**Afrika Statistika**
Vol. 11(1), 2016, pages 933–942.
DOI: http://dx.doi.org/10.16929/as/2016.933.84

Afrika Statistika

ISSN 2316-090X

# Association measures and estimation of copula parameters

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Received March 18, 2016; Accepted May 10, 2016



**Abstract.** We apply the inversion method of estimation, with several combinations of two among the four most popular association measures, to estimate the parameters of copulas in the case of bivariate distributions. We carry out a simulation study with two examples, namely Farlie-Gumbel-Morgenstern and Marshall-Olkin two-parameter copulas to make comparisons between the obtained estimators, with respect to bias and root of the mean squared error.

**Résumé.** Nous appliquons la méthode d'inversion, avec plusieurs combinaisons de deux parmi les quatre mesures d'association les plus populaires, pour estimer les paramètres de copules dans le cas de distributions bivariées. Nous réalisons une étude de simulation sur deux exemples, à savoir les copules à deux paramètres de Farlie-Gumbel-Morgenstern et de Marshall-Olkin, pour faire des comparaisons entre les estimateurs, en matière du biais et de la racine de l'erreur quadratique moyenne.

**Key words:** Association measures; Copula; Dependence; Inversion method; Sklar's theorem.
**AMS 2010 Mathematics Subject Classification :** 62H12; 62H20; 62F07.

## 1. Introduction

Dependence relations between random variables (rv's) are one of the most important issues that got a great deal of interest in probability and statistics. Such dependence is appropriately modelled by a very useful tool for handling multivariate distributions with given univariate marginals, known as copula. As mentioned by Fisher (1997), the copulas are of interest to statisticians for two main reasons: first, they represent a way of studying scale-free measures of dependence and second, they may be considered as a starting point for constructing families of multivariate distributions. The concept of copulas has become very

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useful in real life applications as diverse as risk management, reliability, survival analysis, finance, actuarial and medical sciences. An exhaustive list of copula applications can be found in, for instance, Balakrishnan and Lai (2009), pages 55-58, with full details. The notion of copula is used by Brahimi *et al.* (2010) to analyze the distortion risk measures of the sum of two or more insurance losses, where the dependence structure is a very significant factor. A copula is a mean of linking a multivariate distribution function (df) with its margins. Indeed, if $X^{(1)}, ..., X^{(d)}$ are $d \geq 2$ rv's with joint df $F$ and margins $F_i$, $i = 1, ..., d$, then according to Sklar's Theorem Sklar (1959), there exists a copula $C : [0,1]^d \to [0,1]$ such that

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)).$$

In other words, the copula $C$ is the joint df of the $(0,1)$-uniform rv's $U_i := F_i\left(X^{(i)}\right)$, $i = 1, ..., d$. It is defined on $[0,1]^d$ by

$$C(u_1, ..., u_d) = F\left(F_1^{\leftarrow}(u_1), ..., F_d^{\leftarrow}(u_d)\right),$$

where $F_i^{\leftarrow}(s) := \inf\{x \in \mathbb{R} \backslash F_i(x) \geq s\}$, $0 \leq s \leq 1$, denotes the generalized inverse or quantile function of df $F_i$. Note that if all the margins $F_i$ are continuous, then $C$ is unique. In the sequel and for the sake of simplicity, we will restrict ourselves to the two-dimensional case, i.e. we take $d = 2$. For a full description of copulas and comprehensive details on their properties, we refer the reader to the textbooks of Cherubini *et al.* (2004), Joe (1997) and Nelsen (2006).

In the process of modelling the dependence between two (ore more) rv's, we should look for a copula with appropriate parametric form. For each specific problem, two of the fundamental characteristics required in our search are flexibility and analytical simplicity. This leads to the availability of various families of copulas, among which we may cite, in addition of the trivial ones (product, maximum and minimum), the elliptical copulas (Gauss, Student,...), Archimedien copulas (Gumbel-Hougaard, Clayton, Frank,...) and extreme value copulas. Any copula $C$ is delimited by what is called Hoeffding and Fréchet bounds generally denoted by $W$ and $M$, which are trivial copulas known as minimum and maximum copulas respectively. That is, for $0 \leq u, v \leq 1$, we have

$$W(u,v) \leq C(u,v) \leq M(u,v),$$

where $W(u,v) := \max(u + v - 1, 0)$ and $M(u,v) := \min(u,v)$.

The central issue in statistical modelling is the estimation of the parameters upon which the probability distribution depends. In the case of copulas, there is a variety of estimation procedures available in the literature. Depending on the situation, one may consider parametric, semi-parametric or non-parametric copula inference methods based on independent and identically distributed observations of random vectors with dependent components. The forms of the joint distribution and its marginals play a crucial role in choosing the right estimation approach. For this matter, the authors of Choros *et al.* (2004) provide a detailed survey that may be summarized into:
- Parametric model: exact maximum likelihood methods, inference from likelihoods for margins and inversion method of association measures.
- Semi-parametric model: maximum pseudo-likelihood and canonical maximum likelihood.
- Non-parametric model: empirical copula processes.

We must emphasize that, in this paper, we don't directly deal with the estimation of the copulas themselves, but we are rather concerned with estimating their parameters. In multivariate statistical analysis, the inference on copula parameters represents a major topic that has got a great deal of interest from several authors. For an overview of the different estimation methods, see the introductions of Brahimi and Necir (2012) and Brahimi *et al.* (2015) (and the references therein) where the authors recently applied the notions of moments and bivariate L-moments (BLM's) of copulas to provide new parameter estimators which they compared to the already existing ones. However, a close look at their comparison results shows that the conclusions are not always against the inversion method. Indeed, there are instances where this latter produces estimators of better performance, mainly from the root of the mean squared error (rmse) perspective (this will be confirmed later on in Section 3). This suggests that one should be more or at least equally attracted by this method of copula parameters estimation. In addition to this, we think that the inversion approach still has the advantage because it is based on the concordance coefficients which are familiar quantities, expressed as simple functions of the copula and whose empirical counterparts are implemented in most statistical software such as **R**, for simulations and application needs. All this motivated us to focus on the inversion method and to look for the combination of association measures that yield the most accurate parameter estimates. Another reason of our interest in this method is the question raised in the introduction of Brahimi and Necir (2012) with respect to the choice of the appropriate association measures with which the estimation is to be made. Thus, this work may be seen as an attempt to answer that question. Finally, note that, in our study, we consider the bivariate case with two particular classes of copulas, namely Farlie-Gumbel-Morgenstern (FGM) and Marshall-Olkin (MO) two-parameter copulas, that we briefly describe below.

### *1.1. Marshall-Olkin copula*

This copula, also known as generalized Cuadras-Augé copula, is mainly used in reliability, finance, insurance... It originates from a concrete model assumption which can easily be used to simulate pseudo rv's. It may be recalled that rv's $X$ and $Y$ with a MO copula are obtained from independent and exponentially distributed rv's $Z_1$, $Z_2$ and $Z_{12}$, with respective parameters $\lambda_1$, $\lambda_2$ and $\lambda_{12}$, by $X := \min(Z_1, Z_{12})$ and $Y := \min(Z_2, Z_{12})$. The respective df's of $X$ and $Y$

$$F(x) = 1 - \exp\{-(\lambda_1 + \lambda_{12})\, x\} \text{ and } G(y) = 1 - \exp\{-(\lambda_2 + \lambda_{12})\, y\},$$

are readily obtained. Therefore, MO copula with two parameters $a$ and $b$ is given by

$$C^{MO}_{a,b}(u, v) = \min\left\{vu^{1-a}, uv^{1-b}\right\} = \begin{cases} vu^{1-a}, & u^a \geq v^b, \\ uv^{1-b}, & u^a \leq v^b, \end{cases}$$

where $a := \lambda_{12}/(\lambda_1 + \lambda_{12})$ and $b := \lambda_{12}/(\lambda_2 + \lambda_{12})$ meaning that $0 \leq a, b \leq 1$. This copula has the particularity to possess an absolutely continuous component and a singular one. Note that when $a = b$, we have $\lambda_1 = \lambda_2$ and so $X$ and $Y$ are exchangeable. For more details on MO copulas, we refer to Embrechts *et al.* (2010) and Marshall and Olkin (1967).

*1.2. Farlie-Gumbel-Morgenstern copula*

This is one of the most popular parametric families of copulas, which is particularly applied in insurance, hydrology, the health field,... The FGM family with one parameter $a$ is defined by

$$C_a^{FGM}(u,v) = uv\left[1 + a(1-u)(1-v)\right], \ -1 \leq a \leq 1.$$

The corresponding correlation coefficient is equal to $a/3$, which clearly lies in the interval $[-1/3, 1/3]$. In other words, the maximum degree of dependence that an FGM copula can describe does not exceed one third (in absolute value). This represents a limitation to this family as it does not allow the modeling of high dependences.

**Iterated FGM copula.** For some integer $r \geq 1$, Johnson and Kotz (1977) introduced the $(r-1)$-iterated FGM family with $r$-dimensional parameter $\theta = (\theta_1, ..., \theta_r)$, where $|a| \leq 1$ such us

$$C_\theta^{FGM}(u,v) = uv + \sum_{j=1}^{r} \theta_j \,(uv)^{[j/2]+1} \,(\bar{u}\bar{v})^{[j/2+1/2]},$$

where $\bar{w} := 1 - w$ and $[t]$ denotes the greatest integer less than or equal to $t$. For $r = 2$ and $\theta = (a, b)$, we get the one-iterated FGM family as follows

$$C_{a,b}^{FGM}(u,v) = uv[1 + (1-u)(1-v)(a + buv))],$$

where the valid combinations of $a$ and $b$ are $-1 \leq a \leq 1$ and $-1 - a \leq b \leq (3 - a + \sqrt{9 - 6a - 3a^2})/2$. For further details on FGM distributions, we refer to Bekrizadeh *et al.* (2012), Huang and Kotz (1984) and Lin (1987).

## 2. Association measures

Four of the most common non-parametric measures of association between the components of a continuous random vector $(X, Y)$ are Kendall's tau $\tau$, Spearman's rho $\rho$, Gini's gamma $\gamma$ and Blomqvist's beta $\beta$ (also known as the medial correlation coefficient). These measures, which only depend on the copula $C$ pertaining to the pair $(X, Y)$, are respectively equal to

$$\tau = 1 - 4\int \frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v)dudv, \ \rho = 12\int uvC(u,v)dudv - 3, \tag{1}$$

$$\gamma = 4\left[\int C(u, 1-u)du - \int (u - C(u,u))\,du\right] \text{ and } \beta = 4 \times C\left(\frac{1}{2}, \frac{1}{2}\right) - 1. \tag{2}$$

It is noteworthy that the aforementioned coefficients lie between $-1$ and 1. For a further discussion of their properties, see, for instance, Nelsen (2006). Applying the formulas (1) and (2) to MO and FGM copulas, with two parameters $a$ and $b$, yields the values that we summarize in Table 1, where

$$\lambda := \frac{u_0^{3-a}}{3-a} + \frac{(1-u_0)^{3-b}}{3-ab} - \frac{{u_0}^{2-a}}{2-a} - \frac{(1-u_0)^{2-b}}{2-b} + \frac{1}{2-a} + \frac{1}{2-b} - \frac{1}{3-a} - \frac{1}{3-b},$$

with $u_0$ being solution of $u^a - (1-u)^b = 0$.

| | MO copula | FGM copula |
|---|---|---|
| Kendall's $\tau$ | $ab/(a+b-ab)$ | $(100a+25b+ab)/450$ |
| Spearman's $\rho$ | $3ab/(2a+2b-ab)$ | $(4a+b)/12$ |
| Gini's $\gamma$ | $4\lambda-2+\dfrac{4}{3-min(a,b)}$ | $(4a+b)/15$ |
| Blomqvist's $\beta$ | $2^{\min(a,b)}-1$ | $(4a+b)/16$ |

**Table 1.** Association measures of FGM and MO copulas with two parameters.

### *2.1. Kendall's tau*

Kendall's tau is defined in terms of concordance as follows. A sample of size $n \geq 2$ is drawn from a random vector $(X, Y)$, then there are $\binom{n}{2}$ distinct pairs $(x_i, y_i)$ and $(x_j, y_j)$ in this sample. If $(x_i - x_j)(y_i - y_j) > 0$ the pairs are said to be concordant, otherwise they are discordant. Let $n_c$ be the number of concordant pairs and $n_d$ that of discordant ones. Then the sample Kendall's tau is defined as

$$\tau_{emp} := \frac{2}{n(n-1)}(n_c - n_d).$$

Note that $n_c - n_d = \sum_{j=2}^{n}\sum_{i=1}^{j-1} sign\{(x_i - x_j)(y_i - y_j)\}$, where $sign\{z\} = 1$ if $z > 0$ and $-1$ if $z < 0$. If $(x_i - x_j)(y_i - y_j) = 0$ the pair is neither concordant nor discordant.

### *2.2. Spearman's rho*

The empirical Spearman's rho is defined as the correlation coefficient of the ranks of $X$ and $Y$. It is equal to

$$\rho_{emp} := 1 - \frac{6\sum_{i=1}^{n}(R_{x_i} - R_{y_i})^2}{n(n^2-1)},$$

where $R_{x_i}$ and $R_{y_i}$ are the ranks of $X$ and $Y$ respectively.

### *2.3. Gini's gamma*

The sample Gini's gamma is defined by

$$\gamma_{emp} := \frac{1}{[n^2/2]}\left\{\sum_{i=1}^{n}|R_{x_i} + R_{y_i} - (n+1)| - \sum_{i=1}^{n}|R_{x_i} - R_{y_i}|\right\}.$$

### *2.4. Blomqvist's beta*

The empirical Blomqvist's beta is defined in terms of the four quadrants as follows

$$\beta_{emp} := \frac{n_1 - n_2}{n_1 + n_2},$$

where $n_1$ is the number of points located in either the upper right quadrant $(Q_1)$ or the lower left quadrant $(Q_3)$ and $n_2$ is the number of points located in either the upper left quadrant $(Q_2)$ or the lower right quadrant $(Q_4)$. If we set $\mathbf{I} := [0, 1]$, the we have

$$Q_1 := \{(x, y) \in \mathbf{I}^2; \tfrac{1}{2} < x < 1, \tfrac{1}{2} < y < 1\}, \quad Q_2 := \{(x, y) \in \mathbf{I}^2; 0 < x < \tfrac{1}{2}, \tfrac{1}{2} < y < 1\},$$
$$Q_3 := \{(x, y) \in \mathbf{I}^2; 0 < x < \tfrac{1}{2}, 0 < y < \tfrac{1}{2}\}, \quad Q_4 := \{(x, y) \in \mathbf{I}^2; \tfrac{1}{2} < x < 1, 0 < y < \tfrac{1}{2}\}.$$

In other words, $Q_1$ and $Q_3$ are defined by the inequality $\left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right) > 0$, whereas $Q_2$ and $Q_4$ satisfy the inequality $\left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right) < 0$.

## 3. Simulation comparative study

This study is intended to evaluate and compare, with respect to bias and rmse, the copula parameter estimators obtained by the inversion procedure using all possible combinations of two association measures chosen among Kendall's tau, Spearman's rho, Gini's gamma and Blomqvist's beta. To this end, we make use of several packages of the statistical software **R.** Copula values are generated by means of the package **copula** and parameter estimates are calculated using the package **BB,** which permits to solve systems of nonlinear equations. Also, the package **asbio** was needed for the computation of the empirical Blomqvist's beta. On the other hand, for comparison purposes as well, we use the package **lmomco** to do the computations relative to BLM based estimators. This method of copula parameter estimation was very recently introduced in Brahimi *et al.* (2015), where the authors provide complete details on the notion of BLM's of copulas.

The choice of the parameters $a$ and $b$ is made in such a way that we cover the three dependence cases, namely weak, moderate and strong. For each copula, we generate 1000 samples of size $n = 100$, and for every combination of two association measures we determine the estimate values of $a$ and $b$ and then we compute their biases and rmse's. We do likewise with the BLM estimator. Our overall results, summarized in Tables 2, 3 and 4 for MO copula and Tables 5, 6 and 7 for FGM copula, are taken as the empirical means of the results obtained through all repetitions. The same simulation procedure is repeated for a second sample size $n = 500$. Finally, it is noteworthy that when we speak of strong dependence for FGM copula, we mean the maximum degree of dependence that it can cover. With this regard, we carried out a very large number of simulations and found that the absolute value of this top level does not exceed 0.5 for any association coefficient.

On the light of these results, one may draw two overall conclusions. First, regardless of the combination of association measures and the dependence level, the estimation for MO copula is by far more accurate compared to that related to FGM copula. Second, the first parameter of both copulas is generally more precisely estimated than the second one. We also have some remarks that are specific to each one of the copulas :

- MO copula : for the second parameter, the best estimates at any level of dependence, are those based on Kendall's tau and Spearman's rho. However, as far as the first parameter is concerned, the conclusion is not as clear cut. Indeed, the inversion of Kendall's tau and Spearman's rho does not always yield the best estimate values. For instance, in the cases of strong and weak dependences, the estimators of the first parameter built via Kendall's tau and Blomqvist's beta seem to be the most accurate.

- FGM copula : for both parameters, Kendall's tau and Spearman's rho based estimators perform quite well in all three dependence cases. Moreover, they are better than those obtained through the other combinations of association measures which can often be of extremely poor reliability. For this reason, there must be no hesitation in picking the combination of Kendall's tau and Spearman's rho, when dealing with the estimation of the parameters of FGM copula by the inversion method.

Finally, the separate line at the bottom of each table in the Appendix indicates that, from the rmse viewpoint, the first parameter estimator based on the appropriate couple of association measures outclasses that obtained by means of the BLM's. This would mean that one should not neglect nor underrate the inversion method when it comes to estimate copula parameters.

**Acknowledgements.** We are grateful to the reviewer for his pertinent comments which allowed us to improve our work.

## 4. Appendix

In this section, we summarize the results of the simulations performed in Section 3. The first three tables consist in those related to MO copula whereas the other three concern FGM copula.

| | $a = 0.8$ | | $b = 0.6$ | | $a = 0.8$ | | $b = 0.6$ | |
|---|---|---|---|---|---|---|---|---|
| | bias | rmse | bias | rmse | bias | rmse | bias | rmse |
| $\tau - \rho$ | −0.036 | 0.154 | 0.050 | 0.182 | −0.030 | 0.125 | 0.038 | 0.117 |
| $\tau - \beta$ | 0.030 | 0.006 | 0.220 | 0.226 | 0.026 | 0.036 | 0.219 | 0.222 |
| $\tau - \gamma$ | −0.404 | 0.449 | 1.876 | 1.883 | −0.404 | 0.404 | 1.883 | 1.884 |
| $\rho - \beta$ | 0.045 | 0.065 | 0.243 | 0.248 | 0.042 | 0.052 | 0.242 | 0.244 |
| $\rho - \gamma$ | −0.405 | 0.408 | 1.874 | 1.882 | −0.404 | 0.405 | 1.882 | 1.884 |
| $\beta - \gamma$ | 0.200 | 0.200 | 2.515 | 2.516 | 0.200 | 0.200 | 2.516 | 2.516 |
| $BLM$ | −0.766 | 0.767 | −0.101 | 0.108 | −0.759 | 0.762 | −0.086 | 0.090 |

**Table 2.** Estimaton biases and rmse's for MO copula parameters under strong dependence, based on 1000 samples of sizes 100 (left panel) and 500 (right panel).

| | $a = 0.5$ | | $b = 0.4$ | | $a = 0.5$ | | $b = 0.4$ | |
|---|---|---|---|---|---|---|---|---|
| | bias | rmse | bias | rmse | bias | rmse | bias | rmse |
| $\tau - \rho$ | 0.001 | 0.091 | 0.019 | 0.180 | −0.005 | 0.065 | 0.013 | 0.097 |
| $\tau - \beta$ | 0.064 | 0.098 | 0.406 | 0.480 | 0.068 | 0.107 | 0.408 | 0.457 |
| $\tau - \gamma$ | −0.252 | 0.258 | 1.540 | 1.569 | −0.251 | 0.252 | 1.565 | 1.570 |
| $\rho - \beta$ | 0.073 | 0.114 | 0.363 | 0.448 | 0.076 | 0.111 | 0.388 | 0.442 |
| $\rho - \gamma$ | −0.252 | 0.260 | 1.538 | 1.570 | −0.251 | 0.253 | 1.565 | 1.570 |
| $\beta - \gamma$ | 2.124 | 2.545 | 1.486 | 1.808 | 2.315 | 2.706 | 1.358 | 1.676 |
| $BLM$ | −0.474 | 0.475 | 0.006 | 0.024 | −0.467 | 0.468 | 0.005 | 0.012 |

**Table 3.** Estimation biases and rmse's for MO copula parameters under moderate dependence, based on 1000 samples of sizes 100 (left panel) and 500 (right panel).

| | $a = 0.1$ | | $b = 0.4$ | | $a = 0.1$ | | $b = 0.4$ | |
|---|---|---|---|---|---|---|---|---|
| | bias | rmse | bias | rmse | bias | rmse | bias | rmse |
| $\tau - \rho$ | 0.015 | 0.064 | 0.047 | 0.209 | −0.001 | 0.035 | 0.009 | 0.146 |
| $\tau - \beta$ | 0.025 | 0.041 | 0.475 | 1.156 | 0.009 | 0.019 | 0.366 | 0.714 |
| $\tau - \gamma$ | −0.014 | 0.046 | 0.717 | 1.182 | −0.019 | 0.032 | 0.843 | 1.161 |
| $\rho - \beta$ | 0.047 | 0.086 | 0.738 | 1.498 | 0.022 | 0.049 | 0.540 | 1.528 |
| $\rho - \gamma$ | −0.011 | 0.043 | 0.726 | 1.200 | −0.020 | 0.033 | 0.962 | 1.357 |
| $\beta - \gamma$ | 1.436 | 1.448 | 0.639 | 0.855 | 1.468 | 1.469 | 0.600 | 0.600 |
| $BLM$ | −0.076 | 0.080 | 0.019 | 0.027 | −0.074 | 0.078 | 0.012 | 0.014 |

**Table 4.** Estimation biases and rmse's for MO copula parameters under weak dependence, based on 1000 samples of sizes 100 (left panel) and 500 (right panel).

| | $a = 0.9$ | | $b = 1.5$ | | $a = 0.9$ | | $b = 1.5$ | |
|---|---|---|---|---|---|---|---|---|
| | bias | rmse | bias | rmse | bias | rmse | bias | rmse |
| $\tau - \rho$ | −0.054 | 0.311 | 0.178 | 1.525 | −0.034 | 0.199 | 0.143 | 0.963 |
| $\tau - \beta$ | −0.576 | 1.296 | 2.406 | 5.918 | −0.377 | 0.918 | 1.581 | 4.040 |
| $\tau - \gamma$ | −0.147 | 0.588 | 0.577 | 2.569 | −0.022 | 0.286 | 0.098 | 1.173 |
| $\rho - \beta$ | 0.605 | 10.572 | −2.489 | 43.021 | −0.169 | 4.131 | 0.680 | 16.812 |
| $\rho - \gamma$ | 0.203 | 6.493 | −0.874 | 26.498 | −0.427 | 10.359 | 1.712 | 41.452 |
| $\beta - \gamma$ | 0.491 | 7.131 | −2.031 | 29.341 | 0.168 | 3.778 | −0.664 | 15.322 |
| $BLM$ | −0.008 | 0.449 | 0.007 | 1.417 | −0.003 | 0.202 | 0.027 | 0.615 |

**Table 5.** Estimation biases and rmse's for FGM copula parameters under strong dependence, based on 1000 samples of sizes 100 (left panel) and 500 (right panel).

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| | $a=-0.1$ | | $b=3$ | | $a=-0.1$ | | $b=3$ | |
|---|---|---|---|---|---|---|---|---|
| | bias | rmse | bias | rmse | bias | rmse | bias | rmse |
| $\tau-\rho$ | 0.046 | 0.413 | −0.206 | 1.634 | 0.044 | 0.216 | −0.160 | 0.911 |
| $\tau-\beta$ | −0.283 | 0.965 | 1.236 | 4.797 | −0.165 | 0.652 | 0.733 | 3.039 |
| $\tau-\gamma$ | −0.055 | 0.509 | 0.225 | 2.400 | 0.017 | 0.313 | −0.045 | 1.346 |
| $\rho-\beta$ | 0.431 | 11.626 | −1.753 | 47.316 | −0.120 | 5.007 | 0.495 | 20.383 |
| $\rho-\gamma$ | 0.241 | 9.323 | −0.999 | 37.942 | 0.279 | 6.956 | −1.101 | 27.879 |
| $\beta-\gamma$ | 0.282 | 8.495 | −1.154 | 34.885 | 0.132 | 3.742 | −0.512 | 15.354 |
| $BLM$ | 0.002 | 0.497 | −0.020 | 1.574 | −0.002 | 0.267 | 0.026 | 0.675 |

**Table 6.** Estimation biases and rmse's for FGM copula parameters under moderate dependence, based on 1000 samples of sizes 100 (left panel) and 500 (right panel).

| | $a=0.3$ | | $b=-1.3$ | | $a=0.3$ | | $b=-1.3$ | |
|---|---|---|---|---|---|---|---|---|
| | bias | rmse | bias | rmse | bias | rmse | bias | rmse |
| $\tau-\rho$ | 0.016 | 0.362 | −0.034 | 1.650 | −0.013 | 0.214 | 0.072 | 0.737 |
| $\tau-\beta$ | −0.610 | 1.406 | 2.635 | 6.413 | −0.374 | 0.925 | 1.585 | 4.048 |
| $\tau-\gamma$ | −0.248 | 0.761 | 1.059 | 3.404 | 0.000 | 0.334 | 0.029 | 1.342 |
| $\rho-\beta$ | −0.136 | 11.350 | 0.571 | 46.235 | −0.119 | 4.748 | 0.501 | 19.348 |
| $\rho-\gamma$ | −0.326 | 7.955 | 1.331 | 32.423 | 0.237 | 5.224 | −0.929 | 20.965 |
| $\beta-\gamma$ | −0.342 | 9.199 | 1.409 | 37.595 | −0.029 | 5.280 | 0.145 | 21.349 |
| $BLM$ | 0.011 | 0.504 | −0.023 | 1.582 | −0.001 | 0.217 | 0.023 | 0.665 |

**Table 7.** Estimation biases and rmse's for FGM copula parameters under weak dependence, based on 1000 samples of sizes 100 (left panel) and 500 (right panel).

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