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Divergence Measures Estimation and Its Asymptotic Normality Theory Using Wavelets Empirical Processes II

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Abstract. In Ba *et al.*(2017), a general normal asymptotic theory for divergence measures estimators has been provided. These estimators are constructed from the wavelets empirical process and concerned the general ϕ-divergence measures. In this paper, we first extend the aforementioned results to symmetrized forms of divergence measures. Second, the Tsallis and Renyi divergence measures as well as the Kullback-Leibler measures are investigated in details. The question of the applicability of the results, based on the boundedness assumption is also dealt, leading to future packages.

Key words: Divergence measures estimation; Asymptotic normality; Wavelet theory; wavelets empirical processes; Besov spaces.
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Résumé. (French) Dans Ba textit et al. (2017), une théorie asymptotique normale générale pour les estimateurs de mesures de divergence a été fournie. Ces estimateurs sont construits à partir du processus empirique basé sur les des ondelettes et concernait les mesures générales de divergence phi. Dans cet article, nous étendons d'abord les résultats susmentionnés à des formes symétrisées de mesures de divergence. Deuxièmement, les mesures de divergence Tsallis et Renyi ainsi que les mesures de Kullback-Leibler sont étudiées en détail. La question de l'applicabilité des résultats, basée sur l'hypothèse de densités bornées, est également abordée, conduisant à de futurs programmes informatiques.

1. Introduction

1.1. General Introduction

In this paper, we deal with divergence measures estimation using essentially wavelets density function estimation. A great number of them are based on probability density functions (*pdf*). So let us suppose that we study the discrepancy between two probability measures $\mathbb{Q}$ and $\mathbb{L}$ admitting *pdf*'s $f_{\mathbb{Q}}$ and $f_{\mathbb{L}}$ with respect to a σ-finite measure ν on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, which is usually the Lebesgue measure λ_d (with $\lambda_1 = \lambda$) or a counting measure on $\mathbb{R}^d$. Among the most important divergence measures, are the following

(1) The L_2^2-divergence measure :

$$\mathcal{D}_{L_2}(\mathbb{Q}, \mathbb{L}) = \int_{\mathbb{R}^d} (f_{\mathbb{Q}}(x) - f_{\mathbb{L}}(x))^2 d\nu(x). \tag{1}$$

(2) The family of Renyi's divergence measures indexed by $\alpha > 0$, $\alpha \neq 1$, known under the name of Renyi-α :

$$\mathcal{D}_{R,\alpha}(\mathbb{Q}, \mathbb{L}) = \frac{1}{\alpha - 1} \log \left(\int_{\mathbb{R}^d} f_{\mathbb{Q}}^{\alpha}(x) f_{\mathbb{L}}^{1-\alpha}(x) d\nu(x) \right). \tag{2}$$

(3) The family of Tsallis divergence measures indexed by $\alpha > 0$, $\alpha \neq 1$, also known under the name of Tsallis-α :

$$\mathcal{D}_{T,\alpha}(\mathbb{Q}, \mathbb{L}) = \frac{1}{\alpha - 1} \left(\int_{\mathbb{R}^d} f_{\mathbb{Q}}^{\alpha}(x) f_{\mathbb{L}}^{1-\alpha}(x) - 1 \right) d\nu(x); \tag{3}$$

(4) The Kullback-Leibler divergence measure

$$\mathcal{D}_{KL}(\mathbb{Q}, \mathbb{L}) = \int_{\mathbb{R}^d} f_{\mathbb{Q}}(x) \ \log(f_{\mathbb{Q}}(x)/f_{\mathbb{L}}(x)) \ d\nu(x). \tag{4}$$

The latter, the Kullback-Leibler measure, may be interpreted as a limit case of both the Renyi's family and the Tsallis' one by letting $\alpha \to 1$. As well, for α near 1, the Tsallis family may be seen as derived from $\mathcal{D}_{R,\alpha}(\mathbb{Q}, \mathbb{L})$ based on the first order expansion of the logarithm function in the neighborhood of the unity.

The results presented here are consequences of general results proved in Ba *et al.* (2018). They directly concern symmetrized forms of divergence measures and specific asymptotic forms for the Tsallis and Renyi families and the Kullback-Leibler measure with respect to different statistical procedures. A a consequence, the general introduction of Ba *et al.* (2018), its motivations and its literature review should be reconducted here, what we will avoid and refer the reader to this first part of the paper.

As a matter of rule, from this point, the reader is supposed to have in hand the first ten (10) pages of Ba *et al.* (2018). At this point we are going to present our results of the specific divergence measures. We suppose that we have on our probability space, two independent sequences :

(-) a sequence of independent and identically distributed random variables with common *pdf* $f_{\mathbb{P}_X}$:

$$X_1, X_2, ... \tag{5}$$

(-) a sequence of independent and identically distributed random variables with common *pdf* $f_{\mathbb{P}_Y}$:

$$Y_1, Y_2, ... \tag{6}$$

To make the notations more simple, we write

$$f = f_{\mathbb{P}_X} \text{ and } g = f_{\mathbb{P}_Y}.$$

We focus on using *pdf*'s estimates provided by the wavelets approach. We will deal on the Parzen approach in a forthcoming study. So, we need to explain the frame in which we are going to express our results.

We recall that we are using the wavelets estimators. In the frame of this wavelets theory, for each $n \geq 1$, we fix the resolution level depending on n and denoted by $j = j_n$, and we use the following estimator of the *pdf* f associated to X, based on the sample of size n from X, as defined in (5),

$$f_n(x) = \frac{1}{n} \sum_{i=1}^{n} K_{j_n}(x, X_i). \tag{7}$$

As well, in a two samples problem, we will estimate the *pdf* g associated to Y, based of a sample of size n from Y, as defined in (6), by

$$g_n(x) = \frac{1}{n} \sum_{i=1}^{n} K_{j_n}(x, Y_i). \tag{8}$$

The aforementioned estimator is known under the name linear wavelets estimators. They are fully detailed in the first part. We also denote

$$a_n = \|f_n - f\|_\infty\,,\ b_n = \|g_n - g\|_\infty\,,\ n \geq 1 \qquad (9)$$
$$c_n = a_n \vee b_n,\ c_{n,m} = a_n \vee b_m, n \geq 1,\ m \geq 1.$$

The results hold under the conditions in Theorem 2 in Ba *et al.* (2018) and those conditions concern the wavelets constituents, the sequence of resolution level $(j_n)_{n\geq 1}$ and the functional J in the aforementioned Theorem. We are confident that the reader will smoothly follow the rest of this paper whenever he has already read the first pages of the first part.

2. RESULTS

We begin to give a direct extension of Theorem 2 of the first part concerning symmetrization the functional J. We recall the full asymptotic theory of divergence measures of the form

$$J(f,g) = \int_D \phi(f(x), g(x))dx.$$

I - Direct extensions.

Quite a few number of divergence measures are not symmetrical. Among these non-symmetrical measures are some of the most interesting ones. For such measures, estimators of the form $J(f_n, g)$, $J(f, g_n)$ and $J(f_n, g_n)$ are not equal to $J(g, f_n)$, $J(g_n, f)$ and $J(g_n, f_n)$ respectively.

In one-sided tests, we have to decide whether the hypothesis $f = g$, for g known and fixed, is true based on data from f. In such a case, we may use the statistics one of the statistics $J(f_n, g)$ and $J(g, f_n)$ to perform the tests. We may have information that allows us to prefer one of them over the other. If not, it is better to use both of them, upon the finiteness of both $J(f, g)$ and $J(g, f)$, in a symmetrized form as

$$J_{(s)}(f,g) = \frac{J(f,g) + J(g,f)}{2}. \qquad (10)$$

The same situation applies when we face double-side tests, i.e., testing $f = g$ from data generated from f and from g.

Asymptotic a.s. efficiency.

Theorem 1. *Under the assumptions of Theorem 2 in Ba et al. (2018), we have*

$$\limsup_{n\to+\infty} \frac{\left|J_{(s)}(f_n,g)-J_{(s)}(f,g)\right|}{a_n} \leq \frac{1}{2}\left(A_1+A_4\right) \quad a.s., \tag{11}$$

$$\limsup_{n\to+\infty} \frac{\left|J_{(s)}(f,g_n)-J_{(s)}(f,g)\right|}{a_n} \leq \frac{1}{2}\left(A_2+A_3\right) \quad a.s., \tag{12}$$

$$\limsup_{n\to+\infty} \frac{\left|J_{(s)}(f_n,g_n)-J_{(s)}(f,g)\right|}{c_n} \leq \frac{1}{2}\left(A_1+A_2+A_3+A_4\right), \ a.s.. \tag{13}$$

Asymptotic Normality.

Denote

$$\sigma_{1,4}^2 = \mathbb{V}ar(h_1+h_4)(X)) \ \text{ and } \ \sigma_{2,3}^2 = \mathbb{V}ar(h_2+h_3)(Y)).$$

We have

Theorem 2. *Under the assumptions of Theorem 2 in Ba et al. (2018), we obtain, as* $n \to +\infty$,

$$\sqrt{\frac{n}{\mathbb{V}ar(h_1+h_4)(X)}}\left(J_{(s)}(f_n,g)-J_{(s)}(f,g)\right) \rightsquigarrow \mathcal{N}(0,1), \tag{14}$$

$$\sqrt{\frac{n}{\mathbb{V}ar(h_2+h_3)(X)}}\left(J_{(s)}(f,g_n)-J_{(s)}(f,g)\right) \rightsquigarrow \mathcal{N}(0,1), \tag{15}$$

and as $(n,m) \to (+\infty,+\infty)$,

$$\left(\frac{nm}{m\sigma_{1,4}^2+n\sigma_{2,3}^2}\right)^{1/2}\left(J_{(s)}(f_n,g_m)-J_{(s)}(f,g)\right) \rightsquigarrow \mathcal{N}(0,1). \tag{16}$$

We are going to give special forms of these mains results in a number of corollaries. To handle the Renyi and the Tsallis families, we get general results on the functional

$$\mathcal{I}(\alpha,f,g) = \int_D f^{\alpha}(x)g^{1-\alpha}(x)dx, \alpha > 0.$$

which is used by these families. In turn the treatment of both of them are derived from the $\mathcal{I}$ functional using the delta method. For all these particular cases, we do not give their proofs since they derive from the general cases by straightforward computations.

3. Particular cases

A - Renyi and Tsallis families.

These two families are expressed through the functional

$$\mathcal{I}(\alpha, f, g) = \int_D f^{\alpha}(x) g^{1-\alpha}(x) dx, \alpha > 0, \quad \alpha \neq 1.$$

which of the form of the ϕ-divergence measure with

$$\phi(x, y) = x^{\alpha} y^{1-\alpha}, (x, y) \in \{(f(s), g(s)), \ s \in D\}.$$

So we begin by :

A - (a) - The asymptotic behavior of the functional $\mathcal{I}(\alpha)$.

With a compact domain D and under the boundedness assumption, and under the condition that neither f nor g vanishes on D, all the conditions of Theorem 2 in Ba *et al.* (2018) are satisfied. Particularly, $C2 - \phi$ derives by the application of the Lebesgue Dominated Theorem. Besides ϕ has continuous partial derivatives bounded against zero, of all order. This entails that the functions h_i are all in the required Besov spaces. Then under the conditions on the wavelets, we have the following results.
First, we have

Corollary 1. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any* $\alpha > 0$, $\alpha \neq 1$, *we have*

$$\limsup_{n\to+\infty} \frac{|\mathcal{I}(\alpha, f_n, g) - \mathcal{I}(\alpha, f, g)|}{a_n} \leq \alpha \int_D (f(x)/g(x))^{\alpha-1} dx =: A_1(\alpha), \quad \textit{a.s}$$

$$\limsup_{n\to+\infty} \frac{|\mathcal{I}(\alpha, f, g_n) - \mathcal{I}(\alpha, f, g)|}{b_n} \leq |\alpha - 1| \int_D (f(x)/g(x))^{\alpha} dx =: A_2(\alpha), \quad \textit{a.s}$$

and

$$\limsup_{(n,m)\to(+\infty,+\infty)} \frac{|\mathcal{I}(\alpha, f_n, g_m) - \mathcal{I}(\alpha, f, g)|}{c_{n,m}} \leq A_1(\alpha) + A_2(\alpha), \quad \textit{a.s.}$$

Denote

$$\sigma_1^2(\alpha, f, g) = \alpha^2 \left(\left(\int_D f(x)(f(x)/g(x))^{2\alpha-2} dx \right) - \left(\int_D f(x)(f(x)/g(x))^{\alpha-1} dx \right)^2 \right)$$

and

$$\sigma_2^2(\alpha, f, g) = (\alpha - 1)^2 \left(\left(\int_D g(x)(f(x)/g(x))^{2\alpha} dx \right) - \left(\int_D g(x)(f(x)/g(x))^{\alpha} dx \right)^2 \right)$$

We have

Corollary 2. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any* $\alpha > 0,\ \alpha \neq 1$, *we have as* $n \to +\infty$,

$$\sqrt{n}(\mathcal{I}(\alpha, f_n, g) - \mathcal{I}(\alpha, f, g)) \rightsquigarrow \mathcal{N}(0, \sigma_1^2(f, g)),$$

$$\sqrt{n}(\mathcal{I}(\alpha, f, g_n) - \mathcal{I}(\alpha, f, g)) \rightsquigarrow \mathcal{N}(0, \sigma_2^2(f, g)),$$

and as $(n, m) \to (+\infty, +\infty)$

$$\left(\frac{mn}{n\sigma_2^2(\alpha, f, g) + m\sigma_1^2(\alpha, f, g)}\right)^{1/2} \left(\mathcal{I}(\alpha, f_n, g_m) - \mathcal{I}(\alpha, f, g)\right) \rightsquigarrow \mathcal{N}(0, 1).$$

As to the symmetrized form

$$\mathcal{I}_s(\alpha, f, g) = \frac{\mathcal{I}_s(\alpha, f, g) + \mathcal{I}_s(\alpha, g, f)}{2},$$

we need the supplementary notations:

$$A_3(\alpha, f, g) = \alpha \int_D (g(x)/f(x))^{\alpha-1} dx, \ , \ A_4(\alpha, f, g) = |\alpha - 1| \int_D (g(x)/f(x))^{\alpha} dx,$$

$$\ell_1(\alpha, x, y) = (y/x)^{\alpha}((1-\alpha) + \alpha(x/y)^{2\alpha-1},\ \ell_2(\alpha, x, y) = (x/y)^{\alpha}((1-\alpha) + \alpha(y/x)^{2\alpha-1},$$

$$\sigma_3^2(\alpha, f, g) = \left(\int_D f(x)\ell_1(\alpha, f(x), g(x))^2 dx\right) - \left(\int_D f(x)\ell_1(\alpha, f(x), g(x)) dx\right)^2,$$

$$\sigma_4^2(\alpha, f, g) = \left(\int_D g(x)\ell_2(\alpha, f(x), g(x))^2 dx\right) - \left(\int_D g(x)\ell_2(\alpha, f(x), g(x)) dx\right)^2.$$

We have

Corollary 3. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any* $\alpha > 0,\ \alpha \neq 1$,

$$\limsup_{n\to+\infty} \frac{|\mathcal{I}_{(s)}(\alpha, f_n, g) - \mathcal{I}_{(s)}(\alpha, f, g)|}{a_n} \leq (A_1(\alpha) + A_4(\alpha))/2 =: A_1^{(s)}(\alpha), \quad \textit{a.s.}$$

$$\limsup_{n\to+\infty} \frac{|\mathcal{I}_{(s)}(\alpha, f, g_n) - \mathcal{I}_{(s)}(\alpha, f, g)|}{b_n} \leq (A_2(\alpha) + A_3(\alpha))/2 =: A_2^{(s)}(\alpha) \quad \textit{a.s.}$$

and

$$\limsup_{(n,m)\to(+\infty,+\infty)} \frac{|\mathcal{I}_{(s)}(\alpha, f_n, g_m) - \mathcal{I}_{(s)}(\alpha, f, g)|}{c_{n,m}} \leq A_1^{(s)}(\alpha) + A_2^{(s)}(\alpha) \quad \textit{a.s.}.$$

We also have

Corollary 4. *Let us assume that the conditions of Theorem 2 in Ba et al. (2018) hold. Then for any $\alpha > 0$, $\alpha \neq 1$, we have as $n \to +\infty$,*

$$\sqrt{n}(\mathcal{I}(\alpha, f_n, g) - \mathcal{I}(\alpha, f, g)) \rightsquigarrow \mathcal{N}(0, \sigma_3^2(f, g)),$$

$$\sqrt{n}(\mathcal{I}(\alpha, f, g_n) - \mathcal{I}(\alpha, f, g)) \rightsquigarrow \mathcal{N}(0, \sigma_3^2(f, g)),$$

and as $(n, m) \to (+\infty, +\infty)$,

$$\left(\frac{mn}{n\sigma_2^4(\alpha, f, g) + m\sigma_3^2(\alpha, f, g)}\right)^{1/2} \left(\mathcal{I}_{(s)}(\alpha, f_n, g_m) - \mathcal{I}_{(s)}(\alpha, f, g)\right) \rightsquigarrow \mathcal{N}(0, 1).$$

A - (b) - Tsallis' Family.

The treatment of the asymptotic behaviour of the Tsallis-α, $\alpha > 0$, $\alpha \neq 1$, is obtained from Part (A) by expansions. We first remark that

$$\mathcal{D}_{T,\alpha}(f, g) = \frac{\mathcal{I}(\alpha, f, g)}{\alpha - 1}.$$

We have the following results

Corollary 5. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any $\alpha > 0$, $\alpha \neq 1$, we have*

$$\limsup_{n\to+\infty} \frac{|\mathcal{D}_{T,\alpha}(f_n, g) - \mathcal{D}_{T,\alpha}(f, g)|}{a_n} \leq \frac{A_1(\alpha)}{|\alpha - 1|} =: A_{T,\alpha,1}, \quad \textit{a.s.}$$

$$\limsup_{n\to+\infty} \frac{|\mathcal{D}_{T,\alpha}(f, g_n) - \mathcal{D}_{T,\alpha}(f, g)|}{b_n} \leq \frac{A_2(\alpha)}{|\alpha - 1|} =: A_{T,\alpha,2}, \quad \textit{a.s.}$$

and

$$\limsup_{(n,m)\to(+\infty,+\infty)} \frac{|\mathcal{D}_{T,\alpha}(f_n, g_m) - \mathcal{D}_{T,\alpha}(f, g)|}{c_{n,m}} \leq A_{T,\alpha,1} + A_{T,\alpha,2}, \quad \textit{a.s.}.$$

Denote

$$\sigma_{T,1}^2(\alpha, f, g) = \frac{\sigma_1^2(\alpha, f, g)}{(\alpha - 1)^2} \quad \text{and} \quad \sigma_{T,2}^2(\alpha, f, g) = \frac{\sigma_2^2(\alpha, f, g)}{(\alpha - 1)^2}.$$

We have

Corollary 6. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any* $\alpha > 0$, $\alpha \neq 1$ *we have as* $n \to +\infty$,

$$\sqrt{n}(\mathcal{D}_{T,\alpha}(f_n, g) - \mathcal{D}_{T,\alpha}(f, g)) \rightsquigarrow \mathcal{N}(0, \sigma^2_{T,1}(\alpha, f, g)),$$

$$\sqrt{n}(\mathcal{D}_{T,\alpha}(f, g_n) - \mathcal{D}_{T,\alpha}(f, g)) \rightsquigarrow \mathcal{N}(0, \sigma^2_{T,2}(\alpha, f, g)),$$

and as $(n, m) \to (+\infty, +\infty)$,

$$\left(\frac{mn}{n\sigma^2_{T,2}(\alpha, f, g) + m\sigma^2_{T,1}(\alpha, f, g)}\right)^{1/2} \Big(\mathcal{D}_{T,\alpha}(f_n, g_m) - \mathcal{D}_{T,\alpha}(f, g)\Big) \rightsquigarrow \mathcal{N}(0, 1).$$

As to the symmetrized form

$$\mathcal{D}^{(s)}_{T,\alpha}(f, g) = \frac{\mathcal{D}_{T,\alpha}(f, g) + \mathcal{D}_{T,\alpha}(g, f)}{2},$$

we simply adapt the parameters obtained for the $A - (a)$. We have

$$A_{T,\alpha,3} = A_3(\alpha, f, g)/|\alpha - 1|, \quad A_{T,\alpha,4} = A_4(\alpha, f, g)/|\alpha - 1|.$$

and

$$\sigma^2_{T,3}(\alpha, f, g) = \sigma^2_3(\alpha, f, g)/(\alpha - 1)^2, \quad \sigma^2_{T,4}(\alpha, f, g) = \sigma^2_4(\alpha, f, g)/(\alpha - 1)^2$$

We have

Corollary 7. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any* $\alpha > 0$, $\alpha \neq 1$,

$$\limsup_{n \to +\infty} \frac{|\mathcal{D}^{(s)}_{T,\alpha}(f_n, g) - \mathcal{D}^{(s)}_{T,\alpha}(f, g)|}{a_n} \leq (A_{T,\alpha,1} + A_{T,\alpha,4})/2 =: A^{(s)}_{T,\alpha,1}(\alpha), \quad \text{a.s.}$$

$$\limsup_{n \to +\infty} \frac{|\mathcal{D}^{(s)}_{T,\alpha}(f, g_n) - \mathcal{D}^{(s)}_{T,\alpha}(f, g)|}{b_n} \leq (A_{T,\alpha,2} + A_{T,\alpha,3})/2 =: A^{(s)}_{T,\alpha,2}(\alpha), \quad \text{a.s.}$$

and

$$\limsup_{(n,m) \to (+\infty, +\infty)} \frac{|\mathcal{D}^{(s)}_{T,\alpha}(f_n, g_m) - \mathcal{D}^{(s)}_{T,\alpha}(f, g)|}{c_{n,m}} \leq A^{(s)}_{T,\alpha,1} + A^{(s)}_{T,\alpha,2}, \quad \text{a.s.}.$$

Denote

$$\sigma^2_{T,1,4}(\alpha,f,g) = \sigma^2_{T,1}(\alpha,f,g)+\sigma^2_{T,4}(\alpha,f,g)$$
$$\sigma^2_{T,2,3}(\alpha,f,g) = \sigma^2_{T,2}(\alpha,f,g)+\sigma^2_{T,3}(\alpha,f,g).$$

We have

Corollary 8. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any* $\alpha>0$, $\alpha\neq 1$, *we have as* $n\to+\infty$

$$\sqrt{n}(\mathcal{D}^{(s)}_{T,\alpha}(f_n,g)-\mathcal{D}^{(s)}_{T,\alpha}(f,g)) \rightsquigarrow \mathcal{N}(0,\sigma^2_{T,1,4}(\alpha,f,g)),$$

$$\sqrt{n}(\mathcal{D}^{s}_{T,\alpha}(f,g_n)-\mathcal{D}^{s}_{T,\alpha}(f,g)) \rightsquigarrow \mathcal{N}(0,\sigma^2_{T,2,3}(\alpha,f,g)),$$

and as $(n,m)\to(+\infty,+\infty)$

$$\left(\frac{mn}{n\sigma^2_{T,2,3}(\alpha,f,g)+m\sigma^2_{T,1,4}(\alpha,f,g)}\right)^{1/2}\left(\mathcal{D}^{s}_{T,\alpha}(f_n,g_m)-\mathcal{D}^{s}_{T,\alpha}(f,g)\right) \rightsquigarrow \mathcal{N}(0,1).$$

A - (c) - Renyi's Family.

The treatment of the asymptotic behaviour of the Renyi-α, $\alpha>0$, $\alpha\neq 1$, is obtained from Part (A) by expansions and by the application of the delta method. We first remark that

$$\mathcal{D}_{R,\alpha}(f,g)=\frac{1}{\alpha-1}\log\left(\int_D f^{\alpha}(x)g^{1-\alpha}(x)dx\right)=\frac{\log(\mathcal{I}(\alpha,f,g))}{\alpha-1}.$$

We have the following results

Corollary 9. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any* $\alpha>0$, $\alpha\neq 1$, *we have*

$$\limsup_{n\to+\infty}\frac{|\mathcal{D}_{R,\alpha}(f_n,g)-\mathcal{D}_{R,\alpha}(f,g)|}{a_n}\leq\frac{A_1(\alpha)}{|\alpha-1|\mathcal{I}(\alpha,f,g)}=:A_{R,\alpha,1},\quad a.s.$$

$$\limsup_{n\to+\infty}\frac{|\mathcal{D}_{R,\alpha}(f,g_n)-\mathcal{D}_{R,\alpha}(f,g)}{b_n}\leq\frac{A_2(\alpha)}{|\alpha-1|\mathcal{I}(\alpha,f,g)}=:A_{R,\alpha,2},\quad a.s.$$

and

$$\limsup_{(n,m)\to(+\infty,+\infty)}\frac{|\mathcal{D}_{R,\alpha}(f_n,g_m)-\mathcal{D}_{R,\alpha}(f,g)}{c_{n,m}}\leq A_{R,\alpha,1}+A_{R,\alpha,2},\quad a.s..$$

Denote

$$\sigma^2_{R,1}(\alpha,f,g)=\frac{\sigma^2_1(\alpha,f,g)}{(\alpha-1)^2\mathcal{I}(\alpha,f,g)^2}, \quad \text{and} \quad \sigma^2_{R,2}(\alpha,f,g)=\frac{\sigma^2_2(\alpha,f,g)}{(\alpha-1)^2\mathcal{I}(\alpha,g,f)^2}.$$

We have

Corollary 10. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any $\alpha>0$, $\alpha\neq 1$, we have as $n\to+\infty$,*

$$\sqrt{n}(\mathcal{D}_{R,\alpha}(f_n,g)-\mathcal{D}_{R,\alpha}(f,g))\rightsquigarrow\mathcal{N}(0,\sigma^2_{R,1}(\alpha,f,g)),$$

$$\sqrt{n}(\mathcal{D}_{R,\alpha}(f,g_n)-\mathcal{D}_{R,\alpha}(f,g))\rightsquigarrow\mathcal{N}(0,\sigma^2_{R,2}(\alpha,f,g)),$$

and as $(n,m)\to(+\infty,+\infty)$

$$\left(\frac{mn}{n\sigma^2_{R,2}(\alpha,f,g)+m\sigma^2_{R,1}(\alpha,f,g)}\right)^{1/2}\Big(\mathcal{D}_{R,\alpha}(f_n,g_m)-\mathcal{D}_{R,\alpha}(f,g)\Big)\rightsquigarrow\mathcal{N}(0,1).$$

As to the symmetrized form

$$\mathcal{D}^{(s)}_{R,\alpha}(f,g)=\frac{\mathcal{D}_{R,\alpha}(f,g)+\mathcal{D}_{R,\alpha}(g,f)}{2},$$

we need the suplementary notations

$$A_{R,\alpha,3} = \frac{A_3(\alpha)}{|\alpha-1|\mathcal{I}(\alpha,f,g)}$$

$$A_{R,\alpha,4} = \frac{A_4(\alpha)}{|\alpha-1|\mathcal{I}(\alpha,f,g)}$$

$$\ell_{R,1}(\alpha,x,y)=\frac{1}{2(\alpha-1)}\left(\frac{\alpha(x/y)^{\alpha-1}}{\mathcal{I}(\alpha,f,g)}+\frac{((1-\alpha)(y/x)^{\alpha}}{\mathcal{I}(\alpha,g,f)}\right),$$

$$\ell_{R,2}(\alpha,x,y)=\frac{1}{2(\alpha-1)}\left(\frac{\alpha(y/x)^{\alpha-1}}{\mathcal{I}(\alpha,g,f)+\frac{(1-\alpha)(x/y)^{\alpha}}{\mathcal{I}(\alpha,f,g)}}\right),$$

$$\sigma^2_{R,3}(\alpha,f,g)=\left(\int_D f(x)\ell_{R,1}(\alpha,f(x),g(x))^2dx\right)-\left(\int_D f(x)\ell_{R,1}(\alpha,f(x),g(x))dx\right)^2,$$

$$\sigma^2_{R,4}(\alpha,f,g)=\left(\int_D g(x)\ell_{R,2}(\alpha,f(x),g(x))^2dx\right)-\left(\int_D g(x)\ell_{R,2}(\alpha,f(x),g(x))dx\right)^2.$$

We have

Corollary 11. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any* $\alpha > 0$, $\alpha \neq 1$,

$$\limsup_{n\to+\infty} \frac{|\mathcal{D}^{(s)}_{R,\alpha}(f_n,g) - \mathcal{D}^{(s)}_{R,\alpha}(f_n,g)|}{a_n} \leq (A_{R,\alpha,1} + A_{R,\alpha,4})/2 =: A^{(s)}_{R,\alpha,1}, \quad a.s.$$

$$\limsup_{n\to+\infty} \frac{|\mathcal{D}^{(s)}_{R,\alpha}(f,g_n) - \mathcal{D}^{(s)}_{R,\alpha}(f,g)|}{b_n} \leq (A_{R,\alpha,2} + A_{R,\alpha,3})/2 =: A^{(s)}_{R,\alpha,2}, \quad a.s.$$

and

$$\limsup_{(n,m)\to(+\infty,+\infty)} \frac{|\mathcal{D}^{(s)}_{R,\alpha}(f_n,g_m) - \mathcal{D}^{(s)}_{R,\alpha}(f,g)|}{c_{n,m}} \leq A^{(s)}_{R,\alpha,1} + A^{(s)}_{R,\alpha,2}, \quad a.s..$$

Denote

$$\sigma^2_{R,1,4}(\alpha,f,g) = \sigma^2_{R,1}(\alpha,f,g) + \sigma^2_{R,4}(\alpha,f,g)$$
$$\sigma^2_{R,2,3}(\alpha,f,g) = \sigma^2_{R,2}(\alpha,f,g) + \sigma^2_{R,3}(\alpha,f,g)$$

We also have

Corollary 12. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then for any* $\alpha > 0$, $\alpha \neq 1$, *we have as* $n \to +\infty$,

$$\sqrt{n}(\mathcal{D}^{s}_{R,\alpha}(f_n,g) - \mathcal{D}^{s}_{R,\alpha}(f,g)) \rightsquigarrow \mathcal{N}(0,\sigma^2_{R,1,4}(f,g)),$$

$$\sqrt{n}(\mathcal{D}^{s}_{R,\alpha}(f,g_n) - \mathcal{D}^{s}_{R,\alpha}(f,g)) \rightsquigarrow \mathcal{N}(0,\sigma^2_{R,2,3}(f,g)),$$

and as $(n,m) \to +\infty, \infty)$,

$$\left(\frac{mn}{n\sigma^2_{R,2,3}(\alpha,f,g) + m\sigma^2_{R,1,4}(\alpha,f,g)}\right)^{1/2} \left(\mathcal{D}^{s}_{R,\alpha}(f_n,g_m) - \mathcal{D}^{s}_{R,\alpha}(f,g)\right) \rightsquigarrow \mathcal{N}(0,1).$$

B- Kullback-Leibler Measure.

Here we have

$$\phi(x,y) = x\log(x/y),\ (x,y) \in \{(f(s),g(s)),\ s \in D\}.$$

and the Kulback-Leibler Measure is defined by $\mathcal{D}_{KL}(f,g) = \int_D f(x)\log(f(x)/g(x)dx$.

The preliminary text of Part (A) is still valid. So, we have first :

Corollary 13. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then we have*

$$\limsup_{n\to+\infty} \frac{|\mathcal{D}_{KL}(f_n,g)-\mathcal{D}_{KL}(f,g)|}{a_n} \leq \int_D |1+\log(f(x)/g(x)|dx =: A_{KL,1}(f,g), \quad a.s.$$

$$\limsup_{n\to+\infty} \frac{|\mathcal{D}_{KL}(f,g_n)-\mathcal{D}_{KL}(f,g)|}{b_n} \leq \int_D f(x)/g(x)dx =: A_{KL,2}(f,g), \quad a.s.$$

and

$$\limsup_{(n,m)\to(+\infty,+\infty)} \frac{|\mathcal{D}_{KL}(f_n,g_m)-\mathcal{D}_{KL}(f,g)|}{c_{n,m}} \leq A_{KL,1}+A_{KL,2}, \quad a.s..$$

Denote

$$\sigma^2_{KL,1}(f,g) = \left(\left(\int_D f(x)(1+\log(f(x)/g(x)^2dx\right) - \left(\int_D f(x)(1+\log(f(x)/g(x)dx\right)^2\right)$$

and

$$\sigma^2_{KL,2}(f,g) = \left(\left(\int_D f^2(x)/g(x)dx\right) - 1\right)$$

We have

Corollary 14. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then we have as* $n\to+\infty$ *,*

$$\sqrt{n}(\mathcal{D}_{KL}(f_n,g)-\mathcal{D}_{KL}(f,g)) \rightsquigarrow \mathcal{N}(0,\sigma^2_{KL,1}(f,g)),$$

$$\sqrt{n}(\mathcal{D}_{KL}(f,g_n)-\mathcal{D}_{KL}(f,g)) \rightsquigarrow \mathcal{N}(0,\sigma^2_{KL,2}(f,g)),$$

and as $(n,m)\to(+\infty,+\infty)$

$$\left(\frac{mn}{n\sigma^2_{KL,2}(f,g)+m\sigma^2_{KL,1}(f,g)}\right)^{1/2}\left(\mathcal{D}_{KL}(f_n,g_m)-\mathcal{D}_{KL}(f,g)\right) \rightsquigarrow \mathcal{N}(0,1).$$

As to the symmetrized form

$$\mathcal{D}^s_{KL}(f,g) = \frac{\mathcal{D}_{KL}(f,g)+\mathcal{D}_{KL}(g,f)}{2},$$

we need the supplementary notations:

$$A_{KL,3}(f,g) = \int_D |1+\log(g(x)/f(x)|dx, \ , \ A_{KL,4}(f,g) = \int_D g(x)/f(x)dx,$$

$$\ell_{KL,1}(x,y) = 1 - (y/x) + \log(x/y),\ \ \ell_{KL,2}(x,y) = 1 - (x/y) + \log(y/x),$$

$$\sigma^2_{KL,3}(\alpha, f, g) = \left(\int_D f(x)\ell_{KL,1}(f(x), g(x))^2 dx\right) - \left(\int_D f(x)\ell_{KL,1}(f(x), g(x)) dx\right)^2,$$

and

$$\sigma^2_{KL,4}(f, g) = \left(\int_D g(x)\ell_2(\alpha, f(x), g(x))^2 dx\right) - \left(\int_D g(x)\ell_2(f(x), g(x)) dx\right)^2.$$

We have

Corollary 15. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then,*

$$\limsup_{n\to+\infty} \frac{|\mathcal{D}^s_{KL}(f_n, g) - \mathcal{D}^s_{KL}(f, g)|}{a_n} \leq (A_{KL,1}(f,g) + A_{KL,4})/2 =: A^{(s)}_{KL,1}(f,g), \quad a.s.$$

$$\limsup_{n\to+\infty} \frac{|\mathcal{D}^s_{KL}(f, g_n) - \mathcal{D}^s_{KL}(f, g)|}{b_n} \leq (A_{KL,2}(f,g) + A_{KL,3})/2 =: A^{(s)}_{KL,2}(f,g), \quad a.s.$$

and

$$\limsup_{(n,m)\to(+\infty,+\infty)} \frac{|\mathcal{D}^s_{KL}(f_n, g_m) - \mathcal{D}^s_{KL}(f, g)|}{c_{n,m}} \leq A^{(s)}_{KL,1}(f,g) + A^{(s)}_{KL,2}(f,g), \quad a.s..$$

Denote

$$\begin{aligned}\sigma^2_{KL,1,4}(f,g) &= \sigma^2_{KL,1}(f,g) + \sigma^2_{KL,4}(f,g)\\ \sigma^2_{KL,2,3}(f,g) &= \sigma^2_{KL,2}(f,g) + \sigma^2_{KL,3}(f,g).\end{aligned}$$

We also have

Corollary 16. *Let the assumptions of Theorem 2 in Ba et al. (2018) hold. Then, we have as* $n \to +\infty$*,*

$$\sqrt{n}(\mathcal{D}^s_{KL}(f_n, g) - \mathcal{D}^s_{KL}(f, g)) \rightsquigarrow \mathcal{N}(0, \sigma^2_{KL,1,4}(f,g)),$$

$$\sqrt{n}(\mathcal{D}^s_{KL}(f, g_n) - \mathcal{D}^s_{KL}(f, g)) \rightsquigarrow \mathcal{N}(0, \sigma^2_{KL,2,3}(f,g)),$$

and as $(n,m) \to (+\infty, +\infty)$

$$\left(\frac{mn}{n\sigma^2_{KL,2,3}(f,g) + m\sigma^2_{KL,1,4}(f,g)}\right)^{1/2} \left(\mathcal{D}^s_{KL}(f_n, g_m) - \mathcal{D}^s_{KL}(f, g)\right) \rightsquigarrow \mathcal{N}(0,1).$$

4. Comments and announcements

In papers I and II of this series, the main results on the asymptotic behaviors of empirical divergence measures based on wavelets theory have been established and particularized for important families of divergence measures like Renyi and Tsallis families and for the Kullback-Leibler measures. While the proofs of results in the second paper may be skipped, the proofs of those in paper I are to be thoroughly proved since they serve as a foundation to the whole structure of results. They are stated in Ba *et al.* (2018), to appear, but are already detailed in Ba *et al.* (2017) in *Arxiv*.

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