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A Lynden-Bell integral estimator for the tail index of right-truncated data with a random threshold

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Abstract. By means of a Lynden-Bell integral with deterministic threshold, Worms and Worms [A Lynden-Bell integral estimator for extremes of randomly truncated data. Statist. Probab. Lett. 2016; 109: 106-117] recently introduced an asymptotically normal estimator of the tail index for randomly right-truncated Pareto-type data. In this context, we consider the random threshold case to derive a Hill-type estimator and establish its consistency and asymptotic normality. A simulation study is carried out to evaluate the finite sample behavior of the proposed estimator.

Résumé. Par l'intégrale de Lynden-Bell avec un seuil déterministe, Worms et Worms [A Lynden-Bell integral estimator for extremes of randomly truncated data. Statist. Probab. Lett. 2016; 109: 106-117] a récemment introduit un estimateur asymptotiquement normal de l'indice de queue pour les données de type Pareto tronquées à droit. Dans ce contexte, nous considérons le cas du seuil aléatoire pour obtenir un estimateur de type Hill et établir sa consistance et sa normalité asymptotique. Une étude de simulation est réalisée pour évaluer le comportement de l'estimateur proposé.

Key words: Extreme value index; Heavy-tails; Lynden-Bell estimator; Random truncation.
AMS 2010 Mathematics Subject Classification: 60F17; 62G30; 62G32; 62P05.

1. Introduction

Let $(\mathbf{X}_i, \mathbf{Y}_i)$, $1 \leq i \leq N$ be a sample of size $N \geq 1$ from a couple $(\mathbf{X}, \mathbf{Y})$ of independent random variables (rv's) defined over some probability space $(\Omega, \mathcal{A}, \mathbf{P})$, with continuous

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marginal distribution functions (df's) $\mathbf{F}$ and $\mathbf{G}$ respectively. Suppose that $\mathbf{X}$ is truncated to the right by $\mathbf{Y}$, in the sense that $\mathbf{X}_i$ is only observed when $\mathbf{X}_i \leq \mathbf{Y}_i$. We assume that both survival functions $\overline{\mathbf{F}} := 1 - \mathbf{F}$ and $\overline{\mathbf{G}} := 1 - \mathbf{G}$ are regularly varying at infinity with respective negative indices $-1/\gamma_1$ and $-1/\gamma_2$. That is, for any $x > 0$,

$$\lim_{z\to\infty} \frac{\overline{\mathbf{F}}(xz)}{\overline{\mathbf{F}}(z)} = x^{-1/\gamma_1} \text{ and } \lim_{z\to\infty} \frac{\overline{\mathbf{G}}(xz)}{\overline{\mathbf{G}}(z)} = x^{-1/\gamma_2}. \tag{1}$$

It is well known that, in extreme value analysis, weak approximations are achieved in the second-order framework (see, e.g., de Haan and Ferreira, 2006, page 48). Thus, it seems quite natural to suppose that $\mathbf{F}$ and $\mathbf{G}$ satisfy the second-order condition of regular variation, which we express in terms of the tail quantile functions pertaining to both df's. That is, we assume that for $x > 0$, we have

$$\lim_{t\to\infty} \frac{\mathbb{U}_{\mathbf{F}}(tx)/\mathbb{U}_{\mathbf{F}}(t) - x^{\gamma_1}}{\mathbf{A}_{\mathbf{F}}(t)} = x^{\gamma_1} \frac{x^{\tau_1} - 1}{\tau_1}, \tag{2}$$

and

$$\lim_{t\to\infty} \frac{\mathbb{U}_{\mathbf{G}}(tx)/\mathbb{U}_{\mathbf{G}}(t) - x^{\gamma_2}}{\mathbf{A}_{\mathbf{G}}(t)} = x^{\gamma_2} \frac{x^{\tau_2} - 1}{\tau_2}, \tag{3}$$

where $\tau_1, \tau_2 < 0$ are the second-order parameters and $\mathbf{A}_{\mathbf{F}}$, $\mathbf{A}_{\mathbf{G}}$ are functions tending to zero and not changing signs near infinity with regularly varying absolute values at infinity with indices τ_1, τ_2 respectively. For any df K, the function $\mathbb{U}_K(t) := K^{\leftarrow}(1 - 1/t)$, $t > 1$, stands for the tail quantile function, with $K^{\leftarrow}(u) := \inf\{v : K(v) \geq u\}$, $0 < u < 1$, denoting the generalized inverse of K. From Lemma 3 in Hua and Joe (2011), the second-order conditions (2) and (3) imply that there exist constants $d_1, d_2 > 0$, such that

$$\overline{\mathbf{F}}(x) = d_1 x^{-1/\gamma_1} \ell_1(x) \text{ and } \overline{\mathbf{G}}(x) = d_2 x^{-1/\gamma_2} \ell_2(x), \ x > 0, \tag{4}$$

where $\lim_{x\to\infty} \ell_i(x) = 1$ and $|1 - \ell_i|$ is regularly varying at infinity with tail index $\tau_i\gamma$, $i = 1, 2$. This condition is fulfilled by many commonly used models such as Burr, Fréchet, Generalized Pareto, absolute Student, log-gamma distributions, to name but a few. Also known as heavy-tailed, Pareto-type or Pareto-like distributions, these models take a prominent role in extreme value theory and have important practical applications as they are used rather systematically in certain branches of non-life insurance, as well as in finance, telecommunications, hydrology, etc... (see, e.g., Resnick, 2006).

Let us now denote (X_i, Y_i), $i = 1, ..., n$ to be the observed data, as copies of a couple of rv's (X, Y), corresponding to the truncated sample $(\mathbf{X}_i, \mathbf{Y}_i)$, $i = 1, ..., N$, where $n = n_N$ is a sequence of discrete rv's which, in virtue of the weak law of large numbers, satisfies $n_N/N \overset{\mathbf{P}}{\to} p := \mathbf{P}(\mathbf{X} \leq \mathbf{Y})$, as $N \to \infty$. We denote the joint df of X and Y by $H(x, y) := \mathbf{P}(X \leq x, Y \leq y) = \mathbf{P}(\mathbf{X} \leq \min(x, \mathbf{Y}), \mathbf{Y} \leq y \mid \mathbf{X} \leq \mathbf{Y})$, which is equal to $p^{-1} \int_0^y \mathbf{F}(\min(x, z))\, d\mathbf{G}(z)$. The marginal distributions of the rv's X and Y, respectively denoted by F and G, are given by $F(x) = p^{-1} \int_0^x \overline{\mathbf{G}}(z)\, d\mathbf{F}(z)$ and $G(y) = p^{-1} \int_0^y \mathbf{F}(z)\, d\mathbf{G}(z)$. Since $\mathbf{F}$ and $\mathbf{G}$ are heavy-tailed, then their right endpoints are infinite and thus they are equal. Hence, from Woodroofe (1985), we may write $\int_x^\infty d\mathbf{F}(y)/\mathbf{F}(y) = \int_x^\infty dF(y)/C(y)$, where $C(z) := \mathbf{P}(X \leq z \leq Y)$. Differentiating the previous equation leads to the following crucial equation $C(x)\,d\mathbf{F}(x) = \mathbf{F}(x)\,dF(x)$, whose solution is defined by $\mathbf{F}(x) = \exp\left\{-\int_x^\infty dF(z)/C(z)\right\}$. This leads to Woodroofe's nonparametric estimator (Woodroofe, 1985) of df $\mathbf{F}$, given by

$$\mathbf{F}_n^{(\mathbf{W})}(x) := \prod_{i:X_i>x} \exp\left\{-\frac{1}{nC_n(X_i)}\right\},$$

which is derived only by replacing df's F and C by their respective empirical counterparts $F_n(x) := n^{-1}\sum_{i=1}^n \mathbf{1}(X_i \leq x)$ and $C_n(x) := n^{-1}\sum_{i=1}^n \mathbf{1}(X_i \leq x \leq Y_i)$. There exists a more popular estimator for $\mathbf{F}$, known as Lynden-Bell nonparametric maximum likelihood estimator (Lynden-Bell, 1971), defined by

$$\mathbf{F}_n^{(\mathbf{LB})}(x) := \prod_{i:X_i>x}\left(1-\frac{1}{nC_n(X_i)}\right),$$

which will be considered in this paper to derive a new estimator for the tail index of df $\mathbf{F}$. Note that the tail of df F simultaneously depends on $\overline{\mathbf{G}}$ and $\overline{\mathbf{F}}$ while that of $\overline{G}$ only relies on $\overline{\mathbf{G}}$. By using Proposition B.1.10 in de Haan and Ferreira (2006), to the regularly varying functions $\overline{\mathbf{F}}$ and $\overline{\mathbf{G}}$, we show that both $\overline{F}$ and $\overline{G}$ are regularly varying at infinity as well, with respective indices $-1/\gamma := -(\gamma_1+\gamma_2)/(\gamma_1\gamma_2)$ and $-1/\gamma_2$. In view of the definition of γ, Gardes and Stupfler (2015) derived a consistent estimator, for the extreme value index γ_1, whose asymptotic normality is established in Benchaira *et al.* (2015), under the tail dependence and the second-order conditions of regular variation. Recently, by considering a Lynden-Bell integration with a deterministic threshold $t_n > 0$, Worms and Worms (2016) proposed another asymptotically normal estimator for γ_1 as follows:

$$\widehat{\gamma}_1^{(\mathbf{LB})}(t_n) := \frac{1}{n\overline{\mathbf{F}}_n^{(\mathbf{LB})}(t_n)}\sum_{i=1}^n \mathbf{1}(X_i > t_n)\frac{\mathbf{F}_n^{(\mathbf{LB})}(X_i)}{C_n(X_i)}\log\frac{X_i}{t_n}.$$

Likewise, Benchaira *et al.* (2016a) considered a Woodroofe integration (with a random threshold) to propose a new estimator for the tail index γ_1 given by

$$\widehat{\gamma}_1^{(\mathbf{W})} := \frac{1}{n\overline{\mathbf{F}}_n^{(\mathbf{W})}(X_{n-k:n})}\sum_{i=1}^k \frac{\mathbf{F}_n^{(\mathbf{W})}(X_{n-i+1:n})}{C_n(X_{n-i+1:n})}\log\frac{X_{n-i+1:n}}{X_{n-k:n}},$$

where, given $n = m = m_N$, $Z_{1:m} \leq ... \leq Z_{m:m}$ denote the order statistics pertaining to a sample $Z_1, ..., Z_m$, and $k = k_n$ is a (random) sequence of integers such that, given $n = m$, $1 < k_m < m$, $k_m \to \infty$ and $k_m/m \to 0$ as $N \to \infty$. The consistency and asymptotic normality of $\widehat{\gamma}_1^{(\mathbf{W})}$ are established in Benchaira *et al.* (2016a) through a weak approximation to Woodroofe's tail process

$$\mathbf{D}_n^{(\mathbf{W})}(x) := \sqrt{k}\left(\frac{\overline{\mathbf{F}}_n^{(\mathbf{W})}(X_{n-k:n}x)}{\overline{\mathbf{F}}_n^{(\mathbf{W})}(X_{n-k:n})} - x^{-1/\gamma_1}\right),\ x > 0.$$

More precisely, the authors showed that, under (2) and (3) with $\gamma_1 < \gamma_2$, there exist a function $\mathbf{A}_0(t) \sim \mathbf{A}_{\mathbf{F}}^*(t) := \mathbf{A}_{\mathbf{F}}\left(1/\overline{\mathbf{F}}(\mathbb{U}_F(t))\right)$, $t \to \infty$, and a standard Wiener process

$\{\mathbf{W}(s);\ s\geq 0\}$, defined on the probability space $(\Omega,\mathcal{A},\mathbf{P})$, such that, for $0<\epsilon<1/2-\gamma/\gamma_2$ and $x_0>0$,

$$\sup_{x\geq x_0} x^{(1/2-\epsilon)/\gamma-1/\gamma_2}\left|\mathbf{D}_n^{(\mathbf{W})}(x)-\mathbf{\Gamma}(x;\mathbf{W})-x^{-1/\gamma_1}\frac{x^{\tau_1/\gamma_1}-1}{\gamma_1\tau_1}\sqrt{k}\mathbf{A}_0(n/k)\right|=o_{\mathbf{P}}(1), \quad (5)$$

as $N\rightarrow\infty$, provided that given $n=m$, $\sqrt{k_m}\mathbf{A}_0(m/k_m)=O(1)$, where $\{\Gamma(x;\mathbf{W});\ x>0\}$ is a Gaussian process defined by

$$\begin{aligned}\mathbf{\Gamma}(x;\mathbf{W}) := & \frac{\gamma}{\gamma_1}x^{-1/\gamma_1}\left\{x^{1/\gamma}\mathbf{W}\left(x^{-1/\gamma}\right)-\mathbf{W}(1)\right\}\\ & +\frac{\gamma}{\gamma_1+\gamma_2}x^{-1/\gamma_1}\int_0^1 s^{-\gamma/\gamma_2-1}\left\{x^{1/\gamma}\mathbf{W}\left(x^{-1/\gamma}s\right)-\mathbf{W}(s)\right\}ds.\end{aligned}$$

In view of the previous weak approximation, the authors also proved that if, given $n=m$, $\sqrt{k_m}\mathbf{A}_{\mathbf{F}}^{*}(m/k_m)\rightarrow\lambda$, then $\sqrt{k}(\widehat{\gamma}_1-\gamma_1)\overset{\mathcal{D}}{\rightarrow}\mathcal{N}\left(\frac{\lambda}{1-\tau_1},\sigma^2\right)$, as $N\rightarrow\infty$, where $\sigma^2:=\gamma^2(1+\gamma_1/\gamma_2)\left(1+(\gamma_1/\gamma_2)^2\right)/(1-\gamma_1/\gamma_2)^3$. Recently, Benchaira *et al.* (2016b) followed this approach to introduce a kernel estimator to γ_1 which improves the bias of $\widehat{\gamma}_1^{(\mathbf{W})}$. In this paper, we are interested in Worm's estimator $\widehat{\gamma}_1^{(\mathbf{LB})}(t_n)$, but with a threshold t_n that is assumed to be random and equal to $X_{n-k:n}$. This makes the estimator more convenient for numerical implementation than the one with a deterministic threshold. In other words, we will deal with the following tail index estimator:

$$\widehat{\gamma}_1^{(\mathbf{LB})}:=\frac{1}{n\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n})}\sum_{i=1}^{k}\frac{\mathbf{F}_n^{(\mathbf{LB})}(X_{n-i+1:n})}{C_n(X_{n-i+1:n})}\log\frac{X_{n-i+1:n}}{X_{n-k:n}}.$$

Note that $\mathbf{F}_n^{(\mathbf{LB})}(\infty)=1$ and write $\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n})=\int_{X_{n-k:n}}^{\infty}d\mathbf{F}_n^{(\mathbf{LB})}(y)$. On the other hand, we have $C_n(x)\,d\mathbf{F}_n^{(\mathbf{LB})}(x)=\mathbf{F}_n^{(\mathbf{LB})}(x)\,dF_n(x)$ (see, e.g., Strzalkowska-Kominiak and Stute, 2009), then

$$\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n})=\int_{X_{n-k:n}}^{\infty}\frac{\mathbf{F}_n^{(\mathbf{LB})}(x)}{C_n(x)}dF_n(x)=\frac{1}{n}\sum_{i=1}^{k}\frac{\mathbf{F}_n^{(\mathbf{LB})}(X_{n-i+1:n})}{C_n(X_{n-i+1:n})}.$$

This allows us to rewrite the new estimator into

$$\widehat{\gamma}_1^{(\mathbf{LB})}:=\sum_{i=1}^{k}a_n^{(i)}\log\frac{X_{n-i+1:n}}{X_{n-k:n}},$$

where

$$a_n^{(i)}:=\frac{\mathbf{F}_n^{(\mathbf{LB})}(X_{n-i+1:n})}{C_n(X_{n-i+1:n})}\Big/\sum_{i=1}^{k}\frac{\mathbf{F}_n^{(\mathbf{LB})}(X_{n-i+1:n})}{C_n(X_{n-i+1:n})}.$$

It is worth mentioning that for complete data, we have $n\equiv N$ and $\mathbf{F}_n\equiv F_n\equiv C_n$, it follows that $a_n^{(i)}\equiv k^{-1}$, $i=1,...,k$ and consequently both $\widehat{\gamma}_1^{(\mathbf{LB})}$ and $\widehat{\gamma}_1^{(\mathbf{W})}$ reduce to the classical Hill estimator (Hill, 1975). The consistency and asymptotic normality of $\widehat{\gamma}_1^{(\mathbf{LB})}$ will be achieved through a weak approximation of the corresponding tail Lynden-Bell process that we define by

$$\mathbf{D}_n^{(\mathbf{LB})}(x) := \sqrt{k}\left(\frac{\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n}x)}{\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n})} - x^{-1/\gamma_1}\right),\ x > 0.$$

The rest of the paper is organized as follows. In Section 2, we provide our main results whose proofs are postponed to Section 4. The finite sample behavior of the proposed estimator $\widehat{\gamma}_1^{(\mathbf{LB})}$ is checked by simulation in Section 3, where a comparison with the one recently introduced by Benchaira *et al.* (2016a) is made as well.

2. Main results

We basically have three main results. The first one, that we give in Theorem 1, consists in an asymptotic relation between the above mentioned estimators of the distribution tail, namely $\overline{\mathbf{F}}_n^{(\mathbf{W})}$ and $\overline{\mathbf{F}}_n^{(\mathbf{LB})}$. This in turn is instrumental to the Gaussian approximation of the tail Lynden-Bell process $\mathbf{D}_n^{(\mathbf{LB})}(x)$ stated in Theorem 2. Finally, in Theorem 3, we deduce the asymptotic behavior of the tail index estimator $\widehat{\gamma}_1^{(\mathbf{LB})}$.

Theorem 1. *Assume that both* $\mathbf{F}$ *and* $\mathbf{G}$ *satisfy the second-order conditions* (2) *and* (3) *respectively with* $\gamma_1 < \gamma_2$. *Let* $k = k_n$ *be a random sequence of integers such that, given* $n = m$, $k_m \to \infty$ *and* $k_m/m \to 0$, *as* $N \to \infty$, *then, for any* $x_0 > 0$, *we have*

$$\sup_{x \geq x_0} x^{1/\gamma_1} \frac{\left|\overline{\mathbf{F}}_n^{(\mathbf{W})}(X_{n-k:n}x) - \overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n}x)\right|}{\overline{\mathbf{F}}(X_{n-k:n})} = O_{\mathbf{P}}\left((k/n)^{\gamma_1/\gamma}\right).$$

Theorem 2. *Assume that the assumptions of Theorem 1 hold and given* $n = m$,

$$k_m^{1+\gamma_1/(2\gamma)}/m \to 0, \tag{6}$$

and $\sqrt{k_m}\mathbf{A}_0(m/k_m) = O(1)$, *as* $N \to \infty$. *Then, for any* $x_0 > 0$ *and* $0 < \epsilon < 1/2 - \gamma/\gamma_2$, *we have*

$$\sup_{x \geq x_0} x^{(1/2-\epsilon)/\gamma - 1/\gamma_2}\left|\mathbf{D}_n^{(\mathbf{LB})}(x) - \mathbf{\Gamma}(x;\mathbf{W}) - x^{-1/\gamma_1}\frac{x^{\tau_1/\gamma_1}-1}{\gamma_1\tau_1}\sqrt{k}\mathbf{A}_0(n/k)\right| = o_{\mathbf{P}}(1).$$

Theorem 3. *Assume that* (1) *holds with* $\gamma_1 < \gamma_2$ *and let* $k = k_n$ *be a random sequence of integers such that given* $n = m$, $k_m \to \infty$ *and* $k_m/m \to 0$, *as* $N \to \infty$, *then* $\widehat{\gamma}_1^{(\mathbf{LB})} \overset{\mathbf{P}}{\to} \gamma_1$. *Assume further that the assumptions of Theorem 2 hold, then*

$$\sqrt{k}\left(\widehat{\gamma}_1^{(\mathbf{LB})} - \gamma_1\right) = \frac{\sqrt{k}\mathbf{A}_0(n/k)}{1-\tau_1} - \gamma\mathbf{W}(1) + \frac{\gamma}{\gamma_1+\gamma_2}\int_0^1 (\gamma_2 - \gamma_1 - \gamma\log s)\, s^{-\gamma/\gamma_2 - 1}\mathbf{W}(s)\,ds + o_{\mathbf{P}}(1).$$

If, in addition, we suppose that, given $n = m$, $\sqrt{k_m}\mathbf{A}_{\mathbf{F}}^*(m/k_m) \to \lambda < \infty$, *then*

$$\sqrt{k}\left(\widehat{\gamma}_1^{(\mathbf{LB})} - \gamma_1\right) \overset{\mathcal{D}}{\to} \mathcal{N}\left(\frac{\lambda}{1-\tau_1}, \sigma^2\right),\ \text{as } N \to \infty.$$

$\gamma_1 = 0.6$; $p = 0.55$

		$\widehat{\gamma}_1^{(\mathbf{LB})}$			$\widehat{\gamma}_1^{(\mathbf{W})}$		
N	n	abs bias	rmse	k^*	abs bias	rmse	k^*
100	54	0.0407	0.2381	26	0.0443	0.2328	26
200	109	0.0378	0.2610	36	0.0358	0.2532	37
300	165	0.0352	0.2359	36	0.0323	0.2315	37
500	274	0.0199	0.2290	61	0.0185	0.2238	61
1000	549	0.0074	0.1763	112	0.0068	0.1748	112
3000	1649	0.0036	0.0982	350	0.0037	0.0981	352
5000	2747	0.0007	0.1066	432	0.0007	0.1065	432

Table 1. Estimation results of Lynden-Bell based (leftt pannel) and Woodroofe based (right pannel) estimators of the shape parameter $\gamma_1 = 0.6$ of Burr's model through 1000 right-truncated samples with 45%-truncation rate.

3. Simulation study

In this section, we illustrate the finite sample behavior of $\widehat{\gamma}_1^{(\mathbf{LB})}$ and, at the same time, we compare it with $\widehat{\gamma}_1^{(\mathbf{W})}$. To this end, we consider two sets of truncated and truncation data, both drawn from Burr's model: $\overline{\mathbf{F}}(x) = \left(1 + x^{1/\delta}\right)^{-\delta/\gamma_1}$, $\overline{\mathbf{G}}(x) = \left(1 + x^{1/\delta}\right)^{-\delta/\gamma_2}$, $x \geq 0$, where $\delta, \gamma_1, \gamma_2 > 0$. The corresponding percentage of observed data is equal to $p = \gamma_2/(\gamma_1 + \gamma_2)$. We fix $\delta = 1/4$ and choose the values 0.6 and 0.8 for γ_1 and 55%, 70% and 90% for p. For each couple (γ_1, p), we solve the equation $p = \gamma_2/(\gamma_1+\gamma_2)$ to get the pertaining γ_2-value. We vary the common size N of both samples $(\mathbf{X}_1, ..., \mathbf{X}_N)$ and $(\mathbf{Y}_1, ..., \mathbf{Y}_N)$, then for each size, we generate 1000 independent replicates. Our overall results are taken as the empirical means of the results obtained through all repetitions. To determine the optimal number of top statistics used in the computation of the tail index estimate values, we use the algorithm of Reiss and Thomas (2007), page 137. Our illustration and comparison are made with respect to the estimators absolute biases (abs bias) and the roots of their mean squared errors (rmse). We summarize the simulation results in Tables 1, 2 and 3 for $\gamma_1 = 0.6$ and in Tables 4, 5 and 6 for $\gamma_1 = 0.8$. After the inspection of all the tables, two conclusions can be drawn regardless of the situation. First, the estimation accuracy of both estimators decreases when the truncation percentage increases and this was quite expected. Second, we notice that the newly proposed estimator $\widehat{\gamma}_1^{(\mathbf{LB})}$ and $\widehat{\gamma}_1^{(\mathbf{W})}$ behave equally well.

4. Proofs

4.1. Proof Theorem 1

For $x \geq x_0$ we have

$$\mathbf{F}_n^{(\mathbf{W})}(X_{n-k:n}x) = \exp\left\{-\int_{X_{n-k:n}x}^{\infty} \frac{dF_n(y)}{C_n(y)}\right\}.$$

We show that the latter exponent is negligible in probability uniformly over $x \geq x_0$. Indeed, note that both $F_n(y)/F(y)$ and $C(y)/C_n(y)$ are stochastically bounded from above on

$\gamma_1 = 0.6$; $p = 0.7$

		$\widehat{\gamma}_1^{(\mathbf{LB})}$			$\widehat{\gamma}_1^{(\mathbf{W})}$		
N	n	abs bias	rmse	k^*	abs bias	rmse	k^*
100	69	0.0158	0.2451	25	0.0144	0.2428	25
200	140	0.0095	0.1871	39	0.0089	0.1866	39
300	210	0.0085	0.1590	61	0.0082	0.1587	61
500	348	0.0074	0.1294	76	0.0072	0.1293	76
1000	699	0.0063	0.1014	124	0.0062	0.1014	124
3000	2096	0.0053	0.0962	246	0.0053	0.0962	246
5000	3498	0.0036	0.0984	400	0.0036	0.0984	400

Table 2. Estimation results of Lynden-Bell based (left-panel) and Woodroofe based (right-panel) estimators of the shape parameter $\gamma_1 = 0.6$ of Burr's model through 1000 right-truncated samples with 30%-truncation rate.

$\gamma_1 = 0.6$; $p = 0.9$

		$\widehat{\gamma}_1^{(\mathbf{LB})}$			$\widehat{\gamma}_1^{(\mathbf{W})}$		
N	n	abs bias	rmse	k^*	abs bias	rmse	k^*
100	90	0.0073	0.1779	21	0.0070	0.1778	21
200	180	0.0066	0.1208	54	0.0064	0.1208	54
300	270	0.0055	0.1133	88	0.0056	0.1133	88
500	450	0.0050	0.0864	125	0.0050	0.0863	125
1000	898	0.0030	0.0614	189	0.0029	0.0614	189
3000	2702	0.0016	0.0494	398	0.0016	0.0494	398
5000	4496	0.0010	0.0112	467	0.0010	0.0112	467

Table 3. Estimation results of Lynden-Bell based (left-panel) and Woodroofe based (right-panel) estimators of the shape parameter $\gamma_1 = 0.6$ of Burr's model through 1000 right-truncated samples with 10%-truncation rate.

$\gamma_1 = 0.8$; $p = 0.55$

		$\widehat{\gamma}_1^{(\mathbf{LB})}$			$\widehat{\gamma}_1^{(\mathbf{W})}$		
N	n	abs bias	rmse	k^*	abs bias	rmse	k^*
100	55	0.0570	0.3330	30	0.0636	0.3167	31
200	110	0.0401	0.3604	33	0.0347	0.3453	35
300	164	0.0252	0.2563	69	0.0272	0.2530	71
500	276	0.0227	0.1807	112	0.0216	0.1794	113
1000	551	0.0148	0.1795	196	0.0142	0.1788	197
3000	1647	0.0124	0.1794	525	0.0121	0.1783	525
5000	2751	0.0075	0.1260	688	0.0074	0.1259	688

Table 4. Estimation results of Lynden-Bell based (left-panel) and Woodroofe based (right-panel) estimators of the shape parameter $\gamma_1 = 0.8$ of Burr's model through 1000 right-truncated samples with 45%-truncation rate.

$\gamma_1 = 0.8;\ p = 0.7$

		$\widehat{\gamma}_1^{(\mathbf{LB})}$			$\widehat{\gamma}_1^{(\mathbf{W})}$		
N	n	abs bias	rmse	k^*	abs bias	rmse	k^*
100	69	0.0217	0.3827	28	0.0195	0.3787	28
200	139	0.0203	0.2918	59	0.0194	0.2905	59
300	210	0.0189	0.1857	66	0.0184	0.1852	66
500	348	0.0143	0.1593	113	0.0140	0.1591	113
1000	700	0.0049	0.1205	230	0.0049	0.1204	230
3000	2100	0.0037	0.0886	449	0.0038	0.0886	449
5000	3500	0.0031	0.0857	500	0.0031	0.0857	500

Table 5. Estimation results of Lynden-Bell based (left-panel) and Woodroofe based (right-panel) estimators of the shape parameter $\gamma_1 = 0.8$ of Burr's model through 1000 right-truncated samples with 30%-truncation rate.

$\gamma_1 = 0.8;\ p = 0.9$

		$\widehat{\gamma}_1^{(\mathbf{LB})}$			$\widehat{\gamma}_1^{(\mathbf{W})}$		
N	n	abs bias	rmse	k^*	abs bias	rmse	k^*
100	89	0.0380	0.1833	38	0.0369	0.1827	38
200	179	0.0345	0.1383	80	0.0342	0.1383	80
300	269	0.0173	0.1014	99	0.0175	0.1013	99
500	450	0.0108	0.0927	143	0.0106	0.0926	143
1000	899	0.0021	0.0729	260	0.0021	0.0729	260
3000	2697	0.0013	0.0591	443	0.0013	0.0591	443
5000	4500	0.0001	0.0309	997	0.0001	0.0309	997

Table 6. Estimation results of Lynden-Bell based (left-panel) and Woodroofe based (right-panel) estimators of the shape parameter $\gamma_1 = 0.8$ of Burr's model through 1000 right-truncated samples with 10%-truncation rate.

$y < X_{n:n}$ (see, e.g., Shorack and Wellner, 1986, page 415 and Strzalkowska-Kominiak and Stute, 2009, respectively), it follows that

$$-\int_{X_{n-k:n}x}^{\infty} \frac{dF_n(y)}{C_n(y)} = O_{\mathbf{P}}(1) \int_{X_{n-k:n}x}^{\infty} \frac{dF(y)}{C(y)}. \tag{7}$$

By a change of variables we have

$$\int_{X_{n-k:n}x}^{\infty} \frac{d\overline{F}(y)}{C(y)} = \frac{\overline{F}(X_{n-k:n})}{C(X_{n-k:n})} \left(\int_{x}^{\infty} \frac{C(X_{n-k:n})}{C(X_{n-k:n}t)} d\frac{\overline{F}(X_{n-k:n}t)}{\overline{F}(X_{n-k:n})} \right). \tag{8}$$

Recall that $X_{n-k:n} \overset{\mathbf{P}}{\to} \infty$ and that $\overline{F}$ is regularly varying at infinity with index $-1/\gamma$. On the other hand, from Assertion (i) of Lemma A2 Benchaira *et al.* (2016a) we deduce that $1/C$ is also regularly varying at infinity with index $1/\gamma_2$. Thus, we may apply Potters inequalities, see e.g. Proposition B.1.10 in de Haan and Ferreira (2006), to both $\overline{F}$ and $1/C$ to write: for all large N, any $t \geq x_0$ and any sufficiently small $\delta, \nu > 0$, with large probability,

$$\left| \frac{\overline{F}(X_{n-k:n}t)}{\overline{F}(X_{n-k:n})} - t^{-1/\gamma} \right| < \delta t^{-1/\gamma \pm \nu} \text{ and } \left| \frac{C(X_{n-k:n})}{C(X_{n-k:n}t)} - t^{1/\gamma_2} \right| < \delta t^{1/\gamma_2 \pm \nu}, \tag{9}$$

where $t^{\pm a} := \max(t^a, t^{-a})$. These two inequalities may be rewritten, into

$$\frac{\overline{F}(X_{n-k:n}t)}{\overline{F}(X_{n-k:n})} = t^{-1/\gamma}\left(1 + o_{\mathbf{P}}\left(t^{\pm\nu}\right)\right) \text{ and } \frac{C(X_{n-k:n})}{C(X_{n-k:n}t)} = t^{1/\gamma_2}\left(1 + o_{\mathbf{P}}\left(t^{\pm\nu}\right)\right),$$

uniformly on $t \geq x_0$. This leads to

$$\int_x^\infty \frac{C(X_{n-k:n})}{C(X_{n-k:n}t)} d\frac{\overline{F}(X_{n-k:n}t)}{\overline{F}(X_{n-k:n})} = -\frac{\gamma_1}{\gamma} x^{-1/\gamma_1}\left(1 + o_{\mathbf{P}}\left(x^{\pm\nu}\right)\right). \tag{10}$$

In view of (4), Benchaira *et al.* (2016a) showed, in Lemma A1, that $\overline{F}(y) = (1 + o(1)) c_1 y^{-1/\gamma}$ and $\overline{G}(y) = (1 + o(1)) c_2 y^{-1/\gamma_2}$ as $y \to \infty$, for some constants $c_1, c_2 > 0$. In other words, $\mathbb{U}_F(s) = (1 + o(1))(c_1 s)^\gamma$as $s \to \infty$, and $C(y) = (1 + o(1)) c_2 y^{-1/\gamma_2}$ as $y \to \infty$. On the other hand, from Lemma A4 in Benchaira *et al.* (2016a), we have $X_{n-k:n} = (1 + o_{\mathbf{P}}(1))\mathbb{U}_F(n/k)$, it follows that $X_{n-k:n} = (1 + o_{\mathbf{P}}(1)) c_1^\gamma (k/n)^{-\gamma}$. Note that $1 - \gamma/\gamma_2 = \gamma/\gamma_1$, hence

$$\frac{\overline{F}(X_{n-k:n})}{C(X_{n-k:n})} = (1 + o_{\mathbf{P}}(1)) c_1^{\gamma/\gamma_2} c_2^{-1} (k/n)^{\gamma/\gamma_1}. \tag{11}$$

Plugging results (10) and (11) in equation (8) yields

$$\int_{X_{n-k:n}x}^\infty \frac{d\overline{F}(y)}{C(y)} = (k/n)^{\gamma/\gamma_1} c_1^{\gamma/\gamma_2} c_2^{-1} \gamma_1 x^{-1/\gamma_1}\left(1 + o_{\mathbf{P}}\left(x^{\pm\nu}\right)\right). \tag{12}$$

By combining equations (7) and (12), we obtain

$$\int_{X_{n-k:n}x}^\infty \frac{dF_n(y)}{C_n(y)} = O_{\mathbf{P}}(1)(k/n)^{\gamma/\gamma_1} x^{-1/\gamma_1}\left(1 + o_{\mathbf{P}}\left(x^{\pm\nu}\right)\right), \tag{13}$$

which obviously tends to zero in probability (uniformly on $x \geq x_0$). We may now apply Taylor's expansion $e^t = 1 + t + O\left(t^2\right)$, as $t \to 0$, to get

$$\exp\left\{-\int_{X_{n-k:n}x}^\infty \frac{dF_n(y)}{C_n(y)}\right\} = 1 - \int_{X_{n-k:n}x}^\infty \frac{dF_n(y)}{C_n(y)} + O_{\mathbf{P}}\left(\int_{X_{n-k:n}x}^\infty \frac{dF_n(y)}{C_n(y)}\right)^2, \; N \to \infty.$$

In other words, we have

$$\overline{\mathbf{F}}_n^{(\mathbf{W})}(X_{n-k:n}x) = \int_{X_{n-k:n}x}^\infty \frac{dF_n(y)}{C_n(y)} + R_{n1}(x), \; N \to \infty, \tag{14}$$

where $R_{n1}(x) := O_{\mathbf{P}}\left((k/n)^{2\gamma/\gamma_1}\right) x^{-2/\gamma_1}\left(1 + o_{\mathbf{P}}\left(x^{\pm\nu}\right)\right)$. Next, we show that

$$\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n}x) = \int_{X_{n-k:n}x}^\infty \frac{dF_n(y)}{C_n(y)} + R_{n2}(x), \; N \to \infty. \tag{15}$$

Observe that, by taking the logarithm then its exponential in the definition of $\mathbf{F}_n^{(\mathbf{LB})}(x)$, we have

$$\mathbf{F}_n^{(\mathbf{LB})}(X_{n-k:n}x) = \exp\left\{\sum_{i=1}^{n} \mathbf{1}(X_{i:n} > X_{n-k:n}x) \log\left(1 - \frac{1}{nC_n(X_{i:n})}\right)\right\},$$

which may be rewritten into $\exp\left\{n\int_x^\infty \log\left(1 - \frac{1}{nC_n(X_{n-k:n}y)}\right) dF_n(X_{n-k:n}y)\right\}$. To get approximation (15) it suffices to apply successively, in the previous quantity, Taylor's expansions $e^t = 1 + t + O(t^2)$ and $\log(1-t) = -t + O(t^2)$ (as $t \to 0$) with similar arguments as above (we omit further details). Combining (14) and (15) and setting $R_n(x) := R_{n1}(x) - R_{n2}(x)$ yield

$$\overline{\mathbf{F}}_n^{(\mathbf{W})}(X_{n-k:n}x) - \overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n}x) = R_n(x), \ N \to \infty. \tag{16}$$

On the other hand, by once again using Taylor's expansion, we write

$$\overline{\mathbf{F}}(X_{n-k:n}) = \int_{X_{n-k:n}}^\infty \frac{dF(y)}{C(y)} + \widetilde{R}_n(x), \ N \to \infty.$$

From equation (12), we infer that $\overline{\mathbf{F}}(X_{n-k:n}) = c_2^{-1} c_1^{1-\gamma/\gamma_1}(k/n)^{\gamma/\gamma_1}(1 + o_{\mathbf{P}}(1))$, which implies, in view of (16), that

$$x^{1/\gamma_1}\frac{\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n}x) - \overline{\mathbf{F}}_n^{(\mathbf{W})}(X_{n-k:n}x)}{\overline{\mathbf{F}}(X_{n-k:n})} = O_{\mathbf{P}}\left((k/n)^{\gamma/\gamma_1}\right)x^{-1/\gamma_1 \pm \nu}.$$

Observe now that, for a sufficiently small $\nu > 0$, we have $x^{-1/\gamma_1 \pm \nu} = O_{\mathbf{P}}(1)$, uniformly on $x \geq x_0 > 0$, as sought.

4.2. Proof Theorem 2

In a similar way to what is done with $\mathbf{D}_n^{(\mathbf{W})}(x)$, in the proof of Theorem 2.1 in Benchaira *et al.* (2016a), we decompose $k^{-1/2}\mathbf{D}_n^{(\mathbf{LB})}(x)$ into the sum of

$$\mathbf{N}_{n1}(x) := x^{-1/\gamma_1}\frac{\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n}x) - \overline{\mathbf{F}}(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})},$$

$$\mathbf{N}_{n2}(x) := -\frac{\overline{\mathbf{F}}(X_{n-k:n}x)}{\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n})}\frac{\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n}) - \overline{\mathbf{F}}(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})},$$

$$\mathbf{N}_{n3}(x) := \left(\frac{\overline{\mathbf{F}}(X_{n-k:n}x)}{\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n})} - x^{-1/\gamma_1}\right)\frac{\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n}x) - \overline{\mathbf{F}}(xX_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n}x)},$$

and $\mathbf{N}_{n4}(x) := \overline{\mathbf{F}}(X_{n-k:n}x)/\overline{\mathbf{F}}(X_{n-k:n}) - x^{-1/\gamma_1}$. If we let

$$\mathbf{M}_{n1}(x) := x^{-1/\gamma_1}\frac{\overline{\mathbf{F}}_n^{(\mathbf{W})}(X_{n-k:n}x) - \overline{\mathbf{F}}(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})},$$

then, by applying Theorem 1, we have $x^{1/\gamma_1}\mathbf{N}_{n1}(x) = x^{1/\gamma_1}\mathbf{M}_{n1}(x) + x^{-1/\gamma_1}O_{\mathbf{P}}\left((k/n)^{\gamma/\gamma_1}\right)$, uniformly on $x \geq x_0$. By assumption we have $k^{1+\gamma_1/(2\gamma)}/n \overset{\mathbf{P}}{\to} 0$, which is equivalent to $\sqrt{k}\,(k/n)^{\gamma/\gamma_1} \overset{\mathbf{P}}{\to} 0$ as $N \to \infty$, therefore

$$x^{1/\gamma_1}\sqrt{k}\mathbf{N}_{n1}(x) = x^{1/\gamma_1}\sqrt{k}\mathbf{M}_{n1}(x) + o_{\mathbf{P}}\left(x^{-1/\gamma_1}\right). \tag{17}$$

In view of this representation we show that, both $\mathbf{D}_n^{(\mathbf{W})}(x)$ and $\mathbf{D}_n^{(\mathbf{LB})}(x)$ are (weakly) approximated, in the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, by the same Gaussian process $\mathbf{\Gamma}(x; \mathbf{W})$ given in (5). Indeed, for a sufficiently small $\epsilon > 0$, and $0 < \eta < 1/2$, Benchaira *et al.* (2016a) (see the beginning of the proof of Theorem 2.1 therein), showed that

$$x^{1/\gamma_1}\sqrt{k}\mathbf{M}_{n1}(x) = \Phi(x) + o_{\mathbf{P}}\left(x^{(1-\eta)/\gamma \pm \epsilon}\right),$$

where $\Phi(x) := x^{1/\gamma}\left\{\dfrac{\gamma}{\gamma_1}\mathbf{W}\left(x^{-1/\gamma}\right) + \dfrac{\gamma}{\gamma_1+\gamma_2}\int_0^1 t^{-\gamma/\gamma_2-1}\mathbf{W}\left(x^{-1/\gamma}t\right)dt\right\}$. Then by using representation (17), we get $x^{1/\gamma_1}\sqrt{k}\mathbf{N}_{n1}(x) = \Phi(x) + o_{\mathbf{P}}\left(x^{-1/\gamma_1}\right) + o_{\mathbf{P}}\left(x^{(1-\eta)/\gamma\pm\epsilon}\right)$. In particular for $x = 1$, we have

$$\sqrt{k}\left(\frac{\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})} - 1\right) = \sqrt{k}\mathbf{N}_{n1}(1) = \Phi(1) + o_{\mathbf{P}}(1), \tag{18}$$

leading to $\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n})/\overline{\mathbf{F}}(X_{n-k:n}) \overset{\mathbf{P}}{\to} 1$, as $N \to \infty$. By applying Potters inequalities to $\overline{\mathbf{F}}$ (as it was done for $\overline{F}$ in (11)) together with the previous limit, we obtain

$$\frac{\overline{\mathbf{F}}(X_{n-k:n}x)}{\overline{\mathbf{F}}_n^{(\mathbf{LB})}(X_{n-k:n})} = \left(1 + O_{\mathbf{P}}\left(x^{\pm\epsilon}\right)\right)x^{-1/\gamma_1}. \tag{19}$$

By combining (18) and(19), we get $x^{1/\gamma_1}\sqrt{k}\mathbf{N}_{n2}(x) = -\Phi(1) + o_{\mathbf{P}}\left(x^{\pm\epsilon}\right)$. For the third term $\mathbf{N}_{n3}(x)$, we use similar arguments to show that

$$x^{1/\gamma_1}\sqrt{k}\mathbf{N}_{n3}(x) = o_{\mathbf{P}}\left(x^{-1/\gamma_1\pm\epsilon}\right) + o_{\mathbf{P}}\left(x^{-1/\gamma_1+(1-\eta)/\gamma\pm\epsilon}\right).$$

Observe that $x^{1/\gamma_1-(1-\eta_0)/\gamma}o_{\mathbf{P}}\left(x^{-1/\gamma_1\pm\epsilon}\right)$ and $x^{1/\gamma_1-(1-\eta_0)/\gamma}o_{\mathbf{P}}\left(x^{-1/\gamma_1+(1-\eta)/\gamma\pm\epsilon}\right)$ respectively equal $o_{\mathbf{P}}\left(x^{-(1-\eta_0)/\gamma\pm\epsilon}\right)$ and $o_{\mathbf{P}}\left(x^{(\eta-\eta_0)/\gamma\pm\epsilon}\right)$, for $\gamma/\gamma_2 < \eta_0 < \eta < 1/2$, and that both the last two quantities are equal to $o_{\mathbf{P}}(1)$ for any small $\epsilon > 0$ and $x \geq x_0 > 0$. Finally, by following the same steps at the end of the proof of Theorem 2.1 in Benchaira *et al.* (2016a), we get

$$\sqrt{k}\mathbf{N}_{n4}(x) = x^{-1/\gamma_1}\frac{x^{\tau_1/\gamma_1}-1}{\gamma_1\tau_1}\sqrt{k}\mathbf{A}_0(n/k) + o_{\mathbf{p}}\left(x^{-1/\gamma_1+(1-\eta)/\gamma\pm\epsilon}\right).$$

Consequently, we have

$$x^{1/\gamma_1-(1-\eta_0)/\gamma}\left\{\mathbf{D}_n^{(\mathbf{LB})}(x) - \mathbf{\Gamma}(x;\mathbf{W}) - x^{-1/\gamma_1}\frac{x^{\tau_1/\gamma_1}-1}{\gamma_1\tau_1}\sqrt{k}\mathbf{A}_0(n/k)\right\} = o_{\mathbf{P}}(1),$$

uniformly over $x \geq x_{0.}$ Recall that $1/\gamma_1 = 1/\gamma - 1/\gamma_2$, then letting $\eta_0 := 1/2 - \xi$ yields $0 < \xi < 1/2 - \gamma/\gamma_2$ and achieves the proof.

4.3. Proof of Theorem 3

The proof is similar, mutatis mutandis, as that of Corollary 3.1 in Benchaira *et al.* (2016a). Therefore we omit the details.

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