Afrika Statistika

Afrika Statistika
Vol. V (14), 2019, pages 1903-1915.
DOI: http://dx.doi.org/10.16929/as/2019.1903.141

ISSN 2316-090X

On two parametric probability distributions on crisp complete pre-orders

Landry Ngibasona[1], **Bertrand Mbama Engoulou**[1], **Siméon Fotso**[2] **and Louis Aimé Fono**[1,*]

[1]Laboratoire de Mathématiques et Faculté des Sciences - Université de Douala - Cameroun, B.P. 24157 Douala
[2]Département de Mathématiques - Ecole Normale Supérieure de Yaoundé - Université de Yaoundé I-Cameroun, B.P. 47 Yaoundé

Received : February 27, 2018. Accepted : February 28, 2019

Abstract. Mallows and Plackett-Luce parametric probability distributions are the two most used and well-known in the set of complete linear orders of a finite universe. In this paper, we extend those two distributions on the set of complete pre-orders of the universe. For that purpose, by considering a parametric family of metrics on the set of complete pre-orders generalizing Kemeny Distance on pre-orders and Kendall metric on orders, we determine a parametric probability distribution on pre-orders generalizing Mallows Distribution. By considering pre-orders as orders on blocks of equivalent elements, we generalize the Plackett-Luce distribution on complete pre-orders.

Key words: Complete pre-orders; Generalized Mallows distribution; Generalized Plackett-Luce distribution.
AMS 2010 Mathematics Subject Classification : 06A06, 97K50, 91B14.

*Corresponding author: Louis Aimé Fono: lfono2000@yahoo.fr; Tel. +(237) 696-02-29-15
Landry Ngibasona: landryngibasona@yahoo.fr
Siméon Fotso: simeonfotso@yahoo.fr
Bertrand Mbama Engoulou: mbama0479@yahoo.fr

Résumé: Les distributions paramétriques de Mallows et Plackett-Luce sont les distributions de probabilité bien connues et les plus utilisées dans la littérature sur les ordres totaux d'un univers fini. Dans cet article, nous généralisons ces distributions aux pré-ordres totaux de l'univers fini. L'extension de la distribution de Mallows sur les pré-ordres totaux est basée sur une famille de distances paramétriques sur les pré-ordres totaux généralisant la distance de Kemeny sur les pré-ordres et la distance de Kendall sur les ordres. La généralisation de la distribution de Plackett-Luce s'obtient à partir de la ré-écriture d'un pré-ordre total sous forme d'un ordre total de blocs d'éléments équivalents (ex-aequo).

1. Introduction

Given a finite universe $X = \{x_1; \ldots; x_n\}$ with $n = |X| \geq 3$, a complete linear order (total order or permutation) π on X can be considered as an entire ranking (without ties) of elements of X. For $i \in \{1; ...; n\}$ and a permutation π, $\pi(i)$ denotes the element of X which occupies position i in π. For example, if $X = \{a; b; c\}$, the ranking $\pi = bca$ means $\pi(1) = b, \pi(2) = c$ and $\pi(3) = a$ which corresponds to the preferential order $b \succ c \succ a$. The set of complete linear orders of X is S_n with $n!$ elements. One topic of interest is to define probability distributions on S_n. Since a probability distribution P on S_n has $n! - 1$ degrees of freedom, specifying such a distribution in a non-parametric way is not practical. Therefore, parametric distributions have been proposed in the literature, the most important ones being the Mallows and the Plackett-Luce distributions (Mallows (1957); Luce (1959); Plackett (1975); Kamdem *et al.* (2019)). These probability distributions have been used for many applications in Social Choice, Preference Modeling, Machine Learning... etc (Daniel *et al.* (1950); Busa-Fekete *et al.* (2013a,b); Marden (1995)).

A complete pre-order on X is a generalization of a linear order since it is a ranking with possible ties of entire elements of X. In Social Science, it is less demanding and more reasonable to require to individuals or a group to express their preferences on X in term of pre-orders on X than linear orders. In preference modeling, a pre-order with k positions $(k \in \{1; ...; n\})$ on n elements can be seen as an order with k positions: Thus, for $k = 1$, all elements are equivalent, for $k = 2$ there are two ranks (i elements occupy one of the two ranks and the $n - i$ others elements occupy the other rank for $i = 1, ..., n - 1$), for $k = n$, the pre-order becomes a linear order (each element occupies one position and in the later case, we have a linear order, that is, a specific pre-order without ties). In the same vein, Andjiga *et al.* (2014) recently considered a pre-order as an k-blocks orders where elements of a block are indifferent. One open question is to define probability distributions on the set of pre-orders of the universe. These generalizations will be a stepping stone for many applications, even by studying with pre-orders, some questions already solved with those distributions on orders. The contribution of this paper is to extend Mallows and Plackett-Luce distribution from orders to pre-orders.

Let us present the two classical and well- known probability distributions on linear orders before we display our road map. The Mallows distribution belongs to the exponential family and it is specified by the Kendall metric, two parameters, a reference ranking $\pi_0 \in S_n$ and a spread parameter $m \in \mathbb{R}_+$ (Mallows (1957); Kamdem *et al.* (2019)). Under this model, the probability of $\pi \in S_n$ is given by

$$p(\pi) = P_{\pi_0,m}(\pi) = \frac{1}{\phi(m)} \exp\big(-m \cdot d_K(\pi, \pi_0)\big), \tag{1}$$

where $\phi(m)$ is a normalization factor (that depends on m but not on π_0) and d_K denotes the *Kendall distance* on S_n given by (Kendall (1938); Kamdem *et al.* (2019)):

$$\forall \pi, \tau \in S_n, d_K(\pi, \tau) = |\Big\{(i,j)\,|\,1 \le i < j \le n, and\, \big(\pi^{-1}(i) - \tau^{-1}(i)\big)\big(\pi^{-1}(j) - \tau^{-1}(j)\big) < 0\Big\}|,$$

i.e., the number of pairs of alternatives that are ranked differently in π and τ.
The Plackett-Luce (PL) model is specified by a parameter vector $\boldsymbol{v} = (v_1, v_2, \ldots v_n) \in \mathbb{R}^n_+$, in which v_i accounts for the weight or "strength" of the option $x_i \in X$ (Luce (1959); Plackett (1975); Kamdem *et al.* (2019)). The probability assigned by the PL model to a ranking represented by a permutation $\pi \in S_n$ is given by

$$P_{\boldsymbol{v}}(\pi) = \prod_{i=1}^{n} \frac{v_{\pi(i)}}{v_{\pi(i)} + v_{\pi(i+1)} + \ldots + v_{\pi(n)}}. \tag{2}$$

Likewise, the larger the parameter v_i in (2) in comparison to the parameters v_j, $j \neq i$, the higher probability that x_i appears on a top rank. Below, we have Example on a universe with three elements.

Example 1.
1) For $X = \{a; b; c\}$, $\pi_0 = abc$ and $m \geq 0$, we have $\phi(m) = 1 + 2e^{-m} + 2e^{-2m} + e^{-3m}$. The Kendall-distances of the rankings and their probabilities under the Mallows model are as follows:

$\pi \in S_3$	abc	acb	bac	bca	cab	cba
$d_K(\pi_0, \pi)$	0	1	1	2	2	3
$p(\pi)$	$\frac{1}{\phi(m)}$	$\frac{e^{-m}}{\phi(m)}$	$\frac{e^{-m}}{\phi(m)}$	$\frac{e^{-2m}}{\phi(m)}$	$\frac{e^{-2m}}{\phi(m)}$	$\frac{e^{-3m}}{\phi(m)}$

2) For $X = \{a; b; c\}$, $v_a, v_b, v_c \in \mathbb{R}_+$, the PL-distribution is given by the following probabilities:

$$p(abc) = \frac{v_a v_b}{(v_a + v_b + v_c)(v_b + v_c)}, \; p(acb) = \frac{v_a v_c}{(v_a + v_b + v_c)(v_c + v_b)},$$
$$p(bac) = \frac{v_b v_a}{(v_a + v_b + v_c)(v_a + v_c)}, \; p(bca) = \frac{v_b v_c}{(v_a + v_b + v_c)(v_c + v_a)},$$
$$p(cab) = \frac{v_c v_a}{(v_a + v_b + v_c)(v_a + v_b)} \text{ and } p(cba) = \frac{v_c v_b}{(v_a + v_b + v_c)(v_b + v_a)}.$$

The paper is organized as follows. In Section 2, based on a family of parametric metrics on the set of crisp binary relations introduced by Diffo *et al.* (2012), we deduce a sub-family of parameterized metrics on pre-orders generalizing Kendall metric. We display that the usual and well-known Kemeny metric on pre-orders is equivalent to a particular metric of that family. We then use metrics of that family to propose an extension of the Mallow distribution on pre-orders. In Section 3, we first recall in a formal way the formulation of a pre-order in term of an k-order with k the number of blocks of indifferent elements. We then introduce an equivalence relation between pre-orders stipulating equivalence between two pre-orders if they have the same blocks. We first assign a probability distribution on the set of equivalence classes of that relation. By using conditional probability, we determine probability distribution on pre-orders generalizing Plackett-Luce distribution. Section 4 gives some concluding remarks.

2. Generalization of Mallows distribution on complete pre-orders

2.1. Metric on pre-orders

Let P_n the set of complete pre-orders of X (ranking of all elements of X with possible ties) and $P_2(X) = \{\{x;y\},(x,y) \in X^2\}$ the set of subsets of two elements of X. For all $\pi \in P_n$, for all $(x,y) \in X^2$, if π strictly prefer x to y, we denote $x \succ_\pi y$ and if π is indifferent between x and y, we denote $x \sim_\pi y$.

Diffo *et al.* (2012) introduced a family of parametric distances on the set of crisp binary relations of X. Based on these distances, we introduce a sub-family of parametric distances on P_n as an application $d_\alpha : P_n \times P_n \longrightarrow \mathbb{R}_+$ ($\alpha \in [0,1]$) entirely determined by its restrictions $d_\alpha^{\{x;y\}}$ on each element $\{x;y\}$ of $P_2(X)$ as follows: Let $\alpha \in [0,1]$, $d_\alpha^{\{x;y\}}$ is defined on $P_n \times P_n$, it satisfies $d_\alpha^{\{x;y\}}(\pi_1,\pi_2) = d_\alpha^{\{x;y\}}(\pi_2,\pi_1)$ and it measures the distance between π_1 and π_2 on $\{x;y\}$ in the following matrix:

	$x \succ_{\pi_1} y$	$x \sim_{\pi_1} y$	$y \succ_{\pi_1} x$
$x \succ_{\pi_2} y$	0	α	1
$x \sim_{\pi_2} y$	α	0	α
$y \succ_{\pi_2} x$	1	α	0

Let us explain the underling idea of the previous matrix: If we compare two elements x and y by each of the two pre-orders π_1 and π_2, we have three situations: (i) π_1 and π_2 rank x and y in the same way, (ii) one of two pre-orders ranks x indifferent to y and the other pre-order ranks x strictly preferred to y or y strictly preferred to x, and (iii) one of the two pre-orders ranks x strictly preferred to y and the second pre-order ranks y strictly preferred to x. The previous symmetric matrix indicates that the restriction $d_\alpha^{\{x;y\}}$ assigns 0, α and 1 to the first, second and third situations.

Aggregating such local metrics, we consider the following mapping $d_\alpha : P_n^2 \to \mathbb{R}_+$ associating a positive real number to a couple of a complete pre-orders as follows:

$$\forall(\pi_1, \pi_2) \in P_n^2, d_\alpha(\pi_1, \pi_2) = \sum_{\{x;y\} \in P_2(X)} d_\alpha^{\{x;y\}}(\pi_1, \pi_2). \tag{3}$$

The following result establishes a necessary and sufficient condition on the parameter α under which d_α is a distance on P_n.

Proposition 1. *Let $\alpha \in]0; 1]$ and $n \in \mathbb{N} \setminus \{0; 1; 2\}$.*
The application d_α is a distance or metric on P_n if and only if $\alpha \geq \frac{1}{2}$.

Proof of Proposition Let us check the three properties of a metric.

1) Since $\forall \{x; y\} \in P_2(X), d_\alpha^{\{x;y\}}$ is symmetric, thus d_α is symetric.

2) Let us show that $d_\alpha(\pi_1, \pi_2) = 0$ iff $\pi_1 = \pi_2$.
Since $\forall \{x; y\} \in P_2(X), d_\alpha^{\{x;y\}} \geq 0$ (due to the fact that values of the matrix are positive), we have $d_\alpha^{\{x;y\}}(\pi_1, \pi_1) = 0$ iff $\forall \{x; y\} \in P_2(X), d_\alpha^{\{x;y\}} = 0$. With the values of the matrix, that means, $\forall \{x; y\} \in P_2(X), d_\alpha^{\{x;y\}} = 0$ iff ($\forall \{x; y\} \in P_2(X)$, π_1 and π_2 provide the same ranking to x and y, we have $d_\alpha^{\{x;y\}} = 0$ iff $\pi_1 = \pi_2$. Hence the result.

3) Let $\pi_1, \pi_2, \pi_3 \in P_n$. Let us show that $d_\alpha(\pi_1, \pi_3) \leq d_\alpha(\pi_1, \pi_2) + d_\alpha(\pi_2, \pi_3)$ if and only if $\alpha \geq \frac{1}{2}$. Let us consider two non-empty subsets of $P_2(X) : A_\alpha(\pi_1, \pi_2, \pi_3) = \{\{x; y\} \in P_2(X), d_\alpha^{\{x;y\}}(\pi_1, \pi_2) = d_\alpha^{\{x;y\}}(\pi_2, \pi_3) = \alpha$ and $d_\alpha^{\{x;y\}}(\pi_1, \pi_3) = 1\}$ and its complementary $B_\alpha(\pi_1, \pi_2, \pi_3)$. Let us set $a_\alpha(\pi_1, \pi_2, \pi_3)$ the cardinality of $A_\alpha(\pi_1, \pi_2, \pi_3)$.

It is easy to show that

$$\forall \{x; y\} \in B_\alpha(\pi_1, \pi_2, \pi_3), d_\alpha^{\{x;y\}}(\pi_1, \pi_3) \leq d_\alpha^{\{x;y\}}(\pi_1, \pi_2) + d_\alpha^{\{x;y\}}(\pi_2, \pi_3).$$

Thus, the inequality

$$\sum_{\{x;y\} \in B_\alpha(\pi_1,\pi_2,\pi_3)} d_\alpha^{\{x;y\}}(\pi_1, \pi_3) \leq \sum_{\{a;b\} \in B_\alpha(\pi_1,\pi_2,\pi_3)} (d_\alpha^{\{x;y\}}(\pi_1, \pi_2) + d_\alpha^{\{x;y\}}(\pi_2, \pi_3))$$

is true. It remains to show that

$$\sum_{\{x;y\} \in A_\alpha(\pi_1,\pi_2,\pi_3)} d_\alpha^{\{x;y\}}(\pi_1, \pi_3) \leq \sum_{\{x;y\} \in A_\alpha(\pi_1,\pi_2,\pi_3)} (d_\alpha^{\{x;y\}}(\pi_1, \pi_2) + d_\alpha^{\{x;y\}}(\pi_2, \pi_3))$$

iff $\alpha \in [\frac{1}{2}; 1]$

Notice that

$$\forall \alpha \in]0;1], \forall \{x;y\} \in A_\alpha(\pi_1,\pi_2,\pi_3), d_\alpha^{\{x;y\}}(\pi_1,\pi_2) + d_\alpha^{\{x;y\}}(\pi_2,\pi_3) = 2\alpha$$

and

$$d_\alpha^{\{x;y\}}(\pi_1,\pi_3) = 1.$$

Therefore, we have

$$\sum_{\{a;b\} \in A_\alpha(\pi_1,\pi_2,\pi_3)} d_\alpha^{\{x;y\}}(\pi_1,\pi_3) = a_\alpha(\pi_1,\pi_2,\pi_3)$$

and

$$\sum_{\{x;y\} \in A_\alpha(\pi_1,\pi_2,\pi_3)} (d_\alpha^{\{x;y\}}(\pi_1,\pi_2) + d_\alpha^{\{x;y\}}(\pi_2,\pi_3)) = 2\alpha a_\alpha(\pi_1,\pi_2,\pi_3).$$

Thus,

$$\sum_{\{x;y\} \in A_\alpha(\pi_1,\pi_2,\pi_3)} d_\alpha^{\{x;y\}}(\pi_1,\pi_3) \leq \sum_{\{x;y\} \in A_\alpha(\pi_1,\pi_2,\pi_3)} (d_\alpha^{\{x;y\}}(\pi_1,\pi_2) + d_\alpha^{\{x;y\}}(\pi_2,\pi_3))$$

if and only if

$$2\alpha a_\alpha(\pi_1,\pi_2,\pi_3) \leq a_\alpha(\pi_1,\pi_2,\pi_3).$$

Hence the result. □

Throughout the paper, we assume that $\alpha \geq \frac{1}{2}$.

In the following, we recall the most usual distance on pre-orders, namely the Kemeny distance d_{Kem}, and we display a link with our family of parametric metrics d_α.

Let us recall the Kemeny distance d_{Kem}. Let π be a pre-order with the associated score matrix $(a_{xy})_{x,y \in X}$ defined by:

$$a_{xy} = \begin{cases} 1 \text{ if } \pi \text{ ranks the object x ahead or tied with object y} \\ -1 \text{ if } \pi \text{ ranks the object x behind the object y} \\ 0 \text{ if the objects x and y are equal} \end{cases} . \quad (4)$$

The Kemeny distance d_{Kem} is defined by: for two pre-orders π_1 and π_2 with score matrix (a_{xy}) and (b_{xy}) defined in (4),

$$d_{Kem}(\pi_1, \pi_2) = \frac{1}{2} \sum_{x \in X} \sum_{y \in X} |a_{xy} - b_{xy}|. \tag{5}$$

The following result shows that the Kemeny metric is equivalent to the particular case $d_{\frac{1}{2}}$ of d_α.

Proposition 2. *Let π_1 and π_2 be two pre-orders of P_n. Then $d_{Kem}(\pi_1, \pi_2) = 2d_{\frac{1}{2}}(\pi_1, \pi_2)$.*

Proof of . Let π_1 and π_2 be two pre-orders and (a_{xy}) and (b_{xy}) their score matrix. Let us show that $d_{Kem}(\pi_1, \pi_2) = 2d_{\frac{1}{2}}(\pi_1, \pi_2)$. As for d_α and by using the definitions of the score matrix, we can entirely determined d_{Kem} by its restrictions $d_{Kem}^{\{x;y\}}$ on elements $\{x; y\}$ of $P_2(X)$ as follows: $d_{Kem}(\pi_1, \pi_2) = \sum_{\{x;y\} \in P_2(X)} d_{Kem}^{\{x;y\}}(\pi_1, \pi_2)$ where $d_{Kem}^{\{x;y\}}(\pi_1, \pi_2) = \frac{1}{2}(|a_{xy} - b_{xy}| + |a_{yx} - b_{yx}|)$. It is sufficient to show that

$$\forall \{x; y\} \in P_2(X), d_{Kem}^{\{x;y\}}(\pi_1, \pi_2) = 2d_{\frac{1}{2}}^{\{x;y\}}(\pi_1, \pi_2). \tag{6}$$

We will distinguish three cases:

(i) In the first case of similarity between x and y in the two pre-orders: we have $|a_{xy} - b_{xy}| = 0 = |a_{yx} - b_{yx}|$ and $d_{\frac{1}{2}}^{\{x;y\}}(\pi_1, \pi_2) = 0$. Hence the result since $d_{Kem}^{\{x;y\}}(\pi_1, \pi_2) = 0 = 2d_{\frac{1}{2}}^{\{x;y\}}(\pi_1, \pi_2)$.

(ii) In the second situation where there pairwise indifference between x and y in one pre-order and pairwise strict preference in the second, we have on the one hand $|a_{xy} - b_{xy}| = 0$ and $|a_{yx} - b_{yx}| = 2$ or $|a_{xy} - b_{xy}| = 2$ and $|a_{yx} - b_{yx}| = 0$. Thus, $d_{Kem}^{\{x;y\}}(\pi_1, \pi_2) = 1$ and on the other hand $2d_{\frac{1}{2}}^{\{x;y\}}(\pi_1, \pi_2) = 1$. Hence the result.

(iii) In the third situation where the two pre-orders have dissimilarity in comparison of x and y : we have $|a_{xy} - b_{xy}| = 2 = |a_{yx} - b_{yx}|$ and $d_{\frac{1}{2}}(\pi_1, \pi_2) = 1$. Hence the result since $d_{Kem}^{\{x;y\}}(\pi_1, \pi_2) = 2 = 2d_{\frac{1}{2}}^{\{x;y\}}(\pi_1, \pi_2)$. □

2.2. Mallows distribution version for complete pre-orders

Since Mallows probabilistic distribution on complete orders is based on Kendall metric, the following result uses the above metric d_α to generalize the Mallows distribution on complete pre-orders.

Proposition 3. *Let* $\alpha \in [\frac{1}{2}, 1], \pi_0 \in P_n$*, for all* $\theta > 0$*, the family* $(P_{\pi_0, d_\alpha, \theta})$ *defined by:*

$$\begin{cases} \forall \pi \in P_n, P_{\pi_0, d_\alpha, \theta}(\pi) = \frac{e^{-d_\alpha(\pi_0, \pi)\theta}}{\Phi(\theta)} \\ \Phi(\theta) = \sum_{\pi \in P_n} e^{-d_\alpha(\pi_0, \pi)\theta} \end{cases} \tag{7}$$

is the parametric probability distribution on P_n*.*

Let us state an example for $X = \{a; b; c\}$.

Example 2. Let $X = \{a; b; c\}$, $\pi_0 = abc$ (to simplify $\pi = abc$ means $a \succ_\pi b \succ_\pi c$, $\pi = a(bc)$ means $a \succ_\pi b \sim_\pi c$, and $\pi = (abc)$ means $a \sim_\pi b \sim_\pi c$) and $p(\pi) = P_{\pi_0, d_\alpha, \theta}(\pi)$ for all $\pi \in P_3$. Then for all $\alpha \in [\frac{1}{2}, 1]$,

$$\begin{cases} p(cab) = p(bca) = \frac{e^{-2\theta}}{\Phi(\theta)}, p(b(ca)) = p((ca)b) = \frac{e^{-(1+\alpha)\theta}}{\Phi(\theta)}, \\ p(acb) = p(bac) = \frac{e^{-\theta}}{\Phi(\theta)}, p(a(cb)) = p((ab)c) = \frac{e^{-\alpha\theta}}{\Phi(\theta)}, \\ p(abc) = \frac{1}{\Phi(\theta)}, p(c(ba)) = p((bc)a) = \frac{e^{-(2+\alpha)\theta}}{\Phi(\theta)}, \\ p((abc)) = \frac{e^{-3\alpha\theta}}{\Phi(\theta)} \; et \; p(cba) = \frac{e^{-3\theta}}{\Phi(\theta)}. \end{cases} \tag{8}$$

3. Generalization of Plackett-Luce distribution on pre-orders

3.1. Equivalence relation on pre-orders

The first step of our construction is to reconsider a pre-order $\pi \in P_n$ as a specific order of l blocks with $1 \leq l \leq n$, each block being formed of indifferent elements. Such pre-order is called l-order and is denoted as follows $\pi = \pi(l) = \pi_{[1]}\pi_{[2]}...\pi_{[l]}$ where $\pi_{[i]}$ is the i^{th}-block in the l-order π (see Andjiga *et al.* (2014)). Let us state some examples for $X = \{a; b; c; d; e; f\}$. The pre-order $\pi = \pi(4) = a(db)c(ef)$ (π has four positions: a is the first, d and b are the two second, c has the third position and, the fourth or last position is occupied by e and f) is the 4-order denoted $\pi_1 = \pi_{[1]}\pi_{[2]}\pi_{[3]}\pi_{[4]}$ where the four blocks are: $\pi_{[1]} = a, \pi_{[2]} = (db), \pi_{[3]} = c$ and $\pi_{[4]} = (ef)$. The pre-order $\pi = \pi(3) = c(dba)(ef)$ is a 3-order denoted $\pi = \pi(3) = \pi_{[1]}\pi_{[2]}\pi_{[3]}$ where the three blocks are: $\pi_{[1]} = c, \pi_{[2]} = (dba)$ and $\pi_{[3]} = (ef)$. The pre-order $\pi = \pi(1) = \pi_{[1]} = (abcdef)$ is a 1-order with $\pi_{[1]} = (abcdef)$ as the only block. If the number of blocks is the number of elements of the universe, the pre-order becomes an order (n-order is an order).

The second step of our construction is to introduce the equivalence binary relation on P_n, based on the previous formulation, denoted by $\simeq$ and defined by: for all $\pi = \pi(l) = \pi_{[1]}\pi_{[2]}...\pi_{[l]}$ and $\pi' = \pi'(k) = \pi'_{[1]}\pi'_{[2]}...\pi'_{[k]}$, $\pi \simeq \pi'$ iff π and π' have the same blocks. Consequently, $l = k$, that is, π and π' have the same number of blocks. Given $1 \leq l \leq n$ and $\pi = \pi(l) = \pi_{[1]}\pi_{[2]}...\pi_{[l]} \in P_n$, the equivalence class of $\pi = \pi(l)$ is the set $S_{\pi(l)} = \{\pi' \in P_n, \pi \simeq \pi'\}$, that is, the set of pre-orders having the same blocks with $\pi(l)$. It is also the set of permutations of the l blocks $\pi_{[1]}, \pi_{[2]}, ..., \pi_{[l]}$ of π. Then $P_n/_\simeq$ is the set of all equivalence classes of $\simeq$ on P_n. For example, if $X = \{a; b; c\}$, we have $P_3/_\simeq = \{S_{abc}; S_{(bc)a}; S_{(ab)c}; S_{(ac)b}; S_{(abc)}\}$ where the five classes are: $S_{abc} = S_3$ the set of permutations of X, $S_{(bc)a} = \{a(bc);(bc)a\}$, $S_{(ab)c} = \{c(ab);(ab)c\}$, $S_{(ac)b} = \{b(ac);(ac)b\}$ and $S_{(abc)} = \{(abc)\}$. In addition, the subset of orders (permutations) S_n of P_n is the unique equivalent class of pre-orders of n blocks.

Let us end this step by determining the number n_{eq} of equivalence classes for the relation $\simeq$ in P_n, that is, the cardinality of $P_n/_{\simeq}$. For $l \in \{1;...;n\}$, we consider the set $\bar{\pi}(l)$ made up of the pre-orders having l blocks. Notice that $\bar{\pi}(l)$ is the disjoint union of the equivalence classes $(S_{\pi(l)})_{\pi(l)\in P_n}$ and the family of sets $(\bar{\pi}(l))_{l\in\{1;...;n\}}$ is a partition of P_n. The cardinality of $\bar{\pi}(l)$ is given by:

$$| \bar{\pi}(l) |= \sum_{(n_1,n_2,...,n_l)\in A_{n,l}} \frac{n!}{n_1!n_2!...n_l!}$$

where the set $A_{n,l} = \{(n_1, n_2, ..., n_l)$ belongs to $\mathbb{N}^{*l}, n_1 + n_2 + ... + n_l = n\}$. And since $| S_{\pi(l)} |= l!$, each $\bar{\pi}(l)$ has $\frac{|\bar{\pi}(l)|}{l!}$ equivalence classes.

Thus

$$n_{eq} = \sum_{l\in\{1;...;n\}} \frac{|\bar{\pi}(l)|}{l!}. \tag{9}$$

For $X = \{a; b; c\}$, we have

$$n_{eq} = \frac{|\bar{\pi}(1)|}{1!} + \frac{|\bar{\pi}(2)|}{2!} + \frac{|\bar{\pi}(3)|}{3!} = \frac{\frac{3!}{3!}}{1!} + \frac{\frac{3!}{1!2!}}{2!} + \frac{\frac{3!}{2!1!}}{2!} + \frac{\frac{3!}{1!1!1!}}{3!} = 1 + 3 + 1 = 5.$$

3.2. Probability on equivalence classes of pre-orders and Plackett-Luce distribution for pre-orders

The first step of the construction of the probability is to define probability to obtain a pre-order π knowing that there exists $l \in \{1;...;n\}$ such that π belongs to $S_{\pi(l)}$. Due to the structure of $S_{\pi(l)}$ which is the set of permutations of the l-order $\pi(l)$, we apply the Plackett-Luce principle in $S_{\pi(l)}$ as follows: we assign a positive real number α_i^{π} to the block $\pi_{[i]}$ (it may be the cardinal of $\pi_{[i]}$), thus we have a l-uplet $\alpha_{[\pi]} = (\alpha_i^{\pi})_{1\leq i\leq l}$. Then, the probability to have π knowing that it belongs to $S_{\pi(l)}$ is defined by

$$P(\pi/S_{\pi(l)}) = \prod_{1\leq i\leq l} \frac{\alpha_i^{\pi}}{\alpha_i^{\pi} + \alpha_{i+1}^{\pi} + ... + \alpha_l^{\pi}}. \tag{10}$$

It important to notice that if π is an order, the previous probability becomes the well-known Plackett-Luce distribution.

The second step is to define the probability to obtain the set $S_{\pi(l)}$. Due to the formula (9), we assign a positive family $(\beta_{S_{\pi(l)}})_{S_{\pi(l)}\in P_n/\simeq} = (\beta_t)_{1\leq t\leq n_{eq}}$ of weights of $[0;1]$ to those sets satisfying $\sum_{S_{\pi(l)}\in P_n/\simeq} \beta_{S_{\pi(l)}} = \sum_{1\leq t\leq n_{eq}} \beta_t = 1$. Then we set

$$P(S_{\pi(l)}) = \beta_{S_{\pi(l)}}. \tag{11}$$

We now define the probability $P(\pi)$ to obtain the pre-order or the l-order $\pi = \pi(l) = \pi_{[1]}\pi_{[2]}...\pi_{[l]}$ with $l \in \{1; 2; ...; n\}$. Notice that the pre-order $\pi = \pi_{(l)}$ occurred iff the two random events "$A = S_{\pi(l)}$" and "$B = \pi = \pi_{[1]}\pi_{[2]}...\pi_{[l]}$" occurred simultaneously. Thus π is carried out conditionally with the probability $P(\pi) = P(A \cap B) = P(B/A)P(A)$. Using probabilities defined by formulas (10) and (11), we have

$$P(\pi) = \beta_{S_{\pi(l)}} \prod_{1\leq i\leq l} \frac{\alpha_i^\pi}{\alpha_i^\pi + \alpha_{i+1}^\pi + ... + \alpha_l^\pi}. \tag{12}$$

Finally, the construction described in the two previous Subsections is summarized in the following result.

Proposition 4. *The generalized Plackett-Luce (P.L.) probability distribution P on P_n is defined as follows: Let $\pi \in P_n$, l its number of blocks, $\beta_{S_{\pi(l)}} \in [0, 1]$ the probability of occurrence of its equivalence class $S_{\pi(l)}$ and $(\alpha_i^\pi)_{i\in\{1,...,l\}}$ the weights of the l blocks of π. Then*

$$P(\pi) = \beta_{S_{\pi(l)}} \prod_{1\leq i\leq l} \frac{\alpha_i^\pi}{\alpha_i^\pi + \alpha_{i+1}^\pi + ... + \alpha_l^\pi}.$$

Notice, in the particular case where

$$\beta_{S_{\pi(l)}} = \begin{cases} 0 \text{ if } l \in \{1; ...; n-1\} \\ 1 \text{ if } l = n \end{cases} \tag{13}$$

(in others words, we assign 0 to each equivalent class except S_n to which we assign 1), then the previous probability distribution becomes Plackett-Luce distribution on S_n.

Let us illustrate this probability distribution in P_3.

Example 3. Let $X = \{a; b; c\}$ and $P_3/_\simeq$ the set of equivalence classes of $\simeq$ in P_3. The extension of the P.L. probability distribution on P_3 is defined on classes as follows:

(a) In the class S_{abc}, we have $\pi_1 = \pi_1(3)$ (with $l = 3$), $\alpha_1^{\pi_1=abc} = \alpha_a, \alpha_2^{\pi_1=abc} = \alpha_b$ and $\alpha_3^{\pi_1=abc} = \alpha_c$. Thus, the probabilities of the six pre-orders of the class are defined by:

$$\begin{aligned}
P(abc) &= \beta_{S_{abc}} \frac{\alpha_a\alpha_b}{(\alpha_a+\alpha_b+\alpha_c)(\alpha_b+\alpha_c)}, & P(acb) &= \beta_{S_{abc}} \frac{\alpha_a\alpha_c}{(\alpha_a+\alpha_b+\alpha_c)(\alpha_c+\alpha_b)}, \\
P(bac) &= \beta_{S_{abc}} \frac{\alpha_b\alpha_a}{(\alpha_a+\alpha_b+\alpha_c)(\alpha_a+\alpha_c)}, & P(bca) &= \beta_{S_{abc}} \frac{\alpha_b\alpha_c}{(\alpha_a+\alpha_b+\alpha_c)(\alpha_c+\alpha_a)} \\
P(cab) &= \beta_{S_{abc}} \frac{\alpha_c\alpha_a}{(\alpha_a+\alpha_b+\alpha_c)(\alpha_a+\alpha_b)}, & P(cba) &= \beta_{S_{abc}} \frac{\alpha_c\alpha_b}{(\alpha_a+\alpha_b+\alpha_c)(\alpha_b+\alpha_a)}
\end{aligned}$$

(b) In the class $S_{(ab)c}$, we have $\pi_2 = \pi_2(2)$ (with $l = 2$), $\alpha_1^{\pi_2=(ab)c} = \alpha_{(ab)}$ and $\alpha_2^{\pi_2=(ab)c} = \alpha_3^{\pi_1=abc} = \alpha_c$. Thus, the probabilities of the two pre-orders of the class are defined by:

$$P((ab)c) = \beta_{S_{(ab)c}} \frac{\alpha_{(ab)}}{\alpha_{(ab)} + \alpha_c} \; and P(c(ab)) = \beta_{S_{(ab)c}} \frac{\alpha_c}{\alpha_c + \alpha_{(ab)}}.$$

(c)In the class $S_{(bc)a}$, we have $\pi_3 = \pi_3(2)$ (with $l = 2$), $\alpha_1^{\pi_3=(bc)a} = \alpha_{(bc)}$ and $\alpha_2^{\pi_3=(bc)a} = \alpha_1^{\pi_1=abc} = \alpha_a$. Thus, the probabilities of the two pre-orders of the class are defined by:

$$P((bc)a) = \beta_{S_{a(bc)}} \frac{\alpha_{(bc)}}{\alpha_{(bc)} + \alpha_a} \; and \; P(a(bc)) = \beta_{S_{a(bc)}} \frac{\alpha_a}{\alpha_a + \alpha_{(bc)}}.$$

(d) In the class $S_{(ac)b}$, we have $\pi_4 = \pi_4(2)$ (with $l = 2$), $\alpha_1^{\pi_4=(ac)b} = \alpha_{(ac)}$ and $\alpha_2^{\pi_4=(ac)b} = \alpha_2^{\pi_1=abc} = \alpha_b$. Thus, the probabilities of the two pre-orders of the class are defined by:

$$P(b(ac)) = \beta_{S_{(ac)b}} \frac{\alpha_b}{\alpha_b + \alpha_{(ac)}} and P((ac)b) = \beta_{S_{(ac)b}} \frac{\alpha_{(ac)}}{\alpha_{(ac)} + \alpha_b}.$$

(e) And the probability to have the unique pre-order $\pi_5 = \pi_5(1) = (abc)$ of the class $S_{(abc)}$ is $\beta_{S_{(abc)}}$.

Notice that parameters of our proposed P.L. probability distribution are obtained in general case. We have not studied particular cases. When one faces to the practical use of that P.L. probability distribution, then the usual and well-known parametric estimation method in Statistics can be used to evaluate these parameters (if there exists data on probabilities of occurrence of pre-orders of X). In addition, we suggest to explore specificities (particularities) on classes or blocks presented by the practical problem. These particularities can permit to reduce the number of parameters or to provide restrictions or links on parameters. For example: (i) assign the same parameter to classes having the same number of pre-orders or to pre-orders having the same number of blocks (ii) assign zero as parameters to useless classes or useless blocks (iii) parameters of classes can depend on number of some blocks, (iv) parameters on blocks can depend or related to those of its individual elements. The most higher (and valuable) such specificities are, the easier the estimation of parameters.

4. Concluding remarks

We determine two families of parametric probability distributions on the set of complete pre-orders of a finite universe. The first one generalizes the well-known Mallow distribution while the second family generalizes Plackett-Luce distribution on the set of complete orders of the universe. The generalized Mallow distribution is defined by means of a metric on pre-orders evaluating difference between pairwise indifference and pairwise strict order and by means of the usual parameter of the Mallows distribution. This new distance generalizes Kendall distance on pre-oders by assigning any score between $\frac{1}{2}$ and 1 instead of $\frac{1}{2}$.
The generalized Plackett-Luce distribution is defined by means of the new parameter vector of components evaluating weights of each class of pre-orders made up of those having the same blocks and a parameter vector of the weights of the blocks (individual elements and not individual elements) of pre-orders.

An open question is to analyze binary relations generated by such distributions.

Acknowledgement The authors thank Prof Eyke Hüllermeier, DAAD and TWAS. The idea of this work was acquired during the research stay of the corresponding author in the team of Prof Eyke Hüllermeier under the funding of DAAD. The work was finalized under the Research Grant No 17-497RG/MATHS/AF/AC-G-FR3240297728 offers by TWAS to the Laboratory of Mathematics-University of Douala. We thank Anonymous referee for his valuable comments.

References

Kendall M.G. (1938) A new measure of rank correlation. *Biometrika*, 30 (1-2), 81-93.

Mallows C.L. (1957) Non null ranking models. *Biometrika*, 44 (1-2), 114-130.

Luce R.D. (1959) *Individual choice behavior: A theoretical analysis*, Wiley, New York.

Plackett R.L. (1975) The analysis of permutations. *J. of the Royal Stat. Soc. Series C*, 24 (2), 193-202.

Daniel H.E. (1950) Rank correlation and population models. *J. Roy. Statist. Soc. B. Met.* 12, 171-191.

Diffo L., Tchantcho B. and Moulen J.(2012) Comparing influence theories in voting games under locally generated measures of dissatisfaction. *Int J. Game theory*, 41 (3): 719-731.

Andjiga N.G., Mekuko A.Y. and Moyouwou I. (2014) Metric rationalization of social welfare functions. *Mathematical Social Sciences*, 72: 14-23.

Kamdem V., Fotso S., Fono L.A. and Hüllermeier E. (2019) On choice functions generated by Mallows and Plackett-Luce Relations. *New Mathematics and Natural Computation*, DOI 10.1142/S179300571950011X.

Busa-Fekete R., Hüllermeier E. and Szoreny B. (2013) Preference-based elicitation using statical models: the case of Mallows, *Proc- ICML-14, 31st International Conference on Machine Learning-Beijing-China*, 2013.

Kemeny J.G.(1959) Mathematics without numbers. *Daedalus* 88 (4), 577-591.

emeny J.G. and Snell J.L.(1962) *Mathematical models in the social sciences*. Vol. 9. Ginn Boston.

Busa-Fekete R., Szoreny B., Weng P., Cheng W. and Hüllermeier E. (2013) Pac rank elicitation through adaptative sampling of stochastic pairwise preferences. *Proc-AAAI-14, 28th Conference on Artificial intelligence, Quebec, Canada*, 2013.

Marden J.R. (1995) *Analysing and Modeling rank data*. Chapman and Hall-UK.