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R–transform associated with asymptotic negative spectral moments of Jacobi ensemble

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Abstract. . We derive an explicit formula for the R–transform of inverse Jacobi matrix $I + W_1^{-1}W_2$, where $W_1, W_2 \sim \mathcal{W}_p(I, n_i)$, $i = 1, 2$ are independent and I is $p \times p$ dimensional identity matrix using property of asymptotic freeness of Wishart and deterministic matrices. Procedure can be extended to other sets of the asymptotically free independent matrices. Calculations are illustrated with some simulations on fixed size matrices.

Résumé. Nous dérivons une formule explicite pour une transformée en R de $I + W_1^{-1}W_2$, où $W_1, W_2 \sim \mathcal{W}_p(I, n_i)$, $i = 1, 2$, et I correspondent à une matrice identité de dimension $p \times p$ en utilisant les propriétés de la loi asymptotique de Wishart et des matrices déterministes. La procédure peut être étendue à d'autres ensembles de matrices indépendantes asymptotiquement libres. Les calculs sont illustrés avec des simulations sur les matrices de dimension fixe.

Key words: Jacobi ensemble; R–transform; S–transform; Negative spectral moments; Spectral moments; Wishart matrix; Marčenko-Pastur law; asymptotical freeness;
AMS 2010 Mathematics Subject Classification : 15A52, 60B20, 46L54.

1. Introduction

Jacobi ensembles are commonly used in statistics, particularly they find application for multivariate analysis of variance (MANOVA), see Muirhead (1982). Jacobi matrix J given as a function of independent Wishart matrices, denoted by W_i, are also commonly used in statistics, Johnstone (2008). For example, often relation of Wishart matrix to sample-covariance matrix is of the interest. Moreover, number of test statistics considered in classical

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multivariate analysis are given as a function of one or several Wishart matrices. The results obtained in the article are for asymptotic spectral distribution of inverse Jacobi matrix, where asymptotic is given under proportional increase of matrix size in both of rows and columns, i.e. $p/n \to c$ for $p, n \to \infty$, where p stands for the number of rows and n for number of column of underlying matrix X such that $W = XX'$. The spectral properties of set of all eigenvalues or maximal (minimal) eigenvalues of Jacobi matrix are of interest due to application in hypothesis testing.
In this paper we derive an explicit formula for the R–transform of inverse of Jacobi matrix J^{-1} using property of asymptotic freeness between independent Wishart matrices, as well as between Wishart and deterministic matrices. Used R–transform is related to spectral moments of underlying $p \times p$ matrix J^{-1} so provides expectation of $p^{-1}\operatorname{Tr}(J^{-k})$ for all $k \in \mathbb{N}$, i.e. the asymptotic negative spectral moments of Jacobi matrix.

This paper is organized as follows. In Section 2 we give background and introduction to mathematical tools used in derivation of results. Section 3 and 4 provide calculations of asymptotic negative spectral moments of weighted Wishart matrix as well as of Jacobi matrix itself. Conclusions can be found in Section 5. For the related works of author please see Pielaszkiewicz *et al.* (2017) (on $E\big[\prod_{i=0}^{k} Tr\{W^{m_i}\}\big]$, where $W \sim \mathcal{W}_p(I, n)$), Pielaszkiewicz *et al.* (2014) (on cumulant-moments relation formula in free probability) and Pielaszkiewicz *et al.* (2017) (on test for covariance matrix).

2. Preliminaries

Let X be a square $p \times p$ random matrix, $X \in \mathrm{RM}_p(\mathbb{R})$. We denote by $\{m_i\}_{i=1}^{\infty}$ the series of spectral moments $(m_i(X) = \mathbb{E}[p^{-1}\operatorname{Tr}(X^i)])$ and by $\{k_i\}_{i=1}^{\infty}$ the series of free spectral cumulants that can be defined via non–crossing partitions using the following recursive relation

$$k_1 = m_1, \qquad m_k = \sum_{\pi \in NC(k)} k_\pi.$$

The sum is taken over all non–crossing partitions $NC(k)$ of the set $\{1, 2, \ldots, k\}$ and $k_\pi = \prod_{i=1}^{r} k_{V(i)}$, where $\pi = \{V(1), \ldots, V(r)\}$ and $k_V = k_s$, where $V = (v(1), \ldots, v(s))$. Above definition is simplified and hence does not define mixed free cumulants of matrices. In this paper we consider only asymptotically free matrices that have vanishing mixed cumulants which justify such a simplification. For more general statements and combinatorial approach to topic see Nica *et al.* (2006); Speicher (1994), for original papers introducing notion of Free probability we refer to Voiculescu (1985, 1991).

Furthermore, we define Stieltjes and R-transform as follows.

Definition 1. Let μ be a probability measure on $\mathbb{R}$. Then, the Stieltjes (Cauchy-Stieltjes) transform of μ is given by

$$\mathrm{G}_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x} d\mu(x),$$

for all $z \in \{z : z \in \mathbb{C}, \Im(z) > 0\}$, where $\Im(z)$ denotes the imaginary part of the complex z.

Defined in such a way the Stieltjes transform can be inverted on any interval. It can also be given as a series of free moments $\{m_i\}_{i=1}^{\infty}$.

Theorem 1. *Let m_k be kth spectral moment of the matrix as defined earlier. Then, a formal power series representing the Stieltjes transform is given by*

$$\mathrm{G}_\mu(z) = \frac{1}{z}\left(1 + \sum_{i=1}^{\infty} z^{-i} m_i\right).$$

Stieltjes transform appears in formulations of a number of results published within Random matrix theory, see for example, Marčenko *et al.* (1967), Girko *et al.* (1994), Silverstein *et al.* (1995), Hachem *et al.* (2007). Related and very convenient tools for studying convolution of measure on algebra of square random matrices (more generally on non–commutative $*$–probability spaces) are the R– and S– transforms (see Nica *et al.* (2006); Couillet *et al.* (2011); Haagerup (1997)). The R–transform linearizes additive free convolution and plays the same role as the log of the Fourier transform in classical probability theory, while the S–transform linearizes the multiplicative convolution of free matrices.

Definition 2. Let μ be a probability measure with compact support and $G_\mu(z)$ the related Stieltjes transform. Then,

$$R_\mu(z) = G_\mu^{-1}(z) - \frac{1}{z} \text{ or equivalently } R_\mu(G_\mu(z)) = z - \frac{1}{G_\mu(z)}$$

defines R-transform $R_\mu(z)$ for underlying the measure μ. If μ denotes the measure associated with the matrix X we equivalently use the notation $R_X(z) := R_\mu(z)$.

R-transform can be expanded to a series of free cumulants as $R(z) = \sum_{i=0}^{\infty} k_{i+1} z^i$. In a similar manner we define series

$$M(z) = 1 + \sum_{i=1}^{\infty} m_i z^i,$$

which is related to series $R(z)$ by

$$1 + zM(z)R(zM(z)) = M(z), \tag{1}$$

see Nica *et al.* (2006); Couillet *et al.* (2011). Moreover, for the R–transform we have mentioned property of linearization of additive convolution

$$R_{A+B}(z) = R_A(z) + R_B(z), \tag{2}$$

for free matrices A and B. Similarly, S–transform, defined by

$$S(z) = \frac{1}{R(zS(z))}, \tag{3}$$

of product of free matrices A and B can be decomposed product of S–transforms by

$$S_{AB}(z) = S_A(z)S_B(z). \tag{4}$$

3. R– and S–transform associated with the inverse of a weighted Wishart matrix

Let us consider a weighted Wishart $p \times p$ matrix $\tilde{W} = \frac{1}{n}W$, where $W \sim \mathcal{W}_p(I, n)$ and I is $p \times p$ identity matrix. Then, the asymptotic spectral distribution of $\tilde{W}$ is known as Marčenko-Pastur law, see reference to early result Marčenko *et al.* (1967) and illustration in Fig. 1. It implies R–transform in form $R(z) = \frac{1}{1-cz}$, where c stands for the limit in Kolmogorov condition, i.e., $p/n \to c$ for $n, p \to \infty$. By equation (1) we obtain $M(z) = \frac{1+zc-z-\sqrt{(z-zc-1)^2-4zc}}{2zc}$ which expanded to the series gives

$$
\begin{aligned}
M(z) &= 1 + z + (c+1)z^2 + \left(c^2 + 3c + 1\right) z^3 + \left(c^3 + 6c^2 + 6c + 1\right) z^4 \\
&+ \left(c^4 + 10c^3 + 20c^2 + 10c + 1\right) z^5 + \left(c^5 + 15c^4 + 50c^3 + 50c^2 + 15c + 1\right) z^6 \\
&+ \left(c^6 + 21c^5 + 105c^4 + 175c^3 + 105c^2 + 21c + 1\right) z^7 + O\left(z^8\right) \\
&\overset{c=0.5}{=} 1. + z + 1.5z^2 + 2.75z^3 + 5.625z^4 + 12.3125z^5 + 28.2188z^6 + 66.8594z^7 + O\left(z^8\right)
\end{aligned}
$$

identifying moments of M-P law.

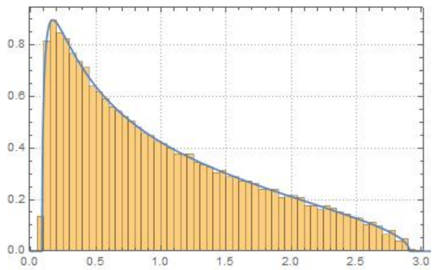


Fig. 1. Asymptotic spectral distribution of Marčenko-Pastur law, c=0.5, n=5000, p=2500.

The negative and positive asymptotic moments for the Marčenko-Pastur law are related by $m_{-k} = m_{k-1}(1-c)^{1-2k}$, see Serdobolskii (2000). We use the relation between m_{-k} and m_{k-1} to obtain a relation between the moment generating functions of $\tilde{W}$ and $\tilde{W}^{-1}$

$$
\begin{aligned}
M_{-}(z) &= 1 + \sum_{i=1}^{\infty} m_{-i} z^i = 1 + \sum_{i=1}^{\infty} m_{i-1}(1-c)^{1-2i} z^i = 1 + (1-c) \sum_{i=1}^{\infty} m_{i-1} \left(\frac{z}{(1-c)^2} \right)^i \\
&= 1 + \frac{z}{1-c} \sum_{i=1}^{\infty} m_{i-1} \left(\frac{z}{(1-c)^2} \right)^{i-1} = 1 + \frac{z}{1-c} M\left(\frac{z}{(1-c)^2} \right).
\end{aligned}
$$

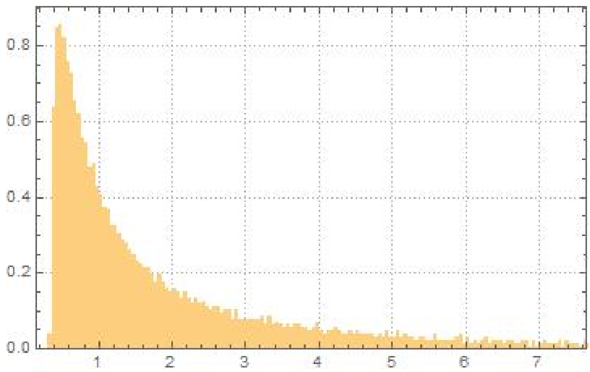
Result, after some technical manipulations, is equivalent to

$$
M(z) = \frac{1}{z(1-c)} (M_{-}(z(1-c)^2) - 1).
$$

Explicit moment generating function for the inverse of a weighted Wishart matrix M_{-} (in the other words negative integer moment generating function for weighted Wishart matrix) is given by

$$\begin{aligned}
M_-(z) &= 1 + \frac{z}{1-c} M\left(\frac{z}{(1-c)^2}\right) \\
&= 1 + \frac{z}{1-c} \frac{1 + \frac{z}{(1-c)^2}c - \frac{z}{(1-c)^2} - \sqrt{(\frac{z}{(1-c)^2} - \frac{z}{(1-c)^2}c - 1)^2 - 4\frac{z}{(1-c)^2}c}}{2\frac{z}{(1-c)^2}c} \\
&= 1 + \frac{(1-c) + \frac{z}{1-c}c - \frac{z}{1-c} - \sqrt{(z-(1-c))^2 - 4zc}}{2c} \\
&= \frac{1 + c - z - \sqrt{(z-(1-c))^2 - 4zc}}{2c} \\
&= \frac{c - \sqrt{(c-1)^2} + 1}{2c} + \frac{\left(\frac{c+1}{\sqrt{(c-1)^2}} - 1\right) z}{2c} + \frac{\sqrt{(c-1)^2} z^2}{(c-1)^4} + \frac{(c+1)z^3}{(c-1)^4 \sqrt{(c-1)^2}} \\
&+ \frac{\sqrt{(c-1)^2}\left(c^2 + 3c + 1\right) z^4}{(c-1)^8} + \frac{\sqrt{(c-1)^2}(c+1)\left(c^2 + 5c + 1\right) z^5}{(c-1)^{10}} + O\left(z^6\right) \\
&\overset{c=0.5}{=} 1. + 2.z + 8.z^2 + 48.z^3 + 352.z^4 + 2880.z^5 + 25216.z^6 + 231168.z^7 + O\left(z^8\right)
\end{aligned}$$

Fig. 2 illustrates the result given above. The histogram generated by inverse of eigenvalues of $\tilde{W} = \frac{1}{n}XX'$, where $X \sim \mathcal{N}_{p,n}(0, I, I)$ is given and the negative spectral moments of simulated matrix are presented in the table. The values for negative spectral moments of $p \times p$ matrix $\tilde{W}$ are relatively close to asymptotic results derived above i.e., $2, 8, 48, 352, \ldots$. The parameter values are fixed to $c = 0.5$, $n = 5000$ and $p = 2500$.



1st Moment (Mean)	2.00117
2st Moment	8.01644
3st Moment	48.1976
4st Moment	354.321
5st Moment	2906.94
6st Moment	25525.6
7st Moment	234701

Fig. 2. Histogram and empirical moments for the spectral distribution of inverse of a weighted $p \times p$ Wishart matrix $\tilde{W}$. Parameters: c=0.5, n=5000, p=2500.

Following equation (1) we can obtain the R–transform for $\tilde{W}^{-1}$ denoted R_-.

$$1 + zM_-(z)R_-(zM_-(z)) = M_-(z) = \frac{1 + c - z - \sqrt{(z-(1-c))^2 - 4zc}}{2c}$$

We use variable substitution $t = z\frac{1+c-z-\sqrt{(z-(1-c))^2-4zc}}{2c}$ that gives $z = \frac{ct+t\pm t\sqrt{c^2-4ct-2c+1}}{2(t+1)}$. Then,

$$\begin{aligned} 1 + tR_-(t) &= t\left(\frac{ct+t\pm t\sqrt{c^2-4ct-2c+1}}{2(t+1)}\right)^{-1} \\ R_-(t) &= \left(\frac{ct+t\pm t\sqrt{c^2-4ct-2c+1}}{2(t+1)}\right)^{-1} - \frac{1}{t} \\ &\overset{c=0.5}{=} 2. + 4.t + 16.t^2 + 80.t^3 + 448.t^4 + 2688.t^5 + 16896.t^6 \\ &\quad +109824.t^7 + 732160.t^8 + O\left(t^9\right). \end{aligned} \tag{5}$$

We execute relation (3) to obtain the S–transform of $\tilde{W}$ and $\tilde{W}^{-1}$ in form

$$S(t) = \frac{1}{ct+1} \qquad \text{and} \qquad S_-(t) = 1 - c - ct. \tag{6}$$

We have finally obtained the R– and S–transforms associated with $\tilde{W}$ and its inverse. In further part of the article the notation $R_{-,\tilde{W}}$ and $S_{-,\tilde{W}}$ will be used to indicate the underlying matrix.

4. R–transform related to asymptotic negative integer spectral moments of Jacobi ensemble

Let us consider inverse, J^{-1}, of Jacobi matrix

$$J = (W_1 + W_2)^{-1}W_1,$$

where W_1 and W_2 are independent, $p \times p$, central Wishart matrices with covariance matrix I, i.e., $W_1 \sim \mathcal{W}_p(I, n_1)$ and $W_2 \sim \mathcal{W}_p(I, n_2)$. Then, the considered matrix can be written in the form $J^{-1} = W_1^{-1}(W_1 + W_2) = I + W_1^{-1}W_2 = I + \frac{n_2}{n_1}\tilde{W}_1^{-1}\tilde{W}_2$. By Mingo *et al.* (1982) W_1 and W_2 are asymptotically free independent under condition $p/n_1 \to c_1$, $p/n_2 \to c_2$, $p, n_1, n_2 \to \infty$. Due to convergence $\frac{n_2}{n_1} \to \frac{c_1}{c_2}$ asymptotic spectral distribution of J^{-1} is identical to limiting results regarding matrix $I + \frac{c_1}{c_2}\tilde{W}_1^{-1}\tilde{W}_2$. R–transform of $I + \frac{c_1}{c_2}\tilde{W}_1^{-1}\tilde{W}_2$ is to be calculated.
As the random matrix $\tilde{W}_1^{-1}\tilde{W}_2$ and deterministic matrix I are asymptotically free (asymptotically free independent, see Couillet *et al.* (2011)) equation (2) provides relation in R–transform

$$R_{-,J}(z) = R_{J^{-1}}(z) = R_{I+\frac{c_1}{c_2}\tilde{W}_1^{-1}\tilde{W}_2}(z) = R_I(z) + \frac{c_1}{c_2}R_{\tilde{W}_1^{-1}\tilde{W}_2}\left(\frac{c_1}{c_2}z\right),$$

where $R_{\tilde{W}_1^{-1}\tilde{W}_2}(\frac{c_1}{c_2}z)$ is obtained using the (4) and (3).

The deterministic matrix I leads to

$$M_I(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

Hence, $R_I(z) = 1$. We use Eq. (4) and (6) to obtain S–transform for matrix $\tilde{W}_1^{-1}\tilde{W}_2$

$$S_{\tilde{W}_1^{-1}\tilde{W}_2}(z) = S_{\tilde{W}_1^{-1}}S_{\tilde{W}_2} = \frac{1-c_1-c_1t}{c_2t+1}$$

and implied R–transform

$$R_{\tilde{W}_1^{-1}\tilde{W}_2}(z) = \frac{1-c_1-c_2z \pm \sqrt{(c_1+c_2z-1)^2-4c_1z}}{2c_1z}.$$

Hence,

$$\begin{aligned} R_{-,J}(z) &= 1+\frac{c_1}{c_2}\frac{1-c_1-c_2\frac{c_1}{c_2}z \pm \sqrt{(c_1+c_2\frac{c_1}{c_2}z-1)^2-4c_1\frac{c_1}{c_2}z}}{2c_1\frac{c_1}{c_2}z} \\ &= \frac{1-c_1+c_1z \pm \sqrt{(c_1+c_1z-1)^2-4c_1^2z/c_2}}{2c_1z} \\ &\overset{c_1=c_2=0.5}{=} 3.+6.z+30.z^2+186.z^3+1290.z^4+9582.z^5+O\left(z^6\right). \end{aligned}$$

Given above closed form of R–transform leads to series identifying negative spectral moments of Jacobi matrix J

$$\begin{aligned} M_{-,J}(z) &= \frac{\pm\sqrt{\left(\frac{1-z}{c_1}-\frac{z}{c_2}+1\right)^2+\frac{4(z-1)}{c_1}}+\frac{z}{c_1}-\frac{1}{c_1}+\frac{z}{c_2}-1}{2(z-1)} \\ &\overset{c_1=c_2=0.5}{=} 1.+3.z+15.z^2+111.z^3+1023.z^4+10623.z^5+O\left(z^6\right). \end{aligned}$$

5. Conclusions

The asymptotic spectral moments and free cumulants of inverse of Jacobi matrix has been derived using its expansion to sum and product of asymptotically free matrices. Knowledge of all asymptotic moments for the compactly supported distribution provides full information about the underling asymptotic distribution. Presented method of analysis of negative spectral moments can be applied to any other combination of the asympotically free matrices. It is also good example of R– and S–transform calculations. Moreover, the fulfillment of the given equality can be treated as a necessary (but not sufficient) condition of asymptotic freeness of the incorporated to Jacobi matrix Wishart and deterministic matrices.

Similar methodology can be applied to the resolvent-type estimators, defined by $p^{-1}\operatorname{Tr}(n^{-1}W+tI)^{-1}$, $t \in \mathbb{R}^+$, where I is $p\times p$ identity matrix can be considered. They have shown great potential in estimating $p^{-1}\operatorname{Tr}(\Sigma+tI)^{-1}$, $t \in \mathbb{R}^+$, a quantity which plays an important role in theoretical development of spectral analysis (Girko (1990); Serdobolskii (1985)), estimation of the precision matrix (Serdobolskii (2008)) and also in more applied settings such as outlier detection (Holgersson *et al.* (2012)).

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