - R(x_{n_{j-1}}))^{n_j-n_{j-1}-1}}{\Gamma(n_j - n_{j-1})} f(x_{n_k}). \blacksquare
\end{aligned}$$

3.3. Examples of distributions of records

A few examples could be useful for studying all the strong records for instance because of the

(A) Re-scaling property. For any sequence $(x_j)_{1\leq j\leq n}$ of n real numbers, $n \geq 2$, and for any increasing function $h : \mathbb{R} \to \mathbb{R}$, the records values of $(h(x_j))_{1\leq j\leq n}$ are the images by h of the record values of $(x_j)_{1\leq j\leq n}$ according the same order and the record times are the same for $(x_j)_{1\leq j\leq n}$ and $(h(x_j))_{1\leq j\leq n}$.

Another interesting rule the following representation.

(B) The Renyi representation. Let X and Y two real-valued random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with *cdf*'s F and G where G is invertible. Then

$$X =_d F^{-1}(G(Y)),$$

where $F^{-1}(u) = \inf\{x \in [lep(F), uep(F)],\ F(x) \geq u\},\ u \in]0,1[$, is the generalized inverse of F and

$$lep(F) = \inf\{x \in \mathbb{R},\ F(x) > 0\},\ uep(F) = \sup\{x \in \mathbb{R},\ F(x) < 1\}$$

By applying Point (A) above to random variable we have :

Proposition 8. *Let us consider sequence iid sequence of random variables $(X_j)_{1\leq j\leq n}$ with common cdf F, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let us suppose that F is strictly increasing. Then for any iid sequence of random variables $(Y_j)_{1\leq j\leq n}$ with common cdf G, we have the following equalities in distribution of the two sequences*

$$\{X_1, \cdots, X_n\} =_d \{Y_1, \cdots, Y_n\}, \tag{19}$$

and (U(H,j) is the j-th record time from the cdf H),

$$\{U(F,1), \cdots, U(F,1)\} =_d \{U(G,1), \cdots, U(G,1)\}, \tag{20}$$

and finally by the re-scaling property for the record values,

$$\{X^{(1)}, \cdots, X^{(n)}\} =_d \{F^{-1}(G(Y^{(1)})), \cdots, F^{-1}(G(Y^{(n)}))\}. \tag{21}$$

An important example concerns the case where G is the standard exponential *cdf* : $F(x) = 1 - exp(-x)$, $x \geq 0$. We get

Proposition 9. *Let us consider sequence iid sequence of random variables $(X_j)_{1\leq j\leq n}$ with common cdf F, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let us suppose that F is strictly increasing and G is invertible. Then for for any iid sequence of iid standard exponential random variables $(E_j)_{1\leq j\leq n}$, we have the equality in distribution of the records times of the two sequences*

$$\{U(F,1), \cdots, U(F,1)\} =_d \{U(E,1), \cdots, U(F,1)\}, \tag{22}$$

the re-scaling property

$$\{X^{(1)}, \cdots, X^{(n)}\} =_d \{F^{-1}(1 - exp(-E^{(1)})), \cdots, F^{-1}(1 - exp(-E^{(n)})\}. \tag{23}$$

Because of Formula (23), a huge number of problems on records are studied through it and the Records theory becomes a theoretical study on functions F^{-1} and on exponential random variables.

Now, two interesting and useful examples are given, one for absolutely continuous random variable and one for the discrete random variables : the exponential records we are giving right now and the geometric records to be stated in page 2048 after the discrete records are presented.

(C) Exponential records. Let us consider a sequence of independent standard exponential random variables $(X_j)_{1\leq j\leq n}$, $n \geq 2$, of intensity $\theta > 0$, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, By Theorem 5 (page 5) and Formula (ADR) therein, we have

$$f_{(X^{(1)},\cdots,X^{(n)})}(x_1,\cdots,x_n) = \theta^n e^{-\theta x_n} 1_{(0\leq x_1\leq\cdots\leq x_n)}. \tag{24}$$

We see that $(X^{(1)},\cdots,X^{(n)})$ are the arrival times of a Poisson stochastic process of intensity λ and hence inter-arrival (for $X^{(0)} = 0$)

$$X^{(j)} - X^{(j-1)},\ j \in \{1,\cdots,n\},$$

are independent and identically distributed following θ-exponential law.

3.4. Probability laws of strong records from a discrete random variable

(I) General formula in the iid case.

Here, we suppose that X, X_1, X_2, ... is a sequence of independent and identically distributed real-valued random variables, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with a common *discrete pdf* f given on the strictly support $\mathcal{V}_X = \{x_j,\ j \in J\} \subset \mathcal{R}$, $J \subset \mathbb{N}$, by

$$f(x) = \mathbb{P}(X = x)1_{\mathcal{V}_X}(x),\ x \in \mathbb{R}, \tag{25}$$

with the condition

$$(\forall x \in \mathcal{V}_X,\ f(x) > 0),\ \ (\forall x \notin \mathcal{V}_X,\ f(x) = 0)\ \ \textit{and}\ \ \left(\sum_{x\in\mathcal{V}_X} f(x) = 1\right).$$

Similarly to the absolutely continuous case, we define for all $x \in \mathbb{R}$,

$$\begin{aligned}F(x) &= \sum_{j\in J,\ x_j\leq x} f(x_j)\\ 1-F(x) &= \sum_{j\in J,\ x_j> x} f(x_j)\\ &= R(x) = \frac{f(x)}{1-F(x)},\quad for\ x \in x \in \mathcal{V}_X\ and\ F(x)<1.\end{aligned}$$

Actually, we mainly use the integration methods, here with respect to the counting measure μ supported by $\mathcal{V}_X$ and defined by

$$\mu = \sum_{j\in J} \delta_{x_j},$$

(where δ_{x_j} is the Dirac measure concentrated on x_j with mass one), with respect to which the probability law $\mathbb{P}_X$ have the Radon-Nikodym derivative f, that is

$$d\mathbb{P}_X = f\ d\mu.$$

Let us introduce a notation that will replace $R(x) = -\log(1-F(x))$ in the discrete case. We denote for $x<y$,

$$J(x,y) =]x,y[\cap \mathcal{V}_X \quad and \quad \#J(x,y) = d(x,y).$$

For two integers $r<s$, we set

$$Jo(r,s,x,y) = \{(t_1,\cdots,t_{r-s-1}) \in J(x,y)^{r-s-1},\ t_1<\cdots<t_{r-s-1}\}$$

We define $R(x,y)=1$ if $d(x,y)=0$ or $r-s-1=0$. Otherwise, we set

$$R(r,s,x,y) = \sum_{(t_1,\cdots,t_{r-s-1})\in Jo(r,s,x,y)} \prod_{h=1}^{r-s-1} r(t_h). \tag{26}$$

In general, similar rules based only on integration with respect to a measure apply. We get the same formulas but the *pfd*'s are with respect ot counting measures. But for pedagogical purposes, redoing the proofs using discrete integration has its own merit. By taking the discrete form of the *pdf*'s in Proposition 5, we have

Proposition 10. *(a) The joint probability law of* $(X^{(1)}, X^{(2)}, \cdots, X^{(n)})^T$, $n\geq 1$, *has pdf with respect to* $\mu^{\otimes n}$ *with support* $\mathcal{V}_X^n$ *with discrete pdf*

$$f^{(n)}(y_1,y_2,\cdots,y_n) = \left(\prod_{i=1}^{n-1} r(y_j)\right) f(y_n), 1_{((y_1<y_2<\cdots<y_n)\cap\mathcal{V}_X^n)}(y). \quad (DDR_2)$$

(b) More over for any k*-tuple* $(n_1,\cdots,n_k)$, $1\leq k\leq n$ *with* $1<n_1<\cdots<n_k$, *the joint discrete pdf with respect to* $\mu^{\otimes k}$ *with support* $\mathcal{V}_X^k$ *is given by :*

$$f^{(n_1,n_2,\dots,n_k)}(y_1,\cdots,y_k) = f(y_n)\prod_{j=1}^{k-1} r(y_{n_j})\prod_{j=1}^{k} R(n_{j-1},n_j,x_{n_{j-1}},x_{n_j}).$$

(c) For each $n \geq 1$, the pdf with respect to the counting measure supported by $\mathcal{V}_X$ is given by

$$f^{(n)}(y) = R(0,n,y_0,y)f(y)1_{\mathcal{V}_X}(y). \quad (DDR_3)$$

Remark. Here, the strict inequality in the domain $(y_1 < \cdots < y_k)$ is due the fact that the y_j's are strong records. Here we have to distinguish between strong and weak records.

Proof. Let us begin by the joint distribution of $(X^{(1)},X^{(2)},\cdots,X^{(n)})^T$, $n \geq 1$. For $y_1 < y_2 < \cdots < y_n$. Let us set

$U(j,j+1)$ as the number of observations strictly between the j-th record $X_{U(j)}$ and the $(j+1)$-th record $X_{U(j+1)}$.

The event

$$A = (X^{(1)} = y_1,\cdots,X^{(n)} = y_n)$$

can be decomposed as

$$A = \sum_{0\leq p_2<+\infty,\cdots,0\leq p_{n-1}<+\infty} A\cap(U(j,j+1) = p_j,\ 2 \leq j\ n-1).$$

Let us consider the cumulative forms of the p_j's: $p_2^* = p_2$, $p_j^* = p_2 + \cdots + p_j$, $2 \leq j \leq n-1$. The event

$$A\cap(U(j,j+1) = p_j,\ 2 \leq j\ n-1)$$

exactly means that the first observation is equal to y_1, the p_2 next observations are less that y_1, the (p_2^*+2)-th observation is equal to y_2, the next p_3 observations are less that y_2, the (p_3^*+3)-th, $\cdots$, the $(p_{n-2}^*+(n-2))$ is equal to y_{n-2}, the p_{n-1} next observations are less that y_{n-1} and finally the $(p_{n-1}^*+(n-1))$-th (the n-th) is equal to y_n. So we have

$$\begin{aligned}&\mathbb{P}(A\cap(U(j,j+1) = p_j,\ 2 \leq j\ n-1))\\ &= f(y_1)F(y_1)^{p_2}f(y_2)F(y_2)^{p_3}\cdots f(y_{n-1})F(y_{n-1})^{p_{n-1}}f(y_n).\end{aligned}$$

By summing over the p_j's, we get

$$\mathbb{P}(A) = \left(\frac{f(y_1)}{1-F(f(y_1))}\frac{f(y_2)}{1-F(f(y_2))}\cdots\frac{f(y_{n-1})}{1-F(f(y_{n-1}))}\right)f(y_n).$$

This puts en end to the proof of Point (1) of the theorem. $\square$

Proof of Point (2). Let $(n_1, \cdots, n_k)$, $1 \leq k \leq n$ with $1 < n_1 < \cdots < n_k$, be a k-tuple and let

$$1 \leq x_{n_1} < x_{n_2} < \cdots < x_{n_{k-1}} < x_{n_k})$$

such that $(x_{n_1}, \cdots, x_{n_k}) \in \mathcal{V}_X^k$. For

$$A = (X^{(n_1)} = x_{n_1}, \cdots, X^{(n_k)} = x_{n_k})$$

The marginal distribution of $(X^{(n_1)}, \cdots, X^{(n_k)})$ is given, for n_0, $y_0 = lep(F)$, by

$$\begin{aligned}\mathbb{P}(A) &= \sum_{x_j,\ j \notin \{n_1, \cdots, n_k\}} \prod_{j=1}^{k-1} r(y_j)\ f(y_n) \\ &\times\ 1_{(y_1 < \cdots < y_{n_1 - 1} < y_{n_1}) < y_{n_1+1} < \cdots < y_{n_2 - 1} < y_{n_2}) < y_{n_2+1} < \cdots < y_{n_k - 1} < y_{n_k})} \\ &= f(y_n) \prod_{j=1}^{k-1} r(y_{n_j}) \prod_{j=1}^{k} \sum_{y_{n_{j-1}+1} < \cdots < y_{n_j - 1}} r(y_{n_{j-1}+1}) \cdots y_{n_j - 1} \\ &\times\ 1_{(y_{n_{j-1}} < y_{n_{j-1}+1} < \cdots < y_{n_j - 1} < y_{n_j})} \\ &= f(y_n) \prod_{j=1}^{k-1} r(y_{n_j}) \prod_{j=1}^{k} R(n_{j-1}, n_j, x_{n_{j-1}}, x_{n_j}).\ \square\end{aligned}$$

Proof of Point (3). Let $n \geq 1$. The distribution of $X^{(n)}$ is the n-th marginal of the joint distribution $(X^{(1)}, X^{(2)}, \cdots, X^{(n)})^T$. So for $n_0 = 0$, $y_0 = lep(F)$and for $y \in \mathcal{V}_X$, we have

$$\begin{aligned}\mathbb{P}(X^{(n)} = y) &= \sum_{y_j,\ 1 \leq j \leq n-1 \notin \{n_1, \cdots, n_k\}} \prod_{j=1}^{k-1} r(y_j)\ f(y) \\ &\times\ 1_{(y_1 < \cdots < y_{n-1} < y)} \\ &= R(0, n, y_0, y) f(y).\end{aligned}$$

(B) Application to geometric records. . Let us consider a sequence of *iid* geometric random variables $(X_j)_{1 \leq j \leq n}$, $n \geq 2$, of probability $p = 1 - q \in]0, 1[$, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let us recall that each X_j follows the probability law of the number of trials which is necessary to have one success in a Bernoulli trial. Hence the common mass discrete *pdf* f is define by $f(k) = \mathbb{P}(X_1 = k) = q^k - 1p$, $k \geq 1$. Hence for all $k \geq 1$,

$$1 - F(k) = \sum_{h > k} f(k) = p \sum_{h=k+1}^{+\infty} q^{h-1} = q^k$$

and

$$r(k) = \frac{r(k)}{1 - F(k)} = (p/q).$$

Now by applying Theorem 10 (page 2046) and Formula (DDR2) therein, we get for $1 \leq < k_1 < \cdots < k_n$

$$f_{(X^{(1)},\cdots,X^{(n)})}(k_1, \cdots, k_n) = (p/q)^{n-1} q^{k_m - 1} p = (p/q)^n q^{k_n}$$

and we have for

$$f_{(X^{(1)},\cdots,X^{(n)})}(k_1, \cdots, k_n) = (p/q) q^{k_n} 1_{(1 \leq < k_1 < \cdots < k_n)}. \tag{27}$$

We also see that $(X^{(1)}, \cdots, X^{(n)})$ are the arrival times of a Bernoulli stochastic process of intensity λ and hence inter-arrival (for $X^{(0)} = 0$)

$$X^{(j)} - X^{(j-1)}, \ j \in \{1, \cdots, n\},$$

are independent and identically distributed following p-geometric laws.

4. Records in ordered spaces

Let us suppose that we have an ordered probability space $(E, \mathcal{B}, \leq)$. The order relation is denoted by $\mathbb{R}$ or by $(\leq)$ and by $x \neg\mathcal{R}\ y$, we mean that x and y are not comparable. As well, by *ordered probability space*, we mean that the σ-algebra $\mathcal{B}$ is compatible with the partial order relation $(\leq)$ in the following sense : the subsets of $(E,\ \mathcal{B})$ or $(E^k, \mathcal{B}^{\otimes}k,\ k \geq$, are measurable :

$$
\begin{aligned}
& x \in \mathcal{B},\ x \in E \\
&]\leftarrow,\ x] = \{y \in E,\ y \leq x\} \in \mathcal{B},\ x \in E \\
&]x,\ \rightarrow] = \{y \in E,\ y \leq x\} \in \mathcal{B},\ x \in E \\
& N_x = \{y \in E,\ y\ x\} \in \mathcal{B},\ x \in E \\
& N = \{(x,y) \in E^2,\ x \neg\mathcal{R}\ y\} \in \mathcal{B}^{\otimes 2} \\
& etc.
\end{aligned}
$$

For example, this holds with $E = \mathbb{R}^d$, $d \geq 1$, endowed with partial order :

$$\left(\mathbb{R}^k \ni (x_1, \cdots, x_d) \leq (y_1, \cdots, y_d) \in \mathbb{R}^d\right) \Leftrightarrow \left(\forall i \in \{1, \cdots, d\},\ x_i \leq y_i\right).$$

As previously, we work a sequence of random variables $(Z_n)_{n\geq 1}$ defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in E.

4.1. *Totally ordered spaces*

The general characterizations of the probability laws for the records values and the record times in Propositions 1, 2 and 6 (in pages 2024, 2024 and 2029 respectuvely) remain valid.

4.2. *Partially ordered spaces*

The definition of records times and records values does not change. But the events based on them will not as simple as *less* or `greater`, since the non comparability will count.

New proofs will not be done again. The results are adapted following the following principles. In a totally ordered space, on $(U(n) = k \textit{ and } U(n+1) = k + \ell)$, $\ell \geq 1$, we have

$$\forall h \in [k+1, k+\ell-1],\ Z_h \leq Z_k \textit{ and } Z_{k+\ell} > Z_k. \tag{28}$$

But for a partial order, (28) becomes

$$\forall h \in [k+1, k+\ell-1], \ \Big((Z_h \leq Z_k) \cup (Z_h \ \neg\mathcal{R} \ Z_k)\Big) \ and \ Z_{k+\ell} > Z_k, \tag{29}$$

where the intersection over $h \in [k+1, k+\ell-1]$ is the empty set for $\ell = 1$. And we denote

$$C_{k,\ell} = \bigcap_{j=k+1}^{k+\ell} \Big((Z_h \leq Z_k) \bigcup (Z_h \ \neg\mathcal{R} \ Z_k)\Big) \bigcap (Z_{k+\ell} > Z_k). \tag{30}$$

In that regard, Proposition 1 becomes:

Proposition 11. *The sequence $(U(n))_{n\geq 1}$ of strong record times is a Markovian chain with non-homogeneous transition probabilities*

$$p_{t,n}(k,j) = \begin{cases} \mathbb{P}\Big((Z_j > X_k) \bigcap \bigcap_{k+1 \leq h \leq j-1} ((Z_h \leq Z_k) \bigcup (Z_h \ \neg\mathcal{R} \ X_k))\Big) & \text{if} \quad j > k, \\ 0 & \text{otherwise.} \end{cases}$$

The proposition 2 remains valid as

Proposition 12. *The sequence $(Z^{(n)})_{n\geq 1}$ of strong records is a Markovian chain with transition probability*

$$p_{r,n}(x, A) = \sum_{k=n}^{+\infty} \mathbb{P}\left((Z_{min(j>k, \ Z_j > x_n)} \in A) / (Z^{(n)} = x) \right).$$

where x is a real number and A a Borel set of $\mathbb{R}$.

The only concerns is on the computations

$$\mathbb{P}\left((Z_{min(j>k, \ Z_j > x_n)} \in A) / (Z^{(n)} = x) \right) \tag{31}$$

$$= \sum_{\ell=1}^{\infty} \mathbb{P}\Big((Z_{k+\ell} \in A) \bigcap \bigcap_{1 \leq < \ell-1} \Big((Z_{k+h} \leq Z_k) \bigcup (Z_{k+h} \ \neg\mathcal{R} \ Z_k) \Big) / (Z^{(n)} = x) \Big), \tag{32}$$

with the convention that

$$\bigcap_{1 \leq < \ell-1} C_\ell = \emptyset,$$

for $\ell = 1$ or $\ell = 2$, whatever be the sets C_ℓ.

Remark. Following Goldie and Resnick (1989), the notation

$$(Z_h > Z_k)^c = \left((Z_h \leq Z_k) \bigcup (Z_h \neg\mathcal{R}\ X_k)\right),\ h \geq 1$$

can be used.

Proposition 6 is now

Proposition 13. *If $U(n)$ is finite, then the joint probability law of*

$$(\Delta_1, \cdots, \Delta_n)$$

is given by

$$\mathbb{P}(\Delta_2 = k_2, ..., \Delta_n = k_n)$$
$$= \int_{(x_1 < x_2 < \cdots < x_n)} \mathbb{P}\left(\bigcap_{1 \leq j \leq n-1} \bigcap_{1+\overline{k}_j \leq h \leq \overline{k}_{j-1}-1} \left((Z_h \leq Z_k) \bigcup (Z_h \neg\mathcal{R}\ Z_k)\right)\right)$$
$$d\mathbb{P}_{(X_{\overline{k}_1}, \cdots, X_{\overline{k}_n})}(x_1, \cdots, x_n),$$

with $k_1 = 1$, $k_j \geq 1$ for $j \in \{2, \cdots, n\}$ and $\overline{k}_j = k_1 + \cdots + k_j$ for $1 \leq j \leq n$.

In a such general case, we cannot go further without knowing the topology of E. But the three results will lead to more precise characterizations and more fine description once that topology holds. The first step to take will concern $E = \mathbb{R}^d$, $d > 1$.

5. Conclusion

This presentation offers a full context of general characterization of the main questions about the probabilistic study of the records and their occurrence times of the sequence of random variables with values in an ordered and measurable space E. The literature has a great deal of fine tune results on records on $\mathbb{R}$. We have checked that our record value and record times characterizations remain valid for their counter-parts on $\mathbb{R}$, mostly for independent random variables and *iid* ones. The basis of further and more general results is set. The step to take should concern $E = \mathbb{R}^d$, $d > 1$, on the steps of the pioneering works of Goldie and Resnick (1989).

Annexe.

1. Direct proof of Formula 17.

$$I_n = \int_{\mathbb{R}^n} \prod_{i=1}^{n} r(x_i) 1_{(x_1<\ldots<x_n<y)} dx_1 \ldots dx_n = \frac{(-\log(1-F(y))^n}{\Gamma(n+1)},$$

where $R(x) = -\log(1-F(x)),\ x \in \mathbb{R}$, and $\Gamma(n) = (n-1)!$, $n \geq 1$.

Let us show it for $n = 1, 2, 3$. The case $n = 1$ is immediate since

$$I_1 = \int_{\mathbb{R}} r(x) 1_{(x<y)} dx = \int_{-\infty}^{y} dR(x) = [[R(x)]_{-\infty}^{y} = R(y). \tag{33}$$

For $n = 2$, we have

$$\begin{aligned}
I_2 &= \int_{-\infty<x_1<x_2<y} r(x_1) \left(\int_{x_1}^{y} r(x_2) dx_2 \right) dx_1 \\
&= \int_{-\infty<x_1<y} r(x_1) \left(\int_{x_1}^{y} dR(x_2) \right) dx_1 \\
&= \int_{-\infty<x_1<y} r(x_1)(R(y) - R(x_1) dx_1 \quad (L2) \\
&= R(y) \int_{-\infty<x_1<y} r(x_1) dx_1 - \int_{-\infty<x_1<y} r(x_1) R(x_1) dx_1 \\
&= R(y) \int_{-\infty<x_1<y} dR(x_1) - \int_{-\infty<x_1<y} d(R(x_1)^2) \\
&= R(y)^2 - R(y)^2/2 = (1/2) R(y)^2.
\end{aligned}$$

For $n = 3$, we use the results in Formula (33) and line $(L2)$ of the last blocs of formulas above to get

$$\begin{aligned}
I_2 &= \int_{-\infty<x_1<x_2<y} r(x_1) r(x_2) \left(R(y) - R(x_2) \right) dx_1 dx_2 \\
&= \int_{-\infty<x_1<y} r(x_1) \left(\int_{x_1}^{y} R(y) r(x_2) - r(x_2) R(x_2) \right) dx_1 \\
&= \int_{-\infty<x_1<y} r(x_1) \int_{x_1}^{y} \left(R(y)^2 - R(y) R(x_1) - (R(y)^2 - R(x_1)^2) \right) dx_1
\end{aligned}$$

$$= \int_{-\infty<x_1<y} \int_{x_1}^{y} \left(R(y)^2 r(x_1) - R(y)r(x_1)R(x_1) - (R(y)^2 r(x_1) - r(x_1)R(x_1)^2)\right) dx_1$$
$$= \int_{-\infty<x_1<y} \int_{x_1}^{y} \left(R(y)^2 dR(x_1) - R(y)d(R(x_1)^2))/2 - (R(y)^2 dR(x_1) - dR(x_1)^3)/3\right)$$
$$= R(y)^3 - (R(y)^3))/2 - R(y)^3 + R(y)^3)/3 = (1/6)R(y)^3.$$

From there, the induction is clear.

2. Direct proof of Formula (18) [see page 2042].

We have to prove that

$$\int_{z<x_1<\ldots<x_{n-1}<y} \left(\prod_{j=1}^{n-1} r(x_i)\right) dx_1 \ldots dx_{n-1} = \frac{(R(y)-R(z))^{n-1}}{\Gamma(n)}.$$

Let us begin by $n=2$, which corresponds to the formula

$$\int_{z<x<y} r(x)dx = (R(y)-R(z)),$$

which is obvious since $r(x) = dR(x)/dx$. Suppose it is true for $n \geq 2$. Let us prove it for $n+1$. We have

$$\int_{z<x_1<\ldots<x_{n-1}<x_n<y} \left(\prod_{j=1}^{n} r(x_i)\right) dx_1 \ldots dx_{n-1}dx_n$$
$$= \int_{z<x_1<y} r(x_1)dx_1 \int_{x_1<x_2<\ldots<x_n<y} \left(\prod_{j=2} nr(x_i)\right) dx_2 \ldots dx_n.$$

The induction hypothesis gives

$$\int_{x_1<x_2<\ldots<x_n<y} \left(\prod_{j=2}^{n} r(x_i)\right) dx_2 \ldots dx_n = \frac{(R(y)-R(x_1))^{n-1}}{\Gamma(n)},$$

and we get

$$\int_{z<x_1<\ldots<x_{n-1}<x_n<y} \left(\prod_{j=1}^{n} r(x_i)\right) dx_1 \ldots dx_{n-1} dx_n$$
$$= \int_{z<x_1<y} r(x_1) \frac{r(x_1)(R(y)-R(x_1))^{n-1}}{\Gamma(n)} dx_1.$$
$$\int_{z<x_1<y} r(x_1) \frac{-d(R(y)-R(x_1))^{n}}{n\Gamma(n)} = \frac{-d(R(y)-R(z))^{n}}{n\Gamma(n+1)}.$$

The proof of Formula (18) is complete.■

Direct proof of Formula (ADR1) in Theorem 5, page 2037.

We define $U(j,j+1)$, for $2 \leq j \leq n-1$, as the number of observations strictly between the j-th record $X_{U(j)}$ and the $(j+1)$-th record $X_{U(j+1)}$. The event

$$A = (Y^{(1)} \in y_1 \pm dy_1/2, \cdots, Y^{(n)} \in y_n \pm dy_n/2)$$

can be decomposed as

$$A = \sum_{0 \leq p_2 < +\infty, \cdots, 0 \leq p_{n-1} < +\infty} A \cap (U(j,j+1) = p_j,\ 2 \leq j\ n-1).$$

Let us accumulate the p_j's as $p_2^* = p_2$, $p_j^* = p_2 + \cdots + p_j$, $2 \leq j \leq n-1$. Since the data are continuous, implying that the equality of observations has probability zero, we may and do suppose that each interval $y_j \pm dy_j/$ contains one observation. So the event

$$A \cap (U(j,j+1) = p_j,\ 2 \leq j\ n-1)$$

exactly means that the first observation is equal to y_1, the p_2 next observation are less that y_1, the $(p_2^* + 2)$-th observation is in $y_1 \pm dy_1/$, the next p_3 are less that y_2, the $(p_3^* + 3)$-th, $\cdots$, the $(p_{n-2}^* + (n-2))$ is in $y_2 \pm dy_2/$, the p_{n-1} next observation are less that y_{n-1} and finally the $(p_{n-1}^* + (n-1))$-th (the n-th) is in $y_n \pm dy_n/$. So we have

$$\mathbb{P}(A \cap (U(j,j+1) = p_j,\ 2 \leq j\ n-1)) = \mathbb{P}(Y^{(1)} \in y_1 \pm dy_1) F(y_1)^{p_2} \mathbb{P}(Y^{(2)} \in y_2 \pm dy_2) F(y_2)^{p_3}$$
$$\cdots\ \mathbb{P}(Y^{(n-1)} \in y_{n-1} \pm dy_{n-1}) F(y_{n-1})^{p_{n-1}} f(y_n) \mathbb{P}(Y^{(n)} \in y_1 \pm dy_n.$$

By summing over the p_j's first, next by dividing by $(dy_1 \cdots dy_n)$ and finally by letting each $dy_J \to 0$, we get

$$\mathbb{P}(A) = \left(\frac{f(y_1)}{1-F(f(y_1))} \frac{f(y_2)}{1-F(f(y_2))} \cdots \frac{f(y_{n-1})}{1-F(f(y_{n-1}))}\right) f(y_n).\ ■$$

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