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Ruin Probabilities in Perturbed Risk Process with Stochastic Investment and Force of Interest

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Abstract. . This work considers a perturbed risk process with investment, where the investments are either into invested in risky and risk-less assets. A third order differential equation for the ruin probability is derived from the resulting integro-differential equation. This equation is further decomposed into two equations describing the contributions of the claim and oscillation to the ruin probability. These two equations are solved separately using suitable transformations as well as theory of Kummer confluence hypergeometric equations. We further investigated these solutions and were able to conclude that the higher the fraction of investment into risky assets, the lower the ruin probability, provided all other parameters are kept constant.

Résumé (Franch) Ce travail considère un processus de risque perturbé avec investissement, où les investissements sont investis dans des actifs risqués et sans risque. En utilisant la théorie des équations hypergéométriques de confluence de Kummer, nous aboutissons à la conclusion que plus la fraction de l'investissement dans les actifs risqués est élevée, plus la probabilité de ruine est grande, les autres paramètres restent constants.

Key words: Risk Reserve; Ruin Probability; Interest; Stochastic Investment; Exponential distribution; Kummer hyper-geometric equation.
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1. Introduction

Consider the perturbed risk process (1)

$$Y(t) = u + \pi(t) - \sum_{i=1}^{N(t)} X_i + \varepsilon(t) \tag{1}$$

where $\pi(t)$ is a non-negative linear premium function, $X_1, X_2, ..., X_{N(t)}$ are independent and identically distributed non-negative random variables denoting the claim size, $N(t)$ is the number of arrivals at time t assumed to be a Poisson process with rate λ and $\epsilon(t)$ is a Gaussian white noise process modeling the uncertainty in both the claim sizes and premiums. Assume that the linear premium function is represented by $pt, p > 0$ and $\epsilon(t)$ is a linear function $\sigma_p \epsilon t$ with values that are independent at every distinct time interval t, where σ_p is a diffusion volatility parameter and ϵ is normally distributed random error with zero mean and unity variance.

Suppose the reserve is continuously invested in both risky and risk-less assets and that the return from the risky assets is govern by the geometric Brownian motion below;

$$dY = aY dt + \sigma Y dW_t. \tag{2}$$

If the interest on fixed returned investment is denoted by a non-negative parameter r, then the income process in a short time interval can be modeled as

$$cdt + \sigma_p dW + rY(t)(1-\varphi) dt + a\varphi Y(t)dt + \sigma_R \varphi Y(t)dW. \tag{3}$$

where φ is the fraction of the reserve invested in markets with stochastic return, W is a Weiner process and σ_R is volatility of the geometric Brownian motion. Thus, the reserve (1) becomes;

$$\begin{aligned} Y(t) = u + ct + r(1-\varphi) \int_0^t Y(s)ds \\ + a\varphi \int_0^t Y(s)ds + \int_0^t (\sigma_p + \sigma_R \varphi Y(s))dW(s) + \sum_{i=1}^{N(t)} X_i \end{aligned} \tag{4}$$

In perturbed risk reserve process, ruin can be caused by a claim or by oscillation (Cai and Yang, 1986). Suppose the time of ruin of the reserve is denoted by the random variable $\tau(u)$, where $\tau(u) = \inf\{t : X_t < 0\}$. Then, the infinite ruin probability of the reserve is given as

$$\psi(u) = P\{\tau(u) < \infty | Y_0 = u\} \tag{5}$$

Since the work of Segerdahl (2005) where the first investigation on risk model with investment income was carried out, several authors have investigated various versions of the model. Segerdahl considered a situation where capital earns interest

at a fixed rate r. Harrison (1977) elaborated on the model by assuming that interest is earned continuously on the firm's assets. A more general form of the model was studied by Delbaen and Haezendonck (1987) where inflation factor was incorporated into the model. Other early works which incorporated return from investment at a fixed rate are Taylor (1979) and Kahane (1979).
One of the earliest works which considered capital invested in risky assets is Paulsen (1993). The work basically considered model with stochastic rate of return on investment as well as stochastic level of inflation. A basic risk process P and return from investment R were considered independent and of the form

$$P_t = y + pt + W_{P,t} - \sum_{i=1}^{N_{P,t}} S_{P,i} \tag{6}$$

$$R_t = rt + W_{R,t} - \sum_{i=1}^{N_{R,t}} S_{R,i}, \tag{7}$$

where $W_{P,t}$ is a Brownian motion. A similar form was also adopted for the inflation. A somewhat generalization of the model was considered by Paulsen (1998), Yuen *et al.* (2004) and Yuen and Wang (2005). These works incorporated the volatility term σ into the return from investment. Paulsen (1997) also considered considered this model incorporating the volatility term in both the reserve and the investment. Another form of the reserve when uncertainty in the reserve is not considered was studied by Belkina *et al.* (2014). Belkina et al. incorporated both risk and risk-less asset into the model and assume that investment in risky assets is governed by Black-Scholes equation and can only be done at a limited leveraging level. Some other researchers who have investigated different variations of this model are Zhang and Xiao (2015), Cai and Yang (1986) and Romera and Runggaldier (2012).

2. Integro-Differential Equation

The derivation of the integro-differential equation for model (4) involves the determination of infinitesimal generator $\mathcal{A}$ defined by

$$(\mathcal{A}f)(x) = \lim_{s\to 0} \frac{(T_s f)(x) - f(x)}{s} \tag{8}$$

where

$$(T_s f)(x) = E[f(X(s))|X(0) = x] = \int f(y)p(s; x, dy) \qquad (s > 0)$$

is a transition operator. One way to do this is splitting the model into a jump process and diffusion process as shown below

$$\begin{aligned} Y(t) &= U(t) \\ &+ \int_0^t (c + r(1-\varphi)Y(s) + a\varphi Y(s))\, ds + (\sigma_p + \sigma_R \varphi \int_0^t Y(s))dW(s) \end{aligned} \tag{9}$$

$$U(t) = u - \sum_{i=1}^{N(t)} X_i \tag{10}$$

Using a different notation, equation (9) can be represented as

$$dY(t) = dU(t) + (c + r(1-\varphi)Y(s) + a\varphi Y(s))\,ds + (\sigma_p + \sigma_R \varphi Y(s))dW(s) \tag{11}$$

Lemma 1. *Suppose a jump process U(t) is defined by equation (10) shown below;*

$$U(t) = u - \sum_{i=1}^{N(t)} X_i$$

Let g be a bounded twice continuously differentiable function, then

$$(\mathcal{A}g)(u) = -\lambda g(u) + \lambda \int_0^{\infty} g(u-x)dF(x) \tag{12}$$

Proof. The conditional expectation of the jump process (10) is

$$E\left[g(U(t))|U(0)=u\right] = E\left[g(u - \sum_{i=1}^{N(t)} X_i)\right] \tag{13}$$

But $N(t)$ is Poisson distributed, thus one obtains

$$\begin{aligned} E\left[g(U(t))|U(0)=u\right] =& E\left[g(u)\right]e^{-\lambda t} + E\left[g(u-X_i)|N(t)=1\right]\lambda t e^{-\lambda t} \\ &+ E\left[g(u - \sum_{i=2}^{N(t)} X_i)|N(t)\geq 2\right] o(t) \end{aligned} \tag{14}$$

Subtracting $g(u)$ from both sides and dividing by t, one obtains;

$$\begin{aligned} \frac{E\left[g(U(t))|U(0)=u\right] - g(u)}{t} =& \frac{E\left[g(u)\right]e^{-\lambda t} - g(u)}{t} \\ &+ E\left[g(u-X_i)|N(t)=1\right]\lambda e^{-\lambda t} \\ &+ E\left[g(u - \sum_{i=2}^{N(t)} X_i)|N(t)\geq 2\right] \frac{o(t)}{t} \end{aligned} \tag{15}$$

Expanding equation (15) further and taking the limits at $t \to 0$ completes the proof.

Theorem 1. *Let g be a bounded twice continuously differentiable function of $Y(t)$ on S. If $Y(t)$ is a diffusion process with drift coefficient $c + r(1-\varphi)Y(t) + a\varphi Y(t)$ and diffusion coefficient $(\sigma_p + \sigma_R\varphi Y(t))^2$ defined on S, then,*

$$\begin{aligned} (\mathcal{A}f)(u) =& \frac{1}{2}(\sigma_p^2 + (\sigma_R\varphi)^2 u^2)g''(u) + (c + r(1-\varphi)u + a\varphi u)g'(u) \\ &- \lambda g(u) + \lambda \int_0^{\infty} g(u-x)dF(x) \end{aligned} \tag{16}$$

Proof. Suppose $Y(s) = y \in S$ is fixed. Let $\eta > 0$ be an arbitrary small number. Since g is continuous, there exist $\epsilon > 0$ such that $|z - y| \leq \epsilon$ whenever $|g''(z) - g''(y)| \leq \eta$ for all z. Thus

$$(T_t g)(y) = E[g(Y(t))1_{|Y(t)-Y(s)|\leq\epsilon}|Y(s) = y] + E[g(Y(t))1_{|Y(t)-Y(s)|>\epsilon}|Y(s) = y] \quad (17)$$

where $0 \leq s < t, \quad 1_{|Y(t)-Y(s)|\leq\epsilon}$ is an indicator function and $(T_t g)(y)$ is a transition operator. Expanding the first term on the RHS of equation (17) into Taylor's series around y and the expanding over the expectation, one obtains;

$$\begin{aligned}(T_t g)(y) =& E[g(y)1_{|Y(t)-Y(s)|\leq\epsilon}|Y(s) = y] \\ &+ E[(Y(t) - y)g'(y)1_{|Y(t)-Y(s)|\leq\epsilon}|Y(s) = y] \\ &+ \frac{1}{2}E[(Y(t) - y)^2 g''(y)1_{|Y(t)-Y(s)|\leq\epsilon}|Y(s) = y] \\ &+ o(t) + E[g(Y(t))1_{|Y(t)-Y(s)|>\epsilon}|Y(s) = y] \quad (18)\end{aligned}$$

Using some properties of diffusion processes and expanding over the outer expectation gives

$$\begin{aligned}(T_t g)(u) =& E\big[E\big[g(y)1_{|Y(t)-Y(s)|\leq\epsilon}|Y(s) = y\big]\big|Y(0) = u\big] \\ &+ E\big[\big(c + r(1-\varphi)Y(s) + a\varphi Y(s)\big)tg'(y)\big|Y(0) = u\big] \\ &+ \frac{1}{2}E\big[\big((\sigma_p^2 + (\sigma_R\varphi Y(s))^2\big)tg''(y) + o(t)\big)\big|Y(0) = u\big] + o(t) \quad (19)\end{aligned}$$

Taking the limit as $t \to 0$, after some computation, gives

$$\begin{aligned}\lim_{t\to 0}\frac{(T_t g)(u) - g(u)}{t} =& \lim_{t\to 0}\frac{E\big[E\big[g(y)1_{|Y(t)-Y(s)|\leq\epsilon}|Y(s) = y\big]\big|Y(0) = u\big] - g(u)}{t} \\ &+ \lim_{t\to 0}E\big[\big(c + r(1-\varphi)Y(s) + a\varphi Y(s)\big)g'(y)\big|Y(0) = u\big] \\ &+ \lim_{t\to 0}\frac{1}{2}E\big[\big((\sigma_p^2 + (\sigma_R\varphi Y(s))^2\big)g''(y) + \frac{o(t)}{t}\big)\big|Y(0) = u\big] \\ &+ \lim_{t\to 0}\frac{o(t)}{t} \quad (20)\end{aligned}$$

But $\lim_{t\to 0}\frac{o(t)}{t} = 0$, $Y(0) = U(0) = u$ and $t \to 0$ implies $s \to 0$, thus

$$\begin{aligned}\lim_{t\to 0}\frac{(T_t g)(u) - g(u)}{t} =& \lim_{t\to 0}\frac{E\big[E\big[g(U)\big|U(0) = u\big] - g(u)\big]}{t} \\ &+ \big(c + r(1-\varphi)u + a\varphi u\big)g'(u) + \frac{1}{2}\big(\sigma_p^2 + (\sigma_R\varphi u)^2\big)g''(u) \quad (21)\end{aligned}$$

Substituting equation (12) in lemma 1 using the definition infinitesimal generator completes the proof.

Using theorem (2.1) on page 968 of Paulsen (1997), the integro-differential equation for the model is

$$\begin{aligned}&\frac{1}{2}(\sigma_p^2 + (\sigma_R\varphi)^2 u^2)\psi''(u) + (c + r(1-\varphi)u + a\varphi u)\psi'(u) \\ &\qquad - \lambda\psi(u) + \lambda\int_0^\infty \psi(u - x)dF(x) = 0 \quad (22)\end{aligned}$$

with the following boundary conditions

$$\begin{aligned}
&\psi(+\infty) = 0 \\
&\psi(0) = 1 \\
&\frac{\sigma_p^2}{2}\psi''(0) + c\psi'(0) - \lambda = 0
\end{aligned} \tag{23}$$

Equation (22) above is a Hamilton Jacobi-Bellman equation (HJB) and we consider the situation when the distribution of claim size $F(x)$ is exponential.

3. Exponential Claim Size

Explicit solutions of similar equations with exponential claim sizes have been derived by Paulsen (1997) and Cai and Yang (1986). We adopt method of derivation similar to those of these authors. A more generalized equation from part (ii) of theorem 2.1 of Paulsen (1997) is given below

$$\begin{aligned}
&\frac{1}{2}(\sigma_p^2 + (\sigma_R\varphi)^2 u^2)q_\alpha''(u) + (c + r(1-\varphi)u + a\varphi u)\, q_\alpha'(u) \\
&\qquad + \lambda \int_0^\infty (q_\alpha(u-x) - q_\alpha(u))dF(x) = \alpha q_\alpha(u)
\end{aligned} \tag{24}$$

where $q_\alpha = E(e^{-\alpha\tau_u})$, with the boundary conditions

$$\begin{aligned}
q_\alpha(u) &= 1 \quad \text{on } u < 0 \\
\lim_{t\to\infty} q_\alpha(u) &= 0 \\
\frac{\sigma_p^2}{2}q_\alpha''(0) + cq_\alpha'(0) - \lambda &= \alpha q_\alpha
\end{aligned}$$

Theorem 2. *Given that the claim size distribution $f(x)$ is exponentially distributed with mean $1/\theta$, then the integro-differential equation (24) is equivalent to the third order differential equation below.*

$$\begin{aligned}
&\frac{1}{2}(\sigma_p^2 + (\sigma_R\varphi)^2 u^2)q_\alpha'''(u) + \left(c + [(\sigma_R\varphi)^2 + \bar{r}]u + \frac{1}{2}\theta(\sigma_p^2 + (\sigma_R\varphi)^2 u^2)\right)q_\alpha''(u) \\
&\qquad + \left(\bar{r} - \lambda - \alpha + c\theta + \bar{r}\theta u\right)q_\alpha'(u) - \alpha\theta q(u) = 0
\end{aligned} \tag{25}$$

where $\bar{r} = r(1-\varphi) + a\varphi$, with boundary conditions given in (23)

Proof. Since $f(x)$ is exponentially distributed, then

$$dF(x) = \theta e^{-\theta x}dx$$

thus, equation (22) becomes

$$\begin{aligned}
&\frac{1}{2}(\sigma_p^2 + (\sigma_R\varphi)^2 u^2)q_\alpha''(u) + (c + r(1-\varphi)u + a\varphi u)\, q_\alpha'(u) \\
&\qquad = -\lambda\theta \int_0^\infty (q_\alpha(u-x) - q_\alpha(u))e^{-\theta x}dx + \alpha q_\alpha(u)
\end{aligned} \tag{26}$$

Simplifying, rearranging and differentiating both sides of (26) gives

$$\frac{1}{2}(\sigma_p^2 + (\sigma_R\varphi)^2\, u^2)q_\alpha'''(u) + \left(c + [(\sigma_R\varphi)^2 + r\,(1-\varphi) + a\varphi]u\right) q_\alpha''(u)$$
$$+\ (r\,(1-\varphi) + a\varphi - \lambda - \alpha)q_\alpha'(u) = -\theta\left(\lambda q_\alpha(u) - \lambda\theta e^{-\theta u}\int_0^u q_\alpha(x)e^{\theta x}dx\right) \tag{27}$$

Substituting equation (26) into (27) one obtains

$$\frac{1}{2}(\sigma_p^2 + (\sigma_R\varphi)^2\, u^2)q_\alpha'''(u)$$
$$+\left(c + [(\sigma_R\varphi)^2 + r\,(1-\varphi) + a\varphi]u + \frac{\theta}{2}(\sigma_p^2 + (\sigma_R\varphi)^2\, u^2)\right) q_\alpha''(u)$$
$$+\left(r\,(1-\varphi) + a\varphi - \lambda - \alpha + c\theta + [r\,(1-\varphi) + a\varphi]\theta u\right)q_\alpha'(u) - \alpha\theta q(u) = 0 \tag{28}$$

substituting $\bar{r} = r\,(1-\varphi) + a\varphi$ completes the proof.

The solution of (25) is very difficult to obtain hence we examine specific cases of the equation.

Case 1: *Consider a case when $\sigma_p = \sigma_R = 0$. The equation (25) takes the form*

$$(c + \bar{r}u)\, q_\alpha''(u) + \left(\bar{r} - \lambda - \alpha + c\theta + \bar{r}\theta u\right)q_\alpha'(u) - \alpha\theta q(u) = 0 \tag{29}$$

with boundary conditions

$$q_\alpha(u) = 1 \quad on\ u < 0$$
$$\lim_{t\to\infty} q_\alpha(u) = 0$$
$$cq_\alpha''(0) + (\bar{r} - \lambda - \alpha + c\theta)q_\alpha'(0) = \alpha\theta$$

Equation (29) can be reduced to hypergeometic equation using appropriate transformation. We utilize the same transformation used by Paulsen (1997). Let

$$u = -x/\theta - c/\bar{r} \qquad \text{and} \qquad v(x) = q_\alpha(u) \qquad \text{then}$$

$$q_\alpha'(u) = -\theta v'(x)$$
$$q_\alpha''(u) = \theta^2 v''(x) \tag{30}$$

Substituting the transformation into equation (29), gives

$$xv''(x) + \left(1 - \frac{\lambda + \alpha}{\bar{r}} - x\right)v'(x) - \left(-\frac{\alpha}{\bar{r}}\right)v(x) = 0 \tag{31}$$

Equation (31) is a confluence hypergeometric equation, hence by Slater (2011),

$$v_0(x) = x^{1-b}e^x M(1-a, 2-b; -x)$$
$$v_1(x) = e^x U(b-a, b; -x) \tag{32}$$

are solutions of equation (31) where $a = -\alpha/\bar{r}$, $b = 1 - (\lambda + \alpha)/\bar{r}$, $M(1-a, 2-b; -x)$ is a confluent hypergeometric function and $U(b-a, b; -x)$ is the second form of confluent hypergeometric function. Thus, we have,

$$\begin{aligned} v_0(x) &= x^{(\lambda+\alpha)/\bar{r}} e^x M\left(1 + \frac{\alpha}{\bar{r}}, 1 + \frac{\lambda+\alpha}{\bar{r}}; -x\right) \\ v_1(x) &= e^x U(1 - \frac{\lambda}{\bar{r}}, 1 - \frac{\lambda+\alpha}{\bar{r}}; -x) \end{aligned} \tag{33}$$

Furthermore, $v_0(x)$ and $v_1(x)$ are two linearly independent solutions. Following the same rational as Cai and Yang (1986), then it can be shown that if $v_0(x)$ is a solution of equation (31) which is linearly independent of $v_1(x)$, then so is

$$v_2(x) = -v_0(x) = -x^{(\lambda+\alpha)/\bar{r}} e^x M\left(1 + \frac{\alpha}{\bar{r}}, 1 + \frac{\lambda+\alpha}{\bar{r}}; -x\right) \tag{34}$$

Thus, the general solution of (31) is

$$\begin{aligned} v(x) &= C_1 v_1(x) + C_2 v_2(x) \\ &= C_1 e^x U(1 - \frac{\lambda}{\bar{r}}, 1 - \frac{\lambda+\alpha}{\bar{r}}; -x) - C_2 x^{(\lambda+\alpha)/\bar{r}} e^x M\left(1 + \frac{\alpha}{\bar{r}}, 1 + \frac{\lambda+\alpha}{\bar{r}}; -x\right) \end{aligned} \tag{35}$$

where C_1 and C_2 are constant. Re-applying the transformation above gives the solution of equation (29) as

$$\begin{aligned} q_\alpha(u) = {} & C_1 e^{-(u\bar{r}+c)\theta/\bar{r}} U\left(1 - \frac{\lambda}{\bar{r}}, 1 - \frac{\lambda+\alpha}{\bar{r}}; \frac{(u\bar{r}+c)\theta}{\bar{r}}\right) \\ & + C_2 \left(\frac{(u\bar{r}+c)\theta}{\bar{r}}\right)^{(\lambda+\alpha)/\bar{r}} e^{-(u\bar{r}+c)\theta/\bar{r}} M\left(1 + \frac{\alpha}{\bar{r}}, 1 + \frac{\lambda+\alpha}{\bar{r}}; \frac{(u\bar{r}+c)\theta}{\bar{r}}\right) \end{aligned} \tag{36}$$

To determine the constants C_1 and C_2, we substitute the boundary condition to get the following two equations

$$C_1 U\left(1 - \frac{\lambda}{\bar{r}}, 1 - \frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) + C_2 \left(\frac{c\theta}{\bar{r}}\right)^{(\lambda+\alpha)/\bar{r}} M\left(1 + \frac{\alpha}{\bar{r}}, 1 + \frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) = e^{-c\theta/\bar{r}} \tag{37}$$

and

$$C_1(cH_{21} + (r - \lambda - \alpha - c\theta)H_{11}) + C_2\left(cH_{22} + (r - \lambda - \alpha - c\theta)H_{12}\right) = \alpha\theta \tag{38}$$

where

$$\begin{aligned} H_{11} = {} & -\theta e^{-c\theta/\bar{r}} U\left(1 - \frac{\lambda}{\bar{r}}, 1 - \frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) \\ & - \theta\left(1 - \frac{\lambda}{\bar{r}}\right) e^{-c\theta/\bar{r}} U\left(2 - \frac{\lambda}{\bar{r}}, 2 - \frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right), \end{aligned} \tag{39}$$

$$
\begin{aligned}
H_{12} = \theta\left(\frac{\lambda+\alpha}{r}\right)\left(\frac{c\theta}{\bar{r}}\right)^{(\lambda+\alpha)/r-1} & e^{-c\theta/\bar{r}} M\left(1+\frac{\alpha}{\bar{r}}, 1+\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) \\
-\theta\left(\frac{c\theta}{\bar{r}}\right)^{(\lambda+\alpha)/r} & e^{-c\theta/\bar{r}} M\left(1+\frac{\alpha}{\bar{r}}, 1+\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) \\
+\theta\left(\frac{c\theta}{r}\right)^{(\lambda+\alpha)/\bar{r}} & \left(\frac{\lambda+\alpha+r}{\alpha+r}\right) e^{-c\theta/\bar{r}} M\left(2+\frac{\alpha}{\bar{r}}, 2+\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right)
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
H_{21} = \theta^2 e^{-c\theta/\bar{r}} & U\left(1-\frac{\lambda}{\bar{r}}, 1-\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) \\
& + 2\theta^2\left(1-\frac{\lambda}{\bar{r}}\right) e^{-c\theta/\bar{r}} U\left(2-\frac{\lambda}{\bar{r}}, 2-\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) \\
& - \theta^2\left(1-\frac{\lambda}{\bar{r}}\right)\left(2-\frac{\lambda}{\bar{r}}\right) e^{-c\theta/\bar{r}} U\left(3-\frac{\lambda}{\bar{r}}, 3-\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right)
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
H_{22} = {} & \theta^2\left(\frac{\lambda+\alpha}{r}\right)\left(\frac{\lambda+\alpha}{r}-1\right)\left(\frac{c\theta}{\bar{r}}\right)^{(\lambda+\alpha)/r-2} e^{-c\theta/\bar{r}} M\left(1+\frac{\alpha}{\bar{r}}, 1+\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) \\
& - 2\theta^2\left(\frac{\lambda+\alpha}{r}\right)\left(\frac{c\theta}{\bar{r}}\right)^{(\lambda+\alpha)/r-1} e^{-c\theta/\bar{r}} M\left(1+\frac{\alpha}{\bar{r}}, 1+\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) \\
& + 2\theta^2\left(\frac{\lambda+\alpha}{r}\right)\left(\frac{c\theta}{\bar{r}}\right)^{(\lambda+\alpha)/r-1} \left(\frac{\lambda+\alpha+r}{\alpha+r}\right) e^{-c\theta/\bar{r}} M\left(2+\frac{\alpha}{\bar{r}}, 2+\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) \\
& + \theta^2\left(\frac{c\theta}{\bar{r}}\right)^{(\lambda+\alpha)/r} e^{-c\theta/\bar{r}} M\left(1+\frac{\alpha}{\bar{r}}, 1+\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) \\
& - 2\theta^2\left(\frac{c\theta}{r}\right)^{(\lambda+\alpha)/\bar{r}} \left(\frac{\lambda+\alpha+r}{\alpha+r}\right) e^{-c\theta/\bar{r}} M\left(2+\frac{\alpha}{\bar{r}}, 2+\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) \\
& + \theta^2\left(\frac{c\theta}{r}\right)^{(\lambda+\alpha)/\bar{r}} \left(\frac{\lambda+\alpha+r}{\alpha+r}\right)\left(\frac{\lambda+\alpha+2r}{\alpha+2r}\right) e^{-c\theta/\bar{r}} M\left(3+\frac{\alpha}{\bar{r}}, 3+\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right)
\end{aligned} \tag{42}
$$

Equations (37) and (38) are then solve simultaneously to give

$$
\begin{aligned}
C_2 = {} & \frac{\alpha\theta U\left(1-\frac{\lambda}{\bar{r}}, 1-\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right) - \left(cH_{21} + (r-\lambda-\alpha-c\theta)H_{11}\right)e^{-c\theta/\bar{r}}}{K} \\
C_1 = {} & \frac{(cH_{22} + (r-\lambda-\alpha-c\theta)H_{12})e^{-c\theta/\bar{r}} - \alpha\theta \mathrm{M}\left(1+\frac{\alpha}{\bar{r}}, 1+\frac{\lambda+\alpha}{\bar{r}}; \frac{c\theta}{\bar{r}}\right)}{K}
\end{aligned} \tag{43}
$$

where

$$K = \left(cH_{22}+(r-\lambda-\alpha-c\theta)H_{12}\right)\mathrm{U}\left(1-\frac{\lambda}{\bar{r}},1-\frac{\lambda+\alpha}{\bar{r}};\frac{c\theta}{\bar{r}}\right) - \left(cH_{21}+(r-\lambda-\alpha-c\theta)H_{11}\right)\mathrm{M}\left(1+\frac{\alpha}{\bar{r}},1+\frac{\lambda+\alpha}{\bar{r}};\frac{c\theta}{\bar{r}}\right)$$

, H_{11}, H_{12}, H_{21} and H_{22} are as defined above.

Case 2 *Consider a case when* $\sigma_p = \alpha = 0$. *The equation* (25) *becomes*

$$\frac{1}{2}\sigma_p^2\psi_s'''(u) + \left(\frac{\theta}{2}\sigma_p^2 + c + \bar{r}u\right)\psi_s''(u) + \left(\bar{r} + c\theta + \bar{r}\theta u\right)\psi_s'(u) = 0 \tag{44}$$

with the boundary conditions

$$\begin{aligned} \psi_s(u) &= 1 \quad \text{on } u < 0 \\ \psi_s(+\infty) &= 0 \\ c\psi_s''(0) + (\bar{r}-\alpha)\psi_s'(0) &= 0 \end{aligned}$$

This equation can be reduced an easily solvable form after some transformation. Firstly, we reduce the third order differential equation (44) to second order differential equation using the transformation

$$\psi'(u) = v(z)e^{-\theta z} \qquad \text{where} \quad z = u + c/r - \theta\sigma^2/2r \tag{45}$$

The resulting equation is given below

$$\frac{1}{2}\sigma_p^2 v''(u) + rzv'(u) + rv(u) = 0 \tag{46}$$

Then applying another transformation,

$$v(z) = h(x) \qquad \text{where} \quad x = -rz^2/\sigma^2 \tag{47}$$

results in the confluence hypergeometric equation below

$$xh''(x) + \left(\frac{1}{2} - x\right)h'(x) - \frac{1}{2}h(x) = 0 \tag{48}$$

Using Slater (2011), the following solutions are obtained

$$\begin{aligned} h_0(x) &= \sqrt{x}\, e^x\, M\left(\frac{1}{2},\frac{3}{2};-x\right) \\ h_1(x) &= e^x\, U\left(0,\frac{1}{2};-x\right) \end{aligned} \tag{49}$$

Clearly, if $h_0(x)$ is a solution of equation (48), so also is $\sqrt{-1}\,h_0(x)$, thus

$$h_2(x) = \sqrt{-x}\, e^x\, M\left(\frac{1}{2},\frac{3}{2};-x\right) \tag{50}$$

is a solution of 48. Therefore the general solution is given by

$$h(x) = C_1 \sqrt{-x}\, e^x\, M\left(\frac{1}{2}, \frac{3}{2}; -x\right) + C_2\, e^x\, U\left(0, \frac{1}{2}; -x\right) \tag{51}$$

where C_1 and C_2 are arbitrary constants. But,

$$v(z) = h(x) = C_1 \frac{\sqrt{r}z}{\sigma}\, e^{-rz^2/\sigma^2}\, M\left(\frac{1}{2}, \frac{3}{2}; \frac{rz^2}{\sigma^2}\right) + C_2\, e^{-rz^2/\sigma^2}\, U\left(0, \frac{1}{2}; \frac{rz^2}{\sigma^2}\right) \tag{52}$$

where z is related to x as given (47), But

$$\begin{aligned}\psi'(u) = v(z)e^{-\theta z} = C_1 \frac{\sqrt{r}(u+\bar{a})}{\sigma}\, e^{-\theta\bar{a}} e^{-(\theta u + r(u+\bar{a})^2/\sigma^2)}\, M\left(\frac{1}{2}, \frac{3}{2}; \frac{r(u+\bar{a})^2}{\sigma^2}\right) \\ + C_2\, e^{-\theta\bar{a}} e^{-(\theta u + r(u+\bar{a})^2/\sigma^2)}\, U\left(0, \frac{1}{2}; \frac{r(u+\bar{a})^2}{\sigma^2}\right)\end{aligned} \tag{53}$$

where $\bar{a} = c/r - \theta\sigma^2/2r$.
Thus,

$$\psi'(u) = C_1^*\, H_1' + C_2^*\, H_2' \tag{54}$$

where

$$\begin{aligned} C_1^* &= C_1 \frac{\sqrt{r}}{\sigma} e^{-\theta\bar{a}}, \\ C_2^* &= C_2\, e^{-\theta\bar{a}}, \\ H_1' &= (u+\bar{a})\, e^{-(\theta u + r(u+\bar{a})^2/\sigma^2)}\, M\left(\frac{1}{2}, \frac{3}{2}; \frac{r(u+\bar{a})^2}{\sigma^2}\right) \quad \text{and} \\ H_2' &= e^{-(\theta u + r(u+\bar{a})^2/\sigma^2)}\, U\left(0, \frac{1}{2}; \frac{r(u+\bar{a})^2}{\sigma^2}\right) \end{aligned}$$

Integrating both sides of (54), using equation (4.12) in Cai and Yang (1986) on obtains

$$\psi(u) = \int_u^\infty \psi'(x)dx = C_1^*\, H_1 + C_2^*\, H_2 \tag{55}$$

where

$$\begin{aligned} H_1(u) &= \int_u^\infty (x+\bar{a})\, e^{-(\theta x + r(x+\bar{a})^2/\sigma^2)}\, M\left(\frac{1}{2}, \frac{3}{2}; \frac{r(x+\bar{a})^2}{\sigma^2}\right) dx \quad \text{and} \\ H_2(u) &= \int_u^\infty e^{-(\theta x + r(x+\bar{a})^2/\sigma^2)}\, U\left(0, \frac{1}{2}; \frac{r(x+\bar{a})^2}{\sigma^2}\right) dx \end{aligned}$$

To determine C_1^* and C_2^* in terms of the parameters we proceed to obtain $\psi(u)''$,

$$\psi''(u) = C_1^*\, H_1'' + C_2^*\, H_2'' \tag{56}$$

where

$$
\begin{aligned}
H_1'' &= \bar{a}\, e^{-(\theta x + r(x+\bar{a})^2/\sigma^2)}\, M\left(\frac{1}{2}, \frac{3}{2}; \frac{r(x+\bar{a})^2}{\sigma^2}\right) \\
&\quad - \frac{(\theta + 2r(x+\bar{a}))(x+\bar{a})}{\sigma^2}\, e^{-(\theta x + r(x+\bar{a})^2/\sigma^2)}\, M\left(\frac{1}{2}, \frac{3}{2}; \frac{r(x+\bar{a})^2}{\sigma^2}\right) \\
&\quad + \frac{2r(x+\bar{a})^2}{3\sigma^2}\, e^{-(\theta x + r(x+\bar{a})^2/\sigma^2)}\, M\left(\frac{3}{2}, \frac{5}{2}; \frac{r(x+\bar{a})^2}{\sigma^2}\right) \\
H_2'' &= -\frac{(\theta + 2r(x+\bar{a}))}{\sigma^2} e^{-(\theta x + r(x+\bar{a})^2/\sigma^2)}\, U\left(0, \frac{1}{2}; \frac{r(x+\bar{a})^2}{\sigma^2}\right)
\end{aligned}
$$

Using the boundary conditions as shown in equation (44) together with equations (54), (55) and (56), one gets

$$
\begin{aligned}
C_1^*\, H_1(0) + C_2^*\, H_2(0) &= 1 \\
C_1^* \left(cH_1''(0) + (\bar{r} - \alpha) H_1'(0)\right) + C_2^* \left(cH_2''(0) + (\bar{r} - \alpha) H_2'(0)\right) &= 0
\end{aligned} \tag{57}
$$

where

$$
\begin{aligned}
H_1'(0) &= \bar{a}\, e^{-r\bar{a}^2/\sigma^2}\, M\left(\frac{1}{2}, \frac{3}{2}; \frac{r\bar{a}^2}{\sigma^2}\right) \\
H_2'(0) &= e^{-r\bar{a}^2/\sigma^2}\, U\left(0, \frac{1}{2}; \frac{r\bar{a}^2}{\sigma^2}\right) \\
H_1''(0) &= \bar{a}\, e^{-r\bar{a}^2/\sigma^2}\, M\left(\frac{1}{2}, \frac{3}{2}; \frac{r\bar{a}^2}{\sigma^2}\right) - \frac{2r\bar{a}^2}{\sigma^2}\, e^{-r\bar{a}^2/\sigma^2}\, M\left(\frac{1}{2}, \frac{3}{2}; \frac{r\bar{a}^2}{\sigma^2}\right) \\
&\quad + \frac{2r\bar{a}^2}{3\sigma^2}\, e^{-r\bar{a}^2/\sigma^2}\, M\left(\frac{3}{2}, \frac{5}{2}; \frac{r\bar{a}^2}{\sigma^2}\right) \\
H_2''(0) &= -\frac{\theta + 2r\bar{a}}{\sigma^2} e^{-r\bar{a}^2/\sigma^2}\, U\left(0, \frac{1}{2}; \frac{r\bar{a}^2}{\sigma^2}\right)
\end{aligned}
$$

Solving equation (57) simultaneously gives

$$
\begin{aligned}
C_1^* &= -\frac{\left(cH_2''(0) + (\bar{r} - \alpha) H_2'(0)\right)}{H_2(0)\left(cH_1''(0) + (\bar{r} - \alpha) H_1'(0)\right) - H_1(0)\left(cH_2''(0) + (\bar{r} - \alpha) H_2'(0)\right)} \\
C_2^* &= \frac{\left(cH_1''(0) + (\bar{r} - \alpha) H_1'(0)\right)}{H_2(0)\left(cH_1''(0) + (\bar{r} - \alpha) H_1'(0)\right) - H_1(0)\left(cH_2''(0) + (\bar{r} - \alpha) H_2'(0)\right)}
\end{aligned} \tag{58}
$$

and

$$ \tag{59} $$

Thus the ruin probability due to oscillation for this model is given as

$$\psi(u) = \frac{H_2(u)\Big(cH_1''(0) + (\bar{r} - \alpha)H_1'(0)\Big) - H_1(u)\Big(cH_2''(0) + (\bar{r} - \alpha)H_2'(0)\Big)}{H_2(0)\Big(cH_1''(0) + (\bar{r} - \alpha)H_1'(0)\Big) - H_1(0)\Big(cH_2''(0) + (\bar{r} - \alpha)H_2'(0)\Big)} \tag{60}$$

where $H_1(u), H_2(u), H_1'(0), H_2'(0), H_1''(0)$ and $H_2''(0)$ are as defined in equations (55) and (57)

The two cases examined above could represent situations where ruin is caused by claim and when ruin is caused by oscillation respectively. For these cases, the claims arrival rate is assumed to be a constant λ. However, it is possible to have arrival rate dependent on time which will lead to Cox processes. The simplest example of such process is the mixed Poisson process where $\lambda(t) = V$ and V is a random process. Such situations are not considered in the present study. It is our intention to consider such process in future studies. Below are numerical examples to show the behaviors of solutions derived from the two cases considered. In each of the examples, a parameter is varied while others are kept constant. The values of the constant parameters are assumed to be $c = 1.1$, $\lambda = 1$, $r = 0.12$, $\theta = 1$, $\varphi = 0.6$, $\sigma_p = \sigma_R = 0.2$, $a = 0.21$ and $\alpha = 0$

Example 1. Suppose φ is varied while other parameters are kept constant. The table 1 below shows the ruin probabilities due to claim for various values of φ. Figure 1 shows the behaviour of these probabilities as the capital increases u and as φ is being varied. Similarly, table 2 shows the ruin probabilities due to oscillation for various values of φ. The graph of the values in table 2 is shown if figure 2.

From figure 1, it can be deduced that the higher the fraction of the resources that is invested in the competing risky investment, the higher the effect of claims on probability of ruin. Expectedly, as the capital u increases, the effect of claims on the ruin probability reduces. Also, considering figure 2, it can also be deducted that increasing the fraction of investment in risky asset increases the ruin probability due to oscillation. Thus, one arrives at that conclusion that ruin probability reduces as the fraction of investment in risky assets increases, provided all other parameters are kept constant.

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Table 1. Ruin probabilities for various values of φ when $\sigma_p = \sigma_R = 0$

u	$\varphi = 0$	$\varphi = 0.2$	$\varphi = 0.4$	$\varphi = 0.6$	$\varphi = 0.8$	$\varphi = 1$
0	1.09E-08	1.19E-07	7.50E-07	3.23E-06	1.06E-05	2.82E-05
0.5	9.13E-09	9.86E-08	6.14E-07	2.61E-06	8.46E-06	2.24E-05
1	7.54E-09	8.04E-08	4.95E-07	2.08E-06	6.67E-06	1.74E-05
1.5	6.17E-09	6.48E-08	3.94E-07	1.64E-06	5.18E-06	1.34E-05
2	4.99E-09	5.17E-08	3.10E-07	1.27E-06	3.98E-06	1.02E-05
2.5	3.99E-09	4.08E-08	2.41E-07	9.75E-07	3.02E-06	7.63E-06
3	3.17E-09	3.18E-08	1.85E-07	7.41E-07	2.26E-06	5.66E-06
3.5	2.49E-09	2.46E-08	1.41E-07	5.57E-07	1.68E-06	4.16E-06
4	1.94E-09	1.89E-08	1.07E-07	4.16E-07	1.24E-06	3.03E-06
4.5	1.50E-09	1.44E-08	8.01E-08	3.07E-07	9.04E-07	2.18E-06
5	1.16E-09	1.09E-08	5.96E-08	2.26E-07	6.55E-07	1.57E-06
5.5	8.81E-10	8.15E-09	4.40E-08	1.64E-07	4.72E-07	1.11E-06
6	6.68E-10	6.07E-09	3.23E-08	1.19E-07	3.37E-07	7.88E-07
6.5	5.03E-10	4.49E-09	2.35E-08	8.56E-08	2.40E-07	5.54E-07
7	3.76E-10	3.31E-09	1.71E-08	6.12E-08	1.69E-07	3.87E-07
7.5	2.80E-10	2.42E-09	1.23E-08	4.36E-08	1.19E-07	2.69E-07
8	2.07E-10	1.76E-09	8.83E-09	3.08E-08	8.33E-08	1.87E-07
8.5	1.53E-10	1.28E-09	6.30E-09	2.17E-08	5.80E-08	1.29E-07
9	1.12E-10	9.20E-10	4.48E-09	1.52E-08	4.03E-08	8.83E-08
9.5	8.17E-11	6.60E-10	3.17E-09	1.06E-08	2.78E-08	6.04E-08
10	5.93E-11	4.72E-10	2.23E-09	7.40E-09	1.91E-08	4.11E-08

Table 2. Ruin probabilities for various values of φ when $\sigma_R = \alpha = 0$

u	$\varphi = 0$	$\varphi = 0.2$	$\varphi = 0.4$	$\varphi = 0.8$	$\varphi = 1$
0.5	5.98E-12	1.81E-11	5.61E-11	5.63E-10	1.82E-09
1	2.62E-25	7.94E-25	2.46E-24	2.46E-23	7.97E-23
1.5	1.30E-39	3.92E-39	1.22E-38	1.23E-37	3.95E-37
2	7.31E-55	2.22E-54	6.87E-54	6.90E-53	2.23E-52
2.5	4.73E-71	1.42E-70	4.38E-70	4.42E-69	1.43E-68
3	3.40E-88	1.03E-87	3.20E-87	3.23E-86	1.04E-85
3.5	2.80E-106	8.50E-106	2.65E-105	2.66E-104	8.57E-104
4	2.64E-125	8.02E-125	2.48E-124	2.49E-123	8.08E-123
4.5	2.81E-145	8.53E-145	2.65E-144	2.66E-143	8.59E-143
5	3.45E-166	1.03E-165	3.19E-165	3.23E-164	1.04E-163
5.5	4.65E-188	1.42E-187	4.41E-187	4.42E-186	1.44E-185
6	7.33E-211	2.22E-210	6.90E-210	6.97E-209	2.24E-208
6.5	1.31E-234	3.92E-234	1.23E-233	1.23E-232	4.00E-232
7	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
7.5	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
8	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
8.5	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
9	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
9.5	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
10	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00

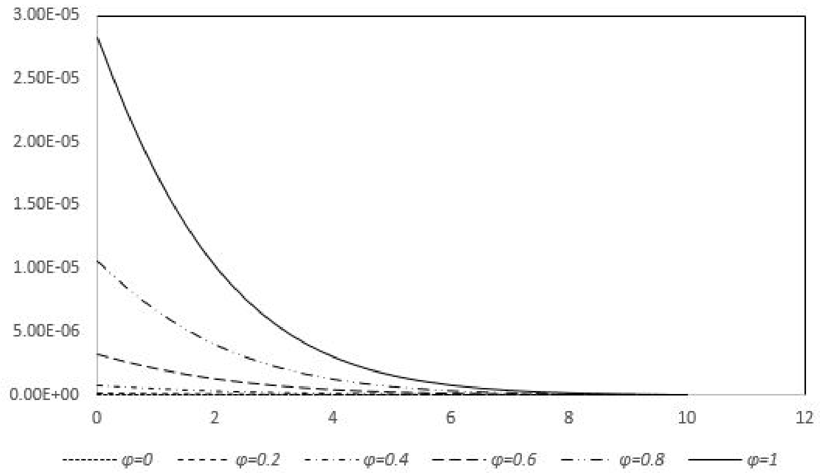

Fig. 1. The behavior of the ruin probability from claims with respect to capital when fraction of investment into risky assets is varied

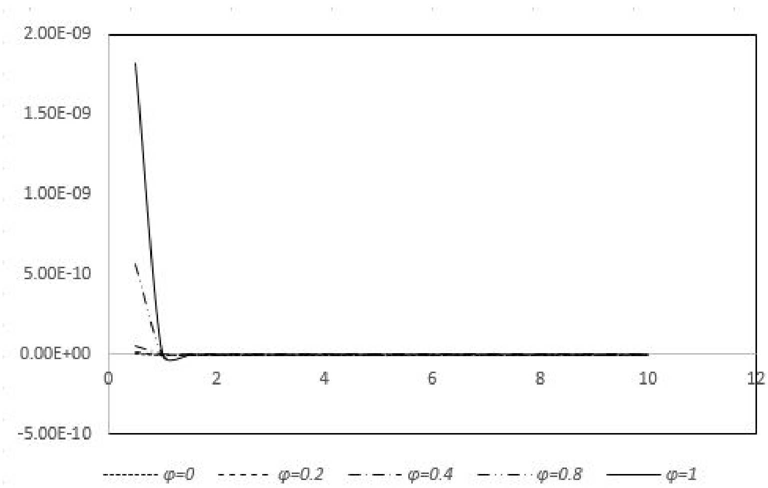

Fig. 2. The behavior of the ruin probability from oscillation with respect to capital when fraction of investment into risky assets is varied

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