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The maximum principle in optimal control of systems driven by martingale measures*

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Abstract. We study the relaxed optimal stochastic control problem for systems governed by stochastic differential equations (SDEs), driven by an orthogonal continuous martingale measure, where the control is allowed to enter both the drift and diffusion coefficient. The set of admissible controls is a set of measure-valued processes. Necessary conditions for optimality for these systems in the form of a maximum principle are established by means of spike variation techniques. Our result extends Peng's maximum principle to the class of measure valued controls.

Résumé. Nous étudions les problèmes de contrôle stochastique relaxés pour des systèmes gouvernés par des équations différentielles stochastiques (EDSs), dirigées par des mesures martingales orthogonales continues, avec un drift et un coefficient de diffusion contrôlé. L'ensemble des contrôles admissibles est constitué de processus à valeurs mesures. On établit des conditions nécessaires d'optimalité en utilisant des preturbations fortes. Notre résultat généralise le principe du maximum de Peng pour la classe de contrôles à valeurs mesures.

Key words: Orthogonal continuous martingale measures; Maximum principle; Optimal control; Relaxed control; Stochastic differential equation.
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1. Introduction

We are interested by optimality necessary conditions for control problems of systems satisfying the stochastic differential equation

$$dx\,(t) = b\,(t, x\,(t)\,, u\,(t))\,dt + \sigma\,(t, x\,(t)\,, u\,(t))\,dB_t, \quad x(0) = x \tag{1}$$

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on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$, where b and σ are deterministic functions, $(B_t, t \geq 0)$ is a Brownian motion, x is the initial state and $u\,(t)$ stands for the control variable. Our control problem consists in minimizing a cost functional of the form

$$J(u) = E\left[\int_0^1 h\left(t, x\left(t\right), u\left(t\right)\right) dt + g(x\left(1\right))\right], \tag{2}$$

over the class $\mathcal{U}$ of admissible controls, that is adapted processes, with values in some compact metric space A, called the action space. A control u^* is called optimal if it satisfies

$$J(u^*) = \inf\left\{J(u), u \in \mathcal{U}\right\}.$$

If, moreover, u^* is in $\mathcal{U}$, it is called strict. Existence of such a strict control or an optimal control in $\mathcal{U}$ follows from the convexity of the image of the action space by the map $\big(b(t,x,.), \sigma^2(t,x,.), h(t,x,.)\big)$, called the Filipov-type convexity condition, see Becker and Mandrekar (1969); El Karoui *et al.* (1987); Fleming (1976); Haussmann (1986); Kushner (1975). Without this convexity condition an optimal control does not necessarily exist in $\mathcal{U}$, this set is not equipped with a compact topology. The idea is then to introduce a larger class $\mathcal{R}$ of control processes, in which the controller chooses at time t a probability measure $q_t(da)$ on the control set $\mathcal{U}$, rather than an element $u_t \in \mathcal{U}$. These are called relaxed controls and have a richer topological structure, for which the control problem becomes solvable and the SDE will have the form

$$dx\left(t\right) = \int_A b\left(t, x\left(t\right), a\right) q_t(da)dt + \int_A \sigma\left(t, x\left(t\right), a\right) M\left(da, dt\right), \quad x(0) = x,$$

where $M(da, dt)$ is an orthogonal continuous martingale measure, whose intensity is the relaxed control $q_t(da)dt$. The corresponding cost is given by

$$J(q) = E\left[\int_0^1 \int_A h\left(t, x\left(t\right), a\right) q_t(da)dt + g(x\left(1\right))\right].$$

The relaxed control problem finds its interest in two essential points. The first is that an optimal solution exists. Fleming (1976) derived an existence result of an optimal relaxed control for systems with uncontrolled diffusion coefficient. The existence of an optimal solution, where the drift and the diffusion coefficients depend explicitly on the relaxed control variable, has been solved by El Karoui *et al.* (1987), see also Haussmann (1986); Haussmann and Lepeltier (1990). The relaxed optimal control in this general case is shown to be Markovian. See also Bahlali *et al.* (2006) for an alternative proof of the existence of an optimal relaxed control based on Skorokhod selection theorem. The second advantage of the use of relaxed controls is that it is a generalization of the strict control problem, in the sense that both control problems have the same value function. Indeed, if $q_t(da) = \delta_{u_t}(da)$ is a Dirac measure charging u_t for each t, we get a strict control as a particular case of the relaxed one.

Motivated by the existence of an optimal relaxed control, various versions of the stochastic maximum principle have been proved. The first result in this direction has been established in Mezerdi and Bahlali (2002), where a stochastic maximum principle for relaxed controls, in the case of uncontrolled diffusion coefficient has been given by using the first order adjoint process (see also Bahlali *et al.*, 2007 the extension to singular control problems). The case of a controlled diffusion coefficient has been treated in Bahlali *et al.* (2006), by using Ekeland's variational principle and an approximation scheme, by using the first and second order adjoint processes. Let us point out that a different relaxation has been used in Bahlali (2008); Ahmed and Charalambous (2013), where the drift and diffusion coefficient have been replaced by their relaxed counterparts. Their relaxed state process is linear in the control variable and is different from ours, in the sense that in our case we relax the infinitesimal generator instead of relaxing directly the state process.

The aim of the present paper is to obtain a Peng-type general stochastic maximum principle for relaxed controls, using directly the spike perturbation. Our method differs from the one used in Bahlali *et al.* (2006), in the sense that we don't use neither the approximation procedure nor Ekeland's variational principle. We use a spike variation method directly on the relaxed optimal control. Then, we derive the variational equation from the state equation and the variational inequality from the inequality

$$J\left(q^{\theta}\right) - J\left(q\right) \geq 0.$$

As for strict controls, the first order expansion of $J\left(q^{\theta}\right)$ is not sufficient to obtain a necessary optimality condition.One has to consider the second-order terms (with respect to the state) in the expansion of $J\left(q^{\theta}\right) - J\left(q\right)$. Although the second-order terms are quadratic with respect to the state variable, a so called second-order variational equation and second-order variational inequality are introduced. By using a suitable predictable representation theorem for martingale measures Overbeck (1995), we obtain the corresponding first and second-order adjoint equations, which are linear backward stochastic differential equations driven by the optimal martingale measure. This could be seen as one of the novelties of this paper.

Our paper is organized as follows. In section 2, first we give some properties of a class of orthogonal martingale measures, formulate the strict and relaxed control problems. In section 3, we obtain a maximum principle of the Pontriagin type for relaxed controls, extending the well known Peng stochastic maximum principle to the class of measure-valued controls.

2. Formulation of the relaxed control problem

2.1. Martingale measures

We start with the definition of a martingale measure introduced by Walsh (1986).

Definition 1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space and $(E, \mathcal{E})$ a Lusin space. $\{M_t\left(A\right), t \geqq 0, A \in \mathcal{E}\}$ is a $\mathcal{F}_t-$martingale measure if and only if:

1) $M_0 = 0, \;\; \forall A \in \mathcal{E}$;

2) $\{M_t\left(A\right), t \geqq 0\}$ is a $\mathcal{F}_t$-martingale, $\forall A \in \mathcal{E}$;

3) $\forall t > 0, M_t(.)$ is a L^2-valued σ-finite measure in the following sense: there exists a non-decreasing sequence $\{E_n\}$ of E with $\cup_n E_n = E$ such that

a) for every $t > 0$, $\sup\limits_{A\in\mathcal{E}_n} E\left[M\left(A,t\right)^2\right] < \infty$, $\mathcal{E}_n = \mathcal{B}\left(E_n\right)$

b) for every $t > 0$, $E\left[M\left(A_j,t\right)^2\right] \to 0$ for all sequence A_j of $\mathcal{E}_n$ decreasing to $\varnothing$.

For $A, B \in \mathcal{E}$, there exists a unique predictable process $\langle M(A), M(B)\rangle_t$, such that

$$M(A,t)M(B,t) - \langle M(A), M(B)\rangle_t \text{ is a martingale.}$$

A martingale measure M is called orthogonal if $M(A,t).M(B,t)$ is a martingale for $A, B \in \mathcal{E}$, $A \cap B = \emptyset$.

If M is an orthogonal martingale measure, one can prove the existence of random σ-finite positive measure $\upsilon(ds, dx)$ on $\mathbb{R} \times E$, $\mathcal{F}_t$-predictable, such that for each A of $\mathcal{A}$ the process $(\upsilon((0,t] \times A))_t$ is predictable and satisfies

$$\forall A \in \mathcal{E}, \forall t > 0, \qquad \upsilon((0,t] \times A) = \langle M(A)\rangle_t \quad P - a.s.$$

υ can be decomposed as follows $\upsilon(dt, da) = q_t(da)dk_t$, where k_t is a random predictable increasing process and $(q_t(da))_{t\geq 0}$ is a predictable family of random σ-finite measure.

We refer to Walsh (1986) and El Karoui and Méléard (1990) for more details and a complete construction of the stochastic integral with respect to orthogonal martingale measures.

2.2. The strict control problem

The systems we wish to control are driven by d-dimensional stochastic differential equations of diffusion type

$$dx(t) = b(t, x(t), u(t))\, dt + \sigma(t, x(t), u(t))\, dB_t, \qquad x(0) = a, \tag{3}$$

where (B_t) is a $d-$dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

For each $t \in [0,1]$, the control u is a measurable $\mathcal{F}_t$-adapted process with values in the action space $\mathbb{A}$, which is a compact metric space.

The infinitesimal generator L corresponding to equation (3), acting on functions f in $C_b^2(\mathbb{R}^d, \mathbb{R})$, is

$$Lf(t,x,u) = \frac{1}{2}\sum_{i,j}\left(a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(t,x,u) + \sum_j \left(b_j \frac{\partial f}{\partial x_j}\right)(t,x,u)$$

where $a_{ij}(t,x,u)$ denotes the generic term of the symmetric matrix $\sigma\sigma^*(t,x,u)$. Let $\mathcal{U}$ denotes the class of admissible controls, that is $\mathcal{F}_t$-adapted processes with values in the action space $\mathbb{A}$.

The function to be minimized over such controls is

$$J(u) = E\left[\int_0^1 h(t, x(t), u(t))\, dt + g(x(1))\right]. \tag{4}$$

$(\mathbf{H_1})$We assume that the coefficients of the control problem satisfy the following hypothesis:

$$\begin{aligned} &b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}^d, \\ &\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{A} \to \mathcal{M}_{d\times k}(\mathbb{R}), \\ &h : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}, \end{aligned}$$

are bounded measurable in (t, x, a).

$(\mathbf{H_2})$ b, σ, h are twice continuously differentiable functions in x with bounded first and second derivatives.

$(\mathbf{H_3})$ $g : \mathbb{R}^d \to \mathbb{R}$ is bounded and twice continuously differentiable with bounded first and second order derivatives.

Under the assumptions above, the controlled equation admits a unique strong solution such that for every $p \geq 1$, $E\left[\sup_{0\leq t\leq T} |x_t|^p\right] < M(p)$.

Definition 2. A strict control is the term $\alpha = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, u(t), x(t), a)$ such that

(1) $a \in \mathbb{R}^d$ is the initial data;

(2) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions;

(3) $u(t)$ is an $\mathbb{A}$-valued process, progressively measurable with respect to $(\mathcal{F}_t)$;

(4) $(x(t))$ is $\mathbb{R}^d$-valued, $\mathcal{F}_t$-adapted, with continuous paths, such that

$$f(x(t)) - f(a) - \int_0^t Lf(s, x(s), u(s))\, ds \quad \text{is a } P-\text{martingale},$$

for every $f \in C_b^2$, where L is the infinitesimal generator of the diffusion $(x(t))$.

The associated controls are called weak controls because of the possible change of the probability space and the Brownian motion with $u(t)$. When path-wise uniqueness holds for the controlled equation it is showed in El Karoui *et al.* (1987), that the weak and strong control problems are equivalent in the sense that they have the same value functions.

2.3. The relaxed control problem

The strict control problem as defined in the last section may fail to have an optimal solution, as shown in the following simple example, taken from deterministic control theory.

Example 1. Minimize the cost function

$$J(u) = \int_0^T x^u(t)^2 dt$$

over the set U_{ad} of open loop controls, that is, measurable functions $u : [0, T] \to \{-1, 1\}$.

Let $x^u(t)$ denote the solution of

$$dx^u(t) = udt, \qquad x(0) = 0.$$

We have $\inf_{u\in\mathcal{U}} J(u) = 0$. Indeed consider the following sequence of controls

$$u_n(t) = (-1)^k; \qquad \text{if } \frac{k}{n} \leq t \leq \frac{k+1}{n}, 0 \leq k \leq n-1.$$

Then clearly $|x^{u_n}(t)| \leq 1/n$ and $|J(u_n)| \leq T/n^2$ which implies that $\inf_{u\in\mathcal{U}} J(u) = 0$. There is however no control u such that $J(u) = 0$.

If this would have been the case, then for every t, $x^u(t) = 0$. This in turn would imply that $u(t) = 0$, which is impossible. The problem is that the sequence (u_n) has no limit in the space of strict controls. This limit, if it exists, will be the natural candidate for optimality. If we identify $u_n(t)$ with the Dirac measure $\delta_{u_n(t)}(da)$ and set $q_n(dt, da) = \delta_{u_n(t)}(da)\,dt$, we get a measure on $[0,1] \times \mathcal{A}$. Then $(q_n(dt, da))_n$ converges weakly to $(1/2)\,dt\,[\delta_{-1} + \delta_1](da)$. This suggests that the set $\mathcal{U}$ of strict controls is too narrow and should be embedded into a wider class with a richer topological structure, for which the control problem becomes solvable. The idea of relaxed control is to replace the A-valued process $(u(t))$ with $P(A)$-valued process (q_t), where $P(A)$ is the space of probability measures equipped with the topology of weak convergence.

In this section, we introduce relaxed controls of SDE as solutions of a martingale problem for a diffusion process whose infinitesimal generator is integrated against the random measures defined over the action space of all controls. Let $\mathcal{V}$ be the set of Radon measures on $[0,1] \times A$ whose projections on $[0,1]$ coincide with the Lebesgue measure dt. Equipped with the topology of stable convergence of measures, $\mathcal{V}$ is a compact metrizable space. Stable convergence is required for bounded measurable functions $h(t,a)$ such that for each fixed $t \in [0,1]$, $h(t,.)$ is continuous.

Definition 3. A relaxed control is the term $q = (\Omega, \mathcal{F}, \mathcal{F}_t, P, B_t, q_t, x(t), a)$ such that

(1) $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a filtered probability space satisfying the usual conditions;

(2) (q_t) is an $P(A)$-valued process, progressively measurable with respect to $(\mathcal{F}_t)$; and such for that for each t, $1_{(0,1]}.q$ is $\mathcal{F}_t$-measurable;

(3) $(x(t))$ is $\mathbb{R}^d$-valued, $\mathcal{F}_t$-adapted, with continuous paths, such that $x(0) = \varrho$ and

$$f(x(t)) - f(a) - \int_0^t \int_A Lf(s, x(s), a)\, q_s(w, da) ds \quad \text{is a } P\text{-martingale, for each } f \in C_b^2(\mathbb{R}^d, \mathbb{R}). \tag{5}$$

We denote by $\mathcal{R}$ the collection of all relaxed controls.

By a slight abuse of notation, we will often denote a relaxed control by q instead of specifying all the components.

The cost function associated to a relaxed control q is defined as

$$J(u) = E\left[\int_0^1 \int_A h(t, x(t), a)\, q_t(da)dt + g(x(1))\right].$$

The set $\mathcal{U}$ of strict controls is embedded into the set $\mathcal{R}$ of relaxed controls by the mapping

$$\Psi : u \in \mathcal{U} \rightarrow \Psi(u)(dt, da) = dt\delta_{u(t)}(da) \in \mathcal{R};$$

where δ_u is the Dirac measure at a single point u. In fact the next lemma, known as the chattering lemma, tells us that any relaxed control is weak limit of sequence of strict controls.

Lemma 1. *(Chattering lemma) Let (q_t) be a predictable with values in the space of probability measure on A. Then there exists a sequence of predictable processes $(u^n(t))$ with values in A such that the sequence of random measures $\left(\delta_{u^n(t)}(da)\,dt\right)$ converge weakly to $q_t(da)\,dt, P - a.s.$*

In the next example, through considering the action space $\mathbb{A}$ to be a finite set of points, hence reducing the problem to controlling a finite-dimensional diffusion process, we will identify the appropriate class of martingale measures that drives the stochastic representation of the coordinate process associated with the solution to the martingale problem (5).

Example 2. Let $\mathbb{A} = \{a_1, a_2, ..., a_n\}$, then every relaxed control $dtq_t\,(da)$ will be a convex combination of the Dirac measures $dtq_t\,(da) = \sum_{i=1}^{n} \alpha_i(t)dt\delta_{a_i}\,(da)$, where for each $i, \alpha_i(t)$ is a real-valued process such that $0 \leq \alpha_i(t) \leq 1$ and $\sum_{i=1}^{n} \alpha_i(t) = 1$. It is shown that the solution of the (relaxed) martingale problem (5) is the law of the solution of the following SDE (see Bahlali, 2008)

$$dx(t) = \sum_{i=1}^{d} b(t, x(t), u_i(t))\alpha_i(t)dt + \sum_{i=1}^{d} \sigma(t, x(t), u_i(t))\alpha_i(t)^{1/2}dB_t^i, \qquad x(0) = a \tag{6}$$

where the B^i's are d-dimensional Brownian motions on an extension of the initial probability space. The process M defined by

$$M\,(A \times [0,t]) = \sum_{i=1}^{d} \int_0^t \alpha_i(s)^{1/2}\delta_{u_i(s)}(A)dB_s^i,$$

is in fact a strongly orthogonal continuous martingale measure (El Karoui and Méléard, 1990; Walsh, 1986) with intensity $q_t(da)dt = \sum \alpha_i(t)\delta_{u_i(t)}\,(da)\,dt$. Thus, the SDE (6) can be expressed in terms of M and q as follows

$$dx(t) = \int_A b(t, x(t), a)q_t(da)dt + \int_A \sigma(t, x(t), a)M(da, dt).$$

The following theorem due to El Karoui and Méléard (1990) gives a pathwise representation of the solution of the martingale problem (5) in terms of strongly orthogonal continuous martingale measure whose intensity are our relaxed control.

Theorem 1. *(1) Let P be the solution of the martingale problem (5). Then P is the law of a d-dimensional adapted and continuous process X defined on an extension of the space $(\Omega, \mathcal{F}, \mathcal{F}_t)$ and solution of the following SDE starting at ϱ*

$$dX^i(t) = \int_A b_i(t, X(t), a)q_t(da)dt + \sum_{k=1}^{d} \int_A \sigma_{i,k}(t, X(t), a)M^k(da, dt), \tag{7}$$

where $M = \left(M^k\right)_{k=1}^{d}$ is a family of d-strongly orthogonal continuous martingale measures with intensity $q_t(da)dt$.

(2) If the coefficients b and σ are Lipschitz in x, uniformly in t and a, the SDE (7) has a unique pathwise solution.

Using the chattering lemma, we get the following result due to Méléard (1992) on approximating continuous orthogonal martingale measures, with given intensity, by a sequence of stochastic integrals with respect to a single Brownian motion.

Proposition 1. *Let M be a continuous orthogonal martingale measure with intensity $q_t(da)\,dt$ on $A\times[0,1]$. Then there exist a sequence of predictable A-valued processes $(u^n(t))$ and a Brownian motion B defined on an extension of $(\Omega,\mathcal{F},P)$ such that for all $t\in[0,T]$ and φ continuous bounded functions from $\mathbb{A}$ to $\mathbb{R}$,*

$$\lim_{n\to+\infty} E\left[\left(M_t(\varphi)-\int_0^t \varphi(u^n(s))\,dB_s\right)^2\right]=0.$$

2.3.1. Approximation and existence of optimal relaxed controls

In order for the relaxed control problem to be truly an extension of the original one, the infimum of the expected cost among relaxed controls must be equal to the infimum among strict controls. This result is based on approximation of a relaxed control by a sequence of strict controls, given by Lemma 1.

The next theorem gives the stability of the controlled stochastic differential equations with respect to the control variable.

Let (q_t) be a relaxed control. We know from Theorem 1 that there exists a family of continuous strongly orthogonal martingale measures $M_t=\left(M_t^k\right)$ such that the state of the system satisfies the following SDE, starting at $X(0)=a$

$$dX(t)=\int_A b(t,X(t),a)q_t(da)dt+\int_A \sigma(t,X(t),a)M(da,dt). \tag{8}$$

Moreover, thanks to Lemma 3.4 in Bahlali (2008) and Proposition 1, there exist a sequence $(u^n(t))$ of strict controls and a Brownian motion B defined on an extension of $(\Omega,\mathcal{F},P)$ such that for each $t\in[0,T]$ and each continuous bounded function φ from $\mathbb{A}$ to $\mathbb{R}$,

$$\lim_{n\to+\infty} E\left[\left(M_t(\varphi)-\int_0^t \varphi(u^n(s))\,dB_s\right)^2\right]=0. \tag{9}$$

Denote by $X^n(t)$ the solution which can be written in relaxed form as

$$\begin{cases} dX^n(t)=\int_A b(t,X^n(t),a)q_t^n(da)dt+\int_A \sigma(t,X^n(t),a)M^n(da,dt) \\ X^n(0)=\varrho, \end{cases} \tag{10}$$

with respect to the martingale measure $M_t^n(A)=\int_0^t \mathbf{1}_A(u^n(s))dB_s$ and $q_t^n(da)=\delta_{u^n(t)}(da)$.

Theorem 2. *Let $X(t)$ and $X^n(t)$ be the diffusion solutions of (8) and (10), respectively. Then*

$$\lim_{n\to+\infty} E\left[\sup_{0\le t\le 1}\left|X^n(t)-X(t)\right|^2\right]=0.$$

Proof. See Bahlali *et al.* (2006).

Corollary 1. *Let $J(u^n)$ and $J(q)$ be the expected costs corresponding, respectively, to u^n and q, where u^n and q are defined as in the last theorem. Then there exists a sub-sequence (u^{n_k}) of (u^n) such that $J(u^{n_k})$ converges to $J(q)$.*

Remark 1. It follows from the last corollary that the strict and relaxed controls are equivalent, in the sense that they have the same value function.

2.4. Predictable representation for orthogonal martingale measures

Let us denote the set of square-integrable martingales over $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ by $\mathbf{M}^2$.

Proposition 2. *Let N be in $\mathbf{M}^2$. Then there exist a unique square integrable predictable process n such that*

$$N_t = N_0 + \int_0^t \int_E n(a,s)M(da,ds) + L_t,$$

where L is an L^2-martingale with $\left\langle L, \int_0^{\cdot} \int_E b(a,s)M(da,ds)\right\rangle = 0$ for every predictable b.

Proof. See Overbeck (1995).

3. The relaxed maximum principle

In this section we establish optimality necessary conditions for relaxed control problems, where the system is described by a SDE driven by an orthogonal continuous martingale measure and the admissible controls are measure-valued processes.

Recall the controlled SDE:

$$dx(t) = \int_A b(t,x(t),a)q_t(da)dt + \int_A \sigma(t,x(t),a)M(da,dt), \qquad x(0) = a \tag{11}$$

where $M(da,dt)$ is orthogonal continuous martingale measure whose intensity is the relaxed control $q_t(da)dt$. The corresponding cost is given by

$$J(q) = E\left[\int_0^1 \int_A h\left(t,x(t),a\right) q_t(da)dt + g(x(1))\right].$$

3.1. Preliminary results

The purpose of the stochastic maximum principle is to find necessary conditions for optimality satisfied by an optimal control. Due to the appearance of the control variable in $\sigma(.,.)$, the usual first order expansion approach can't work. Hence, we introduce a second-order expansion method, we proceed as the classical maximum principle (Peng, 1990).

Suppose that $(x(.), q(.))$ is an optimal solution of the problem and let us introduce the strong perturbed relaxed control in the following way

$$q_t^\theta(A) = \begin{cases} \delta_v(A) & \text{if} \quad t \in E \\ q_t(A) & \text{if} \quad t \in E^c \end{cases}$$

where $E = \{r \leq t \leq r+\theta\}$, $0 \leq r < T$ is fixed and the E^c otherwise, $\theta > 0$ is sufficiently small, and υ is an arbitrary $\mathcal{F}_r$-measurable random variable with values in $\mathcal{U}$, such that

$$\sup_{w\in\Omega} |\upsilon(w)| < \infty.$$

Let x_θ be the trajectory of the control system (11) corresponding to the control $q^\theta_\cdot(A)$, which is the intensity of the orthogonal continuous martingale measures M^θ, we create it of the form

$$M_t^\theta(A) = \int_0^t \int_A \mathbf{1}_{[r,r+\theta]}(s)\delta_\nu(da)dB_s + \int_0^t \int_A \mathbf{1}_{[r,r+\theta]^C}(s)M(da,ds).$$

where $0 \leq r < T$ is fixed, $\theta > 0$ is sufficiently small, and υ is an arbitrary $\mathcal{F}_r$-measurable random variable with values in $\mathcal{U}$.

The variational inequality will be derived from the fact that

$$\lim_{\theta\to 0} \frac{1}{\theta}\left[J(q^\theta(.)) - J(q(.))\right] \geq 0,$$

to this end, we need the following estimation.

Lemma 2. *We assume (**H_1**-**H_3**), then the following estimate holds*

$$E\left[\sup_{0\leq t\leq T} |x_\theta(t) - x(t) - x_1(t) - x_2(t)|^2\right] \leq C(\theta)\theta^2 \tag{12}$$

where $\lim_{\theta\longrightarrow 0} C(\theta) = 0$ *and* $x_1(t)$, $x_2(t)$ *are solutions of the SDEs*

$$\begin{aligned} x_1(t) &= \int_0^t \int_A \left[b(s,x_s,a)q_s^\theta(da) - b(s,x_s,a)q_s(da) + b_x(s,x_s,a)x_1(s)q_s(da)\right] ds \\ &+ \int_0^t \int_A \left[\sigma(s,x_s,a)M^\theta(da,ds) - \sigma(s,x_s,a)M(da,ds) + \sigma_x(s,x_s,a)x_1(s)M(da,ds)\right] \end{aligned} \tag{13}$$

$$\begin{aligned} x_2(t) &= \int_0^t \int_A \left[\left(b_x(s,x_s,a)q_s^\theta(da) - b_x(s,x_s,a)q_s(da)\right)x_1(s)\right] ds \\ &+ \int_0^t \int_A \left[b_x(s,x_s,a)x_2(s)q_s(da) + \tfrac{1}{2}b_{xx}(s,x_s,a)q_s(da)x_1(s)x_1(s)\right] ds \\ &+ \int_0^t \int_A \left[\sigma_x(s,x_s,a)x_1(s)M^\theta(da,ds) - \sigma_x(s,x_s,a)x_1(s)M(da,ds)\right] \\ &+ \int_0^t \int_A \left[\sigma_x(s,x_s,a)x_2(s) + \tfrac{1}{2}\sigma_{xx}(s,x_s,a)x_1(s)x_1(s)\right] M(da,ds). \end{aligned} \tag{14}$$

Remark 2. Equation (13) is called the first-order variational equation. It is the variational equation in the usual sense. (14) is called the second-order variational equation, without this equation we can not derive the variational inequality since σ depends explicitly on the control variable.

Notation

1) For simplicity of the notations, we denote by

$$f\left(t, x(t), q_t\right) = \int_A f\left(t, x(t), a\right) q_t(da),$$

and f stands for b, σ, h and their first and second derivatives.

2) We will generically denote by C_k the positive constants that appear in the estimates below and may differ from line to line and from proof to proof.

The proof is inspired from Yong and Zhou (1999), Theorem 4.4, page 128. We need to show that

$$E\left[\sup_{0\leq t\leq T} \left|x_1(t)\right|^2\right] \leq C_k\theta, \tag{15}$$

$$E\left[\sup_{0\leq t\leq T} \left|x_2(t)\right|^2\right] \leq C_k\theta^2. \tag{16}$$

We can write

$$\begin{aligned}
E\left[\left|x_1(t)\right|^2\right] &\leq 4E\left|\int_0^t \int_A \left[b(s,x_s,a)q_s^\theta(da) - b(s,x_s,a)q_s(da)\right] ds\right|^2 \\
&+4E\left|\int_0^t \int_A \left[\sigma(s,x_s,a)M^\theta(da,ds) - \sigma(s,x_s,a)M(da,ds)\right]\right|^2 \\
&+4E\left|\int_0^t \int_A b_x(s,x_s,a)x_1(s)q_s(da)ds + \int_0^t \int_A \sigma_x(s,x_s,a)x_1(s)M(da,ds)\right|^2 \\
&\leq E(I_1) + E(I_2) + E(I_3)
\end{aligned}$$

Since q^θ is defined as in (15), then

$$\begin{aligned}
E(I_1) &\leq 4E\int_0^t \left|\int_A \left[b(s,x_s,a)\delta_v(da) - b(s,x_s,a)q_s(da)\right] 1_E\right|^2 ds \\
&\leq C_k E\int_r^{r+\theta} \left[\left|b(s,x_s,v)\right|^2 + \int_A \left|b(s,x_s,a)\right|^2 \left|q_s(da)\right|^2\right] ds \\
&\leq C_k \int_r^{r+\theta} E\left[1 + \left|x(t)\right|^2\right] ds \\
&\leq C_k \int_r^{r+\theta} \left[1 + E\left(\sup_{0\leq t\leq T} \left|x(t)\right|^2\right)\right] ds \leq C_k(1+\alpha)\theta.
\end{aligned}$$

$$\begin{aligned}E(I_2) &\leq C_k E\left|\int_r^{r+\theta}\int_A [\sigma(s,x_s,a)\delta_\upsilon(da)dB_s - \sigma(s,x_s,a)M(da,ds)]\right|^2 \\ &\leq C_k E\int_r^{r+\theta}\left[|\sigma(s,x_s,\upsilon)|^2 ds + \int_A |\sigma(s,x_s,a)|^2 q_s(da)ds\right] \\ &\leq C_k \int_r^{r+\theta}\left[1 + E\left(\sup_{0\leq t\leq T}|x(t)|^2\right)\right] ds \leq C_k(1+\alpha)\theta\end{aligned}$$

$$\begin{aligned}E(I_3) &\leq C_k E\left[\int_0^t\int_A |b_x(s,x_s,a)|^2 |x_1(s)|^2 |q_s(da)|^2 ds + \int_0^t\int_A |\sigma_x(s,x_s,a)| |x_1(s)|^2 q_s(da)ds\right] \\ &\leq C_k E\left(\int_0^t |x_1(s)|^2 ds\right) \leq C_k \int_0^t E|x_1(s)|^2 ds\end{aligned}$$

Then, we have

$$E|x_1(s)|^2 \leq C_k E\left(\int_0^t |x_1(s)|^2 ds\right) + C_k(1+\alpha)\theta$$

By Gronwall Lemma and Burkholder-Davis-Gundy's inequality, we have

$$E\left[\sup_{0\leq t\leq T}|x_1(t)|^2\right] \leq C_k\theta.$$

As previously, we have

$$\begin{aligned}E\left[|x_2(t)|^2\right] &\leq 6E\left[\int_0^t\int_A [|b_x(s,x_s,a)x_2(s)q_s(da)ds| + |\sigma_x(s,x_s,a)x_2(s)M(da,ds)|]\right]^2 \\ &+ 3E\left[\int_0^t\int_A [|b_{xx}(s,x_s,a)x_1(s)x_1(s)| q_s(da)ds + |\sigma_{xx}(s,x_s,a)x_1(s)x_1(s)| M(da,ds)]\right]^2 \\ &+ 6E\int_0^t\left(\int_A \left|b_x(s,x_s,a)x_1(s)q_s^\theta(da) - b_x(s,x_s,a)x_1(s)q_s(da)\right|\right)^2 ds \\ &+ 6E\left[\int_0^t\int_A \left|\sigma_x(s,x_s,a)x_1(s)M^\theta(da,ds) - \sigma_x(s,x_s,a)x_1(s)M(da,ds)\right|\right]^2\end{aligned}$$

By (15), we have

$$\begin{aligned}E\left|x_2(s)^2\right| &\leq C_k\left(2\int_0^t E|x_2(s)|^2 ds + 4\int_r^{r+\theta}\theta ds + \int_0^t \theta^2 ds\right) \\ &\leq C_k\left(2\int_0^t E|x_2(s)|^2 ds + 4\int_r^{r+\theta}\theta ds + \int_0^T \theta^2 ds\right) \\ &\leq C_k\int_0^t E|x_2(s)|^2 ds + C_k(4+T)\theta^2\end{aligned}$$

Then by Gronwall's and Burkholder-Davis-Gundy's inequalities, we obtain the inequalities (15) and (16).

As in the proof of Theorem 4.4, page 128, set $x_3 = x_1 + x_2$, we have

$$\begin{aligned} b\left(t, x(t)+x_3(t), q_t^\theta\right) &= b\left(t, x(t), q_t^\theta\right) + b_x\left(t, x(t), q_t^\theta\right) x_3(t) \\ &\quad + \int_0^1 \int_0^1 \lambda b_{xx}\left(t, x(t)+\lambda\theta x_3(t), q_t^\theta\right) d\lambda d\theta x_3(t) x_3(t) \\ \sigma\left(t, x(t)+x_3(t), q_t^\theta\right) &= \sigma\left(t, x(t), q_t^\theta\right) + \sigma_x\left(t, x(t), q_t^\theta\right) x_3(t) \\ &\quad + \int_0^1 \int_0^1 \lambda \sigma_{xx}\left(t, x(t)+\lambda\theta x_3(t), q_t^\theta\right) d\lambda d\theta x_3(t) x_3(t) \end{aligned}$$

than, we can write

$$\begin{aligned} &\int_0^t b\left(s, x(s)+x_3(s), q_s^\theta\right) ds + \int_0^t \int_A \sigma\left(s, x(s)+x_3(s), a\right) M^\theta(da, ds) \\ &= x(t) + x_1(t) + x_2(t) - x(0) + \int_0^t B^\theta(s) ds + \Lambda^\theta(t) \end{aligned}$$

where

$$\begin{aligned} B^\theta(s) &= \frac{1}{2} b_{xx}\left(s, x(s), q_s\right)\left(x_2(s)x_2(s) + 2x_1(s)x_2(s)\right) \\ &\quad + \left(b_x\left(s, x(s), q_s^\theta\right) - b_x\left(s, x(s), q_s\right)\right) x_2(s) \\ &\quad + \textstyle\int_0^1\int_0^1 \left[\lambda b_{xx}\left(s, x(s)+\lambda\theta\left(x_1(s)+x_2(s)\right), q_t^\theta\right)\right] d\lambda d\theta \left(x_1(s)+x_2(s)\right)\left(x_1(s)+x_2(s)\right) \\ &\quad - \textstyle\int_0^1\int_0^1 \left[b_{xx}\left(s, x(s), q_s\right)\right] d\lambda d\theta \left(x_1(s)+x_2(s)\right)\left(x_1(s)+x_2(s)\right) \end{aligned}$$

$$\begin{aligned} \Lambda^\theta(t) &= \frac{1}{2}\textstyle\int_0^t\int_A \sigma_{xx}\left(s, x(s), a\right)\left(x_2(s)x_2(s) + 2x_1(s)x_2(s)\right) M(da, ds) \\ &\quad + \textstyle\int_0^t\int_A \sigma_x\left(s, x(s), a\right) x_2(s) M^\theta(da, ds) - \int_0^t\int_A \sigma_x\left(s, x(s), a\right) x_2(s) M(da, ds) \\ &\quad + \textstyle\int_0^t\int_A\int_0^1\int_0^1 \left[\lambda\sigma_{xx}\left(s, x(s)+\lambda\theta\left(x_1(s)+x_2(s)\right), a\right) d\lambda d\theta \left(x_1(s)+x_2(s)\right)\left(x_1(s)+x_2(s)\right)\right. \\ &\quad \left. M^\theta(da, ds)\right] - \textstyle\int_0^t\int_A\int_0^1\int_0^1 \left[\sigma_{xx}\left(s, x(s), a\right) d\lambda d\theta \left(x_1(s)+x_2(s)\right)\left(x_1(s)+x_2(s)\right) M(da, ds)\right] \end{aligned}$$

and we can drive

$$\begin{aligned} x_\theta(t) - x(t) - x_1(t) - x_2(t) &= \int_0^t \int_A \left[b\left(s, x_\theta(s), a\right) - b\left(s, x(s)+x_1(s)+x_2(s), a\right)\right] q_\theta(da) ds \\ &\quad + \int_0^t \int_A \left[\sigma\left(s, x_\theta(s), a\right) - \sigma\left(s, x(s)+x_1(s)+x_2(s), a\right)\right] M^\theta(da, ds) \\ &\quad + \int_0^t B^\theta(s) ds + \Lambda^\theta(t) \end{aligned}$$

The rest of the proof is similar to that of Theorem 4.4 in Yong and Zhou (1999) and is based on a fine estimate of linear SDEs.

We want now to derive a variational inequality which is become from the Taylor expansion and the cost functional with respect to the perturpation of the control variable.

Since q is an optimal relaxed control and from Lemma 2 we can derive.

Lemma 3. *We assume ($\boldsymbol{H}_1$-$\boldsymbol{H}_3$), then the following estimate holds*

$$\begin{aligned} 0 \leq J\left(q^{\theta}\right) - J\left(q\right) \leq E\left[\int_0^T \left(h\left(t, x(t), q_\theta(t)\right) - h\left(t, x(t), q(t)\right)\right) dt\right] \\ +E\left[g_x(x(T))\left(x_1(T) + x_2(T)\right) + \int_0^T h_x\left(t, x_t, q(t)\right)\left(x_1(t) + x_2(t)\right) dt\right] \\ +\tfrac{1}{2}E\left[g_{xx}(x(T))x_1(T)x_1(T) + \int_0^T h_{xx}\left(t, x(t), q(t)\right) x_1(t)x_1(t)dt\right] + o\left(\theta\right) \end{aligned} \tag{17}$$

Since (x, q) is optimal, we have

$$0 \leq E\left[\int_0^T \left(h\left(t, x_\theta(t), q_\theta(t)\right) - h\left(t, x(t), q(t)\right)\right) dt\right] + E\left[g(x_\theta(T)) - g(x(T))\right]$$

we use (12) to get

$$\begin{aligned} 0 \leq E \textstyle\int_0^T \left[h\left(t, x(t) + x_1(t) + x_2(t), q_\theta(t)\right) - h\left(t, x(t), q(t)\right)\right] dt \\ +E\left[g(x(T) + x_1(T) + x_2(T)) - g(x(T))\right] + o\left(\theta\right) \end{aligned} \tag{18}$$

Then by Taylor expansion at the point x for $h\left(t, x + x_1 + x_2, q_\theta\right)$ and $g\left(x + x_1 + x_2\right)$, we have by (15), (16) and (18) can be rewritten as

$$\begin{aligned} 0 \leq o\left(\theta\right) + \alpha(T) + E \textstyle\int_0^T \left[h\left(t, x(t), q_\theta(t)\right) - h\left(t, x(t), q(t)\right)\right] dt \\ +E \textstyle\int_0^T \left[h_x\left(t, x(t), q(t)\right)\left(x_1(t) + x_2(t)\right)\right] dt \\ +\tfrac{1}{2}E \textstyle\int_0^T \left[h_{xx}\left(t, x(t), q(t)\right) x_1(t)x_1(t)\right] dt \\ +E\left[g_x(x(T)\left(x_1(T) + x_2(T)\right)\right] + \tfrac{1}{2}E\left[g_{xx}(x(T))x_1(T)x_1(T)\right] \end{aligned} \tag{19}$$

where $\alpha(T)$ is given by

$$\begin{aligned} \alpha(T) = E\int_0^T \left[h_x\left(t, x(t), q_\theta(t)\right) - h_x\left(t, x(t), q(t)\right)\right]\left(x_1(t) + x_2(t)\right) dt \\ + \frac{1}{2}E\int_0^T h_{xx}\left(t, x(t), q(t)\right)\left(x_1(t)x_2(t) + x_2(t)x_1(t) + x_2(t)x_2(t)\right) dt \\ + \frac{1}{2}E\int_0^T \left[\left(h_{xx}\left(t, x(t), q_\theta(t)\right) - h_{xx}\left(t, x(t), q(t)\right)\right)\left(x_1(t) + x_2(t)\right)\left(x_1(t) + x_2(t)\right)\right] dt \\ + \frac{1}{2}E\left[g_{xx}(x(T))\left(x_1(T)x_2(T) + x_2(T)x_1(T) + x_2(T)x_2(T)\right)\right] \end{aligned}$$

from the definition of q_θ and the assumption ($\mathbf{H}_1$), using (15), (16) and the Cauchy Schwartz inequality, it holds that

$$\alpha(T) \leq o\left(\theta\right)$$

We use this relation and (19) to complete the proof.

3.2. The adjoint processes and the variational inequality

In this subsection, we will introduce the first and second order adjoint processes involved in the stochastic maximum principle and the associated stochastic Hamiltonian system. These are obtained from the first and second variational equations (13) and (14) as well as (17).

3.2.1. The first order terms

The first order estimation calculate the first order derivatives in (17). The linear term in (13) and (14) may treated in the following way (see Bensoussan, 1982). Let ϕ_1 be the fundamental solution of the linear equation

$$\begin{cases} d\phi_1(t) = \int_A b_x(t, x(t), a)\phi_1(t)q_t(da)dt + \int_A \sigma_x(t, x(t), a)\phi_1(t)M(da, dt) \\ \phi_1(0) = I_d \end{cases}$$

This equation is linear with bounded coefficients, then it have a strong unique solution. Moreover ϕ_1 is invertible and it inverse ψ_1 satisfies

$$\begin{cases} d\psi_1(t) = \int_A \left[\psi_1(t)\sigma_x(t, x(t), a)\sigma_x(t, x(t), a) - \psi_1(t)b_x(t, x(t), a)\right] q_t(da)dt \\ -\int_A \psi_1(t)\sigma_x(t, x(t), a)M(da, dt) \\ \psi_1(0) = I_d. \end{cases}$$

ϕ_1 and ψ_1 satisfy

$$E\left[\sup_{t\in[0,T]} |\phi_1(t)|^2\right] + E\left[\sup_{t\in[0,T]} |\psi_1(t)|^2\right] < \infty.$$

We introduce the following processes

$$\eta_1(t) = \psi_1(t)\left(x_1(t) + x_2(t)\right),$$

and

$$X_1 = \phi_1(T)g_x\left(x(T)\right) + \int_0^T \phi_1(s)\int_A h_x(s, x(s), a)q_s(da)ds$$
$$\zeta_1(t) = E\left(X_1/\mathcal{F}_t\right) - \int_0^t \phi_1(s)\int_A h_x(s, x(s), a)q_s(da)ds$$

then

$$E\left[g_x(x(T))\left(x_1(T) + x_2(T)\right)\right] = E\left[\phi_1(T)g_x\left(x(T)\right)\eta_1(T)\right] = E\left[\eta_1(T)\zeta_1(T)\right]$$

from the orthogonal martingale measure representation (Proposition 2) we have

$$E\left(X_1/\mathcal{F}_t\right) = E\left(X_1\right) + \int_0^t \int_A G_1(a, s)M\left(da, ds\right) + L_t,$$

where L is an L^2-martingale with $\left\langle L, \int_0^\cdot \int_E b(a, s)M(da, ds)\right\rangle = 0$ for every $b \in \mathcal{P}_M$ and such that $E\left[\langle L_t\rangle\right] < \infty$.

Applied Ito's formula to $\eta_1(t)\zeta_1(t)$ and we put

$$p_1(t) = \psi_1^*(t)\zeta_1(t), \tag{20}$$

$$Q_1(t) = \int_A \psi_1^*(t)G_1(t,a)q_t(da) - \int_A \sigma_x^*(t,x(t),a)q_t(da))p_1(t) \tag{21}$$

moreover $p_1(t)$, $Q_1(t)$ satisfy

$$E\left[\sup_{0\le t\le T}|p_1(t)|^2 + \sup_{0\le t\le T}|Q_1(t)|^2\right] < \infty,$$

the process p_1 is called the first adjoint process.

We can derive

$$\begin{aligned}
E\left[g_x(x(T))\left(x_1(T)+x_2(T)\right)\right] = E\int_0^T\int_A p_1(t)\left(b(t,x(t),a)q_t^\theta(da) - b(t,x(t),a)q_t(da)\right)dt \\
+ E\int_0^T\int_A \left[Q_1(t)\left(\sigma(t,x(t),a)q_t^\theta(da) - \sigma(t,x(t),a)q_t(da)\right)\right]dt \\
+ \frac{1}{2}E\int_0^T\int_A p_1(t)b_{xx}(t,x(t),a)x_1(t)x_1(t)q_t(da)dt \\
+ \frac{1}{2}E\int_0^T\int_A Q_1(t)\sigma_{xx}(t,x(t),a)x_1(t)x_1(t)q_t(da)dt \\
- E\int_0^T\int_A h_x(t,x(t),a)\left(x_1(t)+x_2(t)\right)q_t(da)dt \\
+ E\int_0^T\int_A p_1(t)\left[\left(b_x(t,x(t),a)q_t^\theta(da) - b_x(t,x(t),a)q_t(da)\right)\right]x_1(t)dt \\
+ E\int_0^T\int_A \left[Q_1(t)\left(\sigma_x(t,x(t),a)q_t^\theta(da) - \sigma_x(t,x(t),a)q_t(da)\right)\right]x_1(t)dt \\
- E\int_0^T\int_A Q_1(t)\left[\sigma(t,x(t),a) + \sigma_x(t,x(t),a)x_1(t)\right]\delta_\nu(da)1_E(t)dt \\
+ E\int_0^T\int_A \psi_1(t)\left[\sigma(t,x(t),a) + \sigma_x(t,x(t),a)x_1(t)\right]\delta_\nu(da)1_E(t)d\left\langle B_t, L_t\right\rangle \\
+ o(\theta).
\end{aligned}$$

To derive our variational inequality, we need to prove the following estimates in the last equality

$$E\int_0^T\int_A p_1(t)\left[\left(b_x(t,x(t),a)q_t^\theta(da) - b_x(t,x(t),a)q_t(da)\right)\right]x_1(t)dt \le C\theta,$$
$$E\int_0^T\int_A \left[Q_1(t)\left(\sigma_x(t,x(t),a)q_t^\theta(da) - \sigma_x(t,x(t),a)q_t(da)\right)\right]x_1(t)dt \le C\theta,$$
$$E\int_0^T\int_A Q_1(t)\left[\sigma(t,x(t),a) + \sigma_x(t,x(t),a)x_1(t)\right]\delta_\nu(da)1_E(t)dt \le C\theta$$

and

$$E\int_0^T\int_A \psi_1(t)\left[\sigma(t,x(t),a)+\sigma_x(t,x(t),a)x_1(t)\right]\delta_\nu(da)1_E(t)d\left\langle B_t,L_t\right\rangle \leq C\theta.$$

By (15) and applying Young's inequality, the first inequality becomes

$$\begin{aligned}
&E\int_r^{r+\theta} p_1(t)\left[b_x(t,x(t),\nu)-b_x(t,x(t),q)\right]x_1(t)dt\\
&\leq C_k E\int_r^{r+\theta}\left[\left[p_1(t)x_1(t)\right]^2+\left[b_x(t,x(t),\nu)-b_x(t,x(t),q)\right]^2\right]dt\\
&\leq C_k\left(\theta+E\int_r^{r+\theta}\left[1+\sup_{0\leq t\leq T}\left|x(t)\right|^2\right]dt\right)\leq C_k\theta
\end{aligned}$$

For the second and the third estimates, we use the same argument as in the first one. For the fourth term we use Kunita-Watanabe inequality

$$\begin{aligned}
&E\left[\int_r^{r+\theta}\int_A \psi_1(t)\left[\sigma(t,x(t),a)+\sigma_x(t,x(t),a)x_1(t)\right]\delta_\nu(da)d\left\langle B_t,L_t\right\rangle\right]\leq\\
&E\left(\int_r^{r+\theta}\int_A \psi_1^2(t)\left[\sigma(t,x(t),a)+\sigma_x(t,x(t),a)x_1(t)\right]^2\delta_\nu(da)dt\right)^{1/2}\times\\
&E\left(\int_r^{r+\theta}\int_A \delta_\nu(da)d\left\langle L_t,L_t\right\rangle\right)^{1/2}\\
&\leq C_k E\left(\int_r^{r+\theta}\psi_1^2(t)\left[\sigma^2(t,x(t),\nu)+\sigma_x^2(t,x(t),\nu)x_1^2(t)\right]dt\right)^{1/2} E\left(\left\langle L,L\right\rangle_{r+\theta}-\left\langle L,L\right\rangle_r\right)^{1/2}
\end{aligned}$$

Using the same arguments the inequality holds since $E\left[\left\langle L_t\right\rangle\right]<\infty$.

Let us now define the Hamiltonian

$$H\left(t,x,q,p,Q\right)=\int_A h\left(t,x,a\right)q(da)+p\int_A b\left(t,x,a\right)q(da)+Q\int_A \sigma\left(t,x,a\right)q(da),$$

Therefore, we use the value of $E\left[g_x(x(T))\left(x_1(T)+x_2(T)\right)\right]$ and the Hamiltonian definition, (17) can be rewritten

$$\begin{aligned}
&0\leq J\left(q^\theta\right)-J\left(q\right) \qquad (22)\\
&\leq E\int_0^T\int_A\left[H\left(t,x(t),a,p_1(t),Q_1(t)\right)q_t^\theta(da)-H\left(t,x(t),a,p_1(t),Q_1(t)\right)q_t(da)\right]dt\\
&+\frac{1}{2}E\int_0^T\int_A x_1(t)H_{xx}(x(t),a,p_1(t),Q_1(t))x_1^*(t)q_t(da)dt+\frac{1}{2}E\left[x_1(T)g_{xx}(x(T))x_1^*(T)\right]+o\left(\theta\right).
\end{aligned}$$

3.2.2. The second order terms

The second order estimation concerns the second order derivatives in (22). As in Peng (1990), let $Z = x_1 x_1^*$. By Itô's formula we obtain

$$\begin{aligned} dZ(t) &= \int_A \left[Z(t)b_x^*(t,x(t),a) + b_x(t,x(t),a)Z(t)\right] q_t(da)dt + \mathbb{B}_\theta\left(t,x(t),a\right) \\ &+ \int_A \sigma_x(t,x(t),a)Z(t)\sigma_x^*(t,x(t),a)q_t(da)dt + \mathbb{A}_\theta\left(t,x(t),a\right)dt \\ &+ \int_A \left(Z(t)\sigma_x^*(t,x(t),a) + \sigma_x(t,x(t),a)Z(t)\right) M\left(da,dt\right) - \mathbb{B}\left(t,x(t),a\right). \end{aligned} \tag{23}$$

For simplicity of notations, we denote by

$$f\left(t\right) = \int_A f\left(t,x(t),a\right) q_t(da),\ \ f_\theta\left(t\right) = \int_A f\left(t,x(t),a\right) q_t^\theta(da)$$

in $\mathbb{A}_\theta$ and in $\mathbb{B}_\theta$, $\mathbb{B}$ by

$$fdM = \int_A f\left(t,x(t),a\right) M\left(da,dt\right),\ \ f_\theta dM^\theta = \int_A f\left(t,x(t),a\right) M^\theta\left(da,dt\right)$$

f stands for b, σ and their first derivatives.

Then we have

$$\begin{aligned} \mathbb{A}_\theta\left(t\right) q_t(da) &= x_1(t)\left(b_\theta^*(t) - b^*(t)\right) + \left(b^\theta(t) - b(t)\right) x_1^*(t) - \sigma_x(t)x_1(t)\sigma^*(t) - \sigma(t)x_1^*(t)\sigma_x^*(t) \\ &+ \left[\left(\sigma_x(t)x_1(t)\sigma_\theta^*(t) + \sigma_\theta(t)x_1^*(t)\sigma_x^*(t)\right) - \left(\sigma_\theta(t)\sigma^*(t) + \sigma(t)\sigma_\theta^*(t)\right)\right] 1_{E^C}\left(t\right) \\ &+ \sigma_\theta(t)\sigma_\theta^*(t) + \sigma(t)\sigma^*(t), \end{aligned}$$

$$\mathbb{B}_\theta\left(t\right) = \sigma_\theta(t)x_1^*(t) + x_1(t)\sigma_\theta^*(t)dM^\theta, \qquad \mathbb{B}\left(t\right) = \sigma(t)x_1^*(t) + x_1(t)\sigma^*(t)dM$$

we remark that

$$E\int_0^T \mathbb{A}_\theta\left(t\right)dt \le E\int_0^T \left[\left(\sigma_\theta(t)\sigma_\theta^*(t) + \sigma(t)\sigma^*(t)\right) - \left(\sigma_\theta(t)\sigma^*(t) + \sigma(t)\sigma_\theta^*(t)\right) 1_{E^C}\left(t\right)\right]dt + \circ\left(\theta\right)$$

$$E\int_0^T \mathbb{B}_\theta\left(t\right)dM^\theta \le \circ\left(\theta\right) \text{ and } E\int_0^T \mathbb{B}_\theta\left(t\right)dM \le \circ\left(\theta\right).$$

As in the first order estimation, we consider now the following symmetric matrix-valued linear equation associate to (23)

$$\begin{cases} d\phi_2(t) = \int_A \left[\phi_2(t)b_x^*(t,x(t),a) + b_x(t,x(t),a)\phi_2(t) + \sigma_x(t,x(t),a)\phi_2(t)\sigma_x^*(t,x(t),a)\right] q_t(da)dt \\ \qquad\qquad + \int_A \left(\phi_2(t)\sigma_x^*(t,x(t),a) + \sigma_x(t,x(t),a)\phi_2(t)\right) M(da,dt) \\ \phi_2(0) = I_d \end{cases}$$

This equation is linear with bounded coefficients, hence it admit a unique strong solution. Moreover ϕ_2 is invertible and it inverse ψ_2 satisfies

$$\begin{cases} d\psi_2(t) = \int_A \left[(\sigma_x(t,x(t),a) + \sigma_x^*(t,x(t),a))^2 \psi_2(t) - \psi_2(t) b_x^*(t,x(t),a) \right] q_t(da)dt \\ \qquad - \int_A \left[b_x(t,x(t),a)\psi_2(t) + \sigma_x(t,x(t),a)\psi_2(t)\sigma_x^*(t,x(t),a) \right] q_t(da)dt \\ \qquad - \left[\psi_2(t)\sigma_x^*(t,x(t),a) + \sigma_x(t,x(t),a)\psi_2(t) \right] M(da,dt) \\ \psi_2(0) = I_d \end{cases}$$

It is easy to see that ϕ_2 and ψ_2 satisfy

$$E\left[\sup_{t\in[0,T]} |\phi_2(t)|^2 \right] + E\left[\sup_{t\in[0,T]} |\psi_2(t)|^2 \right] < \infty.$$

Using the same arguments as for the first order terms, we introduce the processes $\eta_2(t) = \psi_2(t)Z(t)$ and

$$X_2 = \phi_2^*(T) g_{xx}(x(T)) + \int_0^T \phi_2^*(s) \int_A H_{xx}(s,x(s),a) q_s(da) ds$$

$$\zeta_2(t) = E(X_2/\mathcal{F}_t) - \int_0^t \phi_2^*(s) \int_A H_{xx}(s,x(s),a) q_s(da) ds$$

We remark from these equality that

$$E\left[x_1(T) g_{xx}(x(T)) x_1^*(T) \right] = E\left[\phi_2^*(T) g_{xx}(x(T)) \eta_2(T) \right] = E\left[\eta_2(T)\zeta_2(T) \right]$$

The orthogonal martingale measure representation (Proposition 2) give us

$$E(X_2/\mathcal{F}_t) = E(X_2) + \int_0^t \int_A G_2(a,s) M(da,ds) + L'_t \tag{24}$$

where L' is an L^2-martingale with $\left\langle L', \int_0^\cdot \int_E b(a,s)M(da,ds) \right\rangle = 0$ for every $b \in \mathcal{P}_M$ and such that $E\left[\langle L'_t \rangle\right] < \infty$.

Apply Itô's formula to $\eta_2(t)\zeta_2(t)$, to obtain

$$\begin{aligned} E\left[x_1(T) g_{xx}(x(T)) x_1^*(T) \right] = -E \int_0^T \int_A x_1(t) H_{xx}(t,x(t),a) x_1^*(t) q_t(da) dt + \\ E\int_0^T \int_A tr\left[\left(\sigma(t,x(t),a) q_t^\theta(da) - \sigma(t,x(t),a) q_t(da) \right)^* p_2(t) \times \right. \\ \left. \left(\sigma(t,x(t),a) q_t^\theta(da) - \sigma(t,x(t),a) q_t(da) \right) \right] dt + o(\theta) \end{aligned} \tag{25}$$

where

$$p_2(t) = \psi_2^*(t)\zeta_2(t) \tag{26}$$

the process p_2 is called the second adjoint process.

3.2.3. The adjoint equations and the maximum principle

By applying Ito's formula to the adjoint processes p_1 in (20) and p_2 in (26), we obtain the first and second order adjoint equations, which have the forms

$$\begin{cases} -dp_1(t) = \int_A \left[b_x^*(t,x(t),a)\, p_1(t) + \sigma_x^*(t,x(t),a)\, Q_1(t) + h_x(t,x(t),a)\right] q_t(da)dt \\ \qquad\qquad - \int_A Q_1(t) M(da,dt) - \psi_1^*(t) dL_t \\ p_1(T) = g_x(x(T)). \end{cases} \tag{27}$$

with values in $\mathbb{R}^d$, where L is an L^2-martingale with $\left\langle L, \int_0^{\cdot} \int_E b(a,s) M(da,ds) \right\rangle = 0$ for every $b \in \mathcal{P}_M$, Q_1 is given by (21) with values in $\mathbb{R}^{d\times k}$. The adjoint equation that $p_1(.)$ satisfies is a linear backward stochastic differential equation**.** This BSDE has a unique adapted solution.

Using Itô's formula it is easy to see that p_2 is matrix valued and satisfies

$$\begin{cases} -dp_2(t) = \int_A \left[b_x^*(t,x(t),a)\, p_2(t) + p_2(t) b_x(t,x(t),a) + \right. \\ \qquad\qquad \left. \sigma_x^*(t,x(t),a)\, Q_2(t) + Q_2(t)\sigma_x(t,x(t),a)\right] q_t(da)dt \\ \qquad\qquad + \int_A \left[\sigma_x^*(t,x(t),a)\, p_2(t)\sigma_x(t,x(t),a) + H_{xx}(x(t),a,p_1(t),Q_1(t))\right] q_t(da)dt \\ \qquad\qquad - \int_A Q_2(t) M(da,dt) - \psi_2^*(t) dL'_t \\ p_2(T) = g_{xx}(x(T)), \end{cases} \tag{28}$$

where L' is given by (24) and Q_2 is given by

$$Q_2(t) = \int_A \left[\psi_2^*(t) G_2(t,a) - p_2(t)\sigma_x(t,x(t),a) + \sigma_x^*(t,x(t),a) p_2(t)\right] q_t(da) \tag{29}$$

Note that $p_2(.)$ is also a backward stochastic differential equation with matrix-valued unknowns**.** This BSDE have a unique adapted solution.

Remark 3. $H_{xx}(x(t),q_t,p(t),Q(t))$ is the second derivative of the Hamiltonian H at x and it is given by $H_{xx}(x(t),q_t,p(t),Q(t)) = h_{xx}(t,x(t),q_t) + p(t)b_{xx}(t,x(t),q_t) + Q(t)\sigma_{xx}(t,x(t),q_t)$.

We are ready now to state the main result of this paper.

Theorem 3 (The stochastic maximum principle). *Let q be an optimal control minimizing the cost J over $\mathcal{R}$ and x denotes the corresponding optimal trajectory. Then there are two unique couples of adapted processes (p_1, Q_1) and (p_2, Q_2) which are respectively solutions of the backward stochastic differential equations (27) and (28) such that*

$$\begin{array}{c} 0 \leq H(t,x(t),\nu,p_1(t),Q_1(t)) - H(t,x(t),q_t,p_1(t),Q_1(t)) \\ +\frac{1}{2} tr\left[(\sigma(t,x(t),\nu) - \sigma(t,x(t),q_t))^* p_2(t)(\sigma(t,x(t),\nu) - \sigma(t,x(t),q_t))\right] \end{array} \tag{30}$$

υ is an arbitrary $\mathcal{F}_r$-measurable random variable with values in $\mathcal{U}$, such that

$$\sup_{w\in\Omega} |\upsilon(w)| < \infty.$$

Proof. From (25) and (22) can be rewritten

$$
\begin{aligned}
0 &\leq J\left(q^{\theta}\right) - J\left(q\right) \\
&\leq E\int_0^T \int_A \left[H\left(t, x(t), a, p_1(t), Q_1(t)\right) q_t^{\theta}(da) - H\left(t, x(t), a, p_1(t), Q_1(t)\right) q_t(da)\right] dt + \circ\left(\theta\right) \\
&+\tfrac{1}{2}E\int_0^T \int_A tr\left[\left(\sigma^{\theta}(t, x(t), a)q_t^{\theta}(da) - \sigma(t, x(t), a)q_t(da)\right)^* p_2(t) \times \right. \\
&\qquad \left.\left(\sigma^{\theta}(t, x(t), a)q_t^{\theta}(da) - \sigma(t, x(t), a)q_t(da)\right)\right] dt.
\end{aligned}
$$

This equation is the variational inequation of the second order.
We use the definition of q_{θ}, the last variational inequality becomes

$$
\begin{aligned}
0 &\leq \tfrac{1}{\theta}\left(J\left(q^{\theta}\right) - J\left(q\right)\right) \\
&\leq \tfrac{1}{\theta}E\int_r^{r+\theta} \left[H\left(t, x(t), \nu, p_1(t), Q_1(t)\right) - H\left(t, x(t), q_t, p_1(t), Q_1(t)\right)\right] dt + o\left(\theta\right) \\
&+\tfrac{1}{2\theta}E\int_r^{r+\theta} tr\left[\left(\sigma(t, x(t), \nu) - \sigma\left(t, x(t), q_t\right)\right)^* p_2(t)\left(\sigma(t, x(t), \nu) - \sigma(t, x(t), q_t)\right)\right] dt,
\end{aligned}
$$

Then, the desired result follows by letting θ going to zero.

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