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Statistical properties of the seasonal fractionally integrated separable spatial autoregressive model

Papa Ousmane Cisse[1,2,*], **Abdou Kâ Diongue**[1] **and Dominique Guegan**[2]

[1] Université Gaston Berger, UFR Sciences Appliquées et Technologie, BP 234, Saint-Louis–Sénégal. Laboratoire d'Etudes et de Recherches en Statistiques et Développement (LERSTAD)
[2] Université Paris 1 Panthéon-Sorbonne, Centre d'Économie de la Sorbonne (CES), MSE, 106-112 Boulevard de l'Hôpital, 75013 Paris, France

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Abstract. In this paper we introduce a new model called Fractionally Integrated Separable Spatial Autoregressive processes with Seasonality and denoted Seasonal FISSAR. We focus on the class of separable spatial models whose correlation structure can be expressed as a product of correlations. This new modelling allows taking into account the seasonality patterns observed in spatial data. We investigate the properties of this new model providing stationary conditions, some explicit form of the autocovariance function and the spectral density. We also establish the asymptotic behaviour of the spectral density function near the seasonal frequencies.

Résumé. On introduit une nouvelle classe de processus appelée Processus autoregressif spatiaux, fractionnaires, intégrés et séparables avec saisonnalité. On considère la classe des modè les spatiaux dont la structure de corrélation peut être exprimée comme produit de fonctions de corrélations. Cette nouvelle modé lisation permet de prendre en compte le phénomène de saisonnalité observé dans des données spatiales, bi-dimensionnelles. Nous étudions les propriétés statistiques du modèle proposé telles que les conditions de stationnarité, la fonction d'autocovariance (deux formes) et de la fonction de densité spectrale. Nous établissons aussi l'approximation asymptotique de la fonction de densité spectrale au niveau des fréquences saisonnières.

Key words: Seasonality; Spatial short memory; Seasonal long memory; Two-dimensional data; Separable process; Spatial stationary process; Spatial autocovariance.
AMS 2010 Mathematics Subject Classification : 62M07; 62M30; 62P12; 62P20.

* Corresponding author Papa Ousmane Cisse : cisse.papaousmane@yahoo.fr
Abdou Kâ Diongue: abdou.diongue@ugb.edu.sn
Dominique Guegan : dominique.guegan@univ-paris1.fr

1. Introduction

In recent years many studies have modelled the spatial process. In 1973, Cliff and Ord (1973) give an general presentation on spatial econometrics models and introduce the STAR (Space-Time AutoRegressive) and the Generalized Space-Time AuRegressive (GSTAR) models. The literature on spatial models is relatively abundant, we can also cite the Simultaneous AutoRegression model, SAR (Whittle, 1954), the Conditional AutoRegression model, CAR (Bartlett, 1971; Besag, 1974), the moving average model (Haining, 1978) or the unilateral models (Basu and Reinsel, 1993) among others. Spatial models are currently investigated in many research fields like meteorology (Lim *et al.*, 2002), oceanography (Illig, 2006), agronomy (Whittle, 1954; Lambert *et al.*, 2003), geology (Cressie, 1973), epidemiology (Marshall, 1991), image processing (Jain, 1981), econometrics (Anselin, 1988) and many others in which the data of interest are collected across space. This large domain of applications is due to the richness of the modelling which associates a representation with a geographical component.

Spatial time series modellings concern times series collected with geographical position, in order to use the spatial information in the modelling. Some particularities are included in the modelling: (i) two close data tend to have similar values; (ii) it can exist repetition of values by periodicity (for example, a temperature observed on a site can be observed in the same site after a given period). It is important to explain this repetition and to model it we associate with each direction i and j seasonal parameters s_1 et s_2 respectively.

The studies of spatial data have shown presence of long-range correlation structures (Lim *et al.*, 2002). To deal with this specific feature Boissy *et al.* (2005) had extended the long memory concept from times series to the spatial context and introduced the class of fractional spatial autoregressive model. At the same time Shitan (2008) studies the model called Fractionally Integrated Separable Spatial Autoregressive (FISSAR) model to approximate the dynamics of spatial data when the autocorrelation function decays with a long memory effect.

In another hand some authors have also observed seasonality in some spatial observations: Benth *et al.* (2007) proposed a spatial-temporal model for daily average temperature data. This model includes trend, seasonality and mean reversion. Portmann *et al.* (2009) studied the spatial and seasonal patterns for climate change, temperatures and precipitations. Nobre *et al.* (2011) introduce an spatially varying Autoregressive Processes for satellite data on sea surface temperature for the North Pacific to illustrate how the model can be used to separate trends, cycles, and short-term variability for high-frequency environmental data; a multivariate GSTAR has been developed by Pejman *et al.* (2009) for the study of the water quality.

Thus, it appears natural to incorporate long memory seasonal patterns into the FISSAR model of Shitan (2008) as soon as we work with data collected during several periods or cycles, allowing different seasonal patterns on the spatial locations. In that context common seasonal factors will receive different weights for these different spatial locations (Lopes *et al.*, 2008). Inference problems in spatial location or two-dimensional process have been studied by several authors (for example, Zhu *et al.* (2007) use the maximum likelihood method from spatial random effects). In this work a way to identify and estimate the parameters model in not discussed and this will be the purpose of a companion paper.

In this paper, we focus our attention on the class of separable spatial models whose correlation structure can be expressed as a product of correlations taking into account the seasonality patterns observed in spatial data. Therefore, we consider the Seasonal Fractionally Integrated Separable Spatial Autoregressive model, denoted in the following by Seasonal FISSAR extending at the same time the works Shitan (2008) and Boissy *et al.* (2005). We investigate the properties of this new modelling, providing the stationary conditions, analytic expressions for the autocovariance function and the spectral density function. We also establish the asymptotic mean of the spectral density function. This new modelling will be able to take into account periodic and cyclical behaviours presented in a lot of applications, including the modelling of temperatures, agricultural data, epidemiology when the data are collected during different seasons at different locations, and also financial data to take into account the specific systemic risk observed on the global market (Benirschka and Binkley, 1994; Graaff *et al.*, 2001; Jaworski, 2014).

The paper is organized as follows. The next Section 2 introduces the new class of Seasonal Fractionally Integrated Separable Spatial AutoRegressive model. In Section 3 we investigate some properties of the model, existence, invertibility, causality and stationary conditions. We compute the autocovariance function and provide an analytic expression for the spectral density and its asymptotic behaviour near the seasonal frequencies. In section 4 we provide some illustrations of this new modelling. Some proofs are given in the last section.

2. A new model: The Seasonal FISSAR

We introduce the Seasonal Fractionally Integrated Separable Autoregressive model and establish conditions for its existence and invertibility.

Let $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a sequence of spatial observations in two dimensional regular lattices, they are governed by a Seasonal FISSAR model if:

$$\begin{aligned}(1-\phi_{10}B_1-\phi_{01}B_2+\phi_{10}\phi_{01}B_1B_2)\,(1-\psi_{10}B_1^{s_1}-\psi_{01}B_2^{s_2}+\psi_{10}\psi_{01}B_1^{s_1}B_2^{s_2})\\ \times(1-B_1)^{d_1}\,(1-B_1^{s_1})^{D_1}\,(1-B_2)^{d_2}\,(1-B_2^{s_2})^{D_2}\,X_{ij}=\varepsilon_{ij}\end{aligned} \quad (1)$$

where the integers s_1 and s_2 are respectively the seasonal periods in the i^{th} and j^{th} directions, ϕ_{10}, ϕ_{01},ψ_{10}, ψ_{01} are real numbers and $\{\varepsilon_{ij}\}_{i,j\in\mathbb{Z}_+}$ is a spatial white noise process, mean zero and variance σ_ε^2. The backward shift operators B_1 and B_2 are such that $B_1X_{ij}=X_{i-1,j}$ and $B_2X_{ij}=X_{i,j-1}$. The long memory parameters are denoted d_1 and D_1 for the direction i and for the direction j they are denoted d_2 and D_2.

We specify now the different components of this model in order to understand how we can investigate it, and provide a useful methodology for estimation. First, we provide a part which characterizes the spatial short memory behaviour, second we introduce a new modelling for spatial long memory behaviour with seasonals, extending the work of Shitan (2008).

The spatial short memory behaviour of the variables $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ is explained through the process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$:

$$(1-\phi_{10}B_1)(1-\phi_{01}B_2)(1-\psi_{10}B_1^{s_1})(1-\psi_{01}B_2^{s_2})X_{ij}=W_{ij}. \quad (2)$$

This representation extends the work of (Shitan, 2008) introducing seasonality in the short memory behaviour with the filter $(1-\psi_{10}B_1^{s_1})(1-\psi_{01}B_2^{s_2})$. The process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ has a spatial seasonal long memory behaviour given by:

$$(1-B_1)^{d_1}(1-B_1^{s_1})^{D_1}(1-B_2)^{d_2}(1-B_2^{s_2})^{D_2}W_{ij}=\varepsilon_{ij}. \quad (3)$$

Thus, the Seasonal FISSAR model (1) can be rewritten formally by:

$$\Phi(B_1,B_2)\Psi(B_1^{s_1},B_2^{s_2})X_{ij}=W_{ij}, \quad (4)$$

where

$$\Phi(B_1,B_2)=(1-\phi_{10}B_1)(1-\phi_{01}B_2) \quad (5)$$

and

$$\Psi(B_1^{s_1},B_2^{s_2})=(1-\psi_{10}B_1^{s_1})(1-\psi_{01}B_2^{s_2}). \quad (6)$$

This new modelling is characterized by four operators: two characterizing the short memory behaviour, $(1-B_1^{s_1})^{D_1}$ and $(1-B_2^{s_2})^{D_2}$ and two characterizing the long memory behaviour, $(1-\psi_{10}B_1^{s_1})$ and $(1-\psi_{01}B_2^{s_2})$. They take into account the existence of seasonality in two directions.

We specify now the concept of long memory for stationary processes in two directions. Recall that a stationary process $\{X_t\}_{t\in\mathbb{Z}}$ with spectral density $f_X(.)$, for which it exist a real number $b\in(0,1)$, a constant $C_f>0$ and a frequency $G\in[0,\pi[$ such that $f_x(\omega)\sim C_f\,|\omega-G|^{-b}$, when $\omega\longrightarrow G$, then $\{X_t\}_{t\in\mathbb{Z}}$ has a long memory behaviour (Bisognin and Lopes, 2009). This definition can be extended in dimension two in the following way:

Definition 1. Let $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a stationary process with spectral density $f_X(.,.)$. Suppose there exist real numbers $a,b\in(0,1)$, a constant $C_f>0$ and frequencies λ_1, $\lambda_2\in[0,\pi[$ such that $f_x(\omega_1,\omega_2)\sim C_f\,|\omega_1-\lambda_1|^{-a}\,|\omega_2-\lambda_2|^{-b}$, when $(\omega_1,\omega_2)\longrightarrow(\lambda_1,\lambda_2)$, then $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ has a long memory behaviour.

We investigate now the following properties: (i) existence, (ii) invertibility, (iii) causality and (iv) stationarity for the model (1). We first provide the causal moving average representation of the seasonal FISSAR process (1).

Proposition 1. *Let be the process $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ defined in equation (2). It has the following representation:*

$$X_{ij}=\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\phi_{10}^k\phi_{01}^l\psi_{10}^m\psi_{01}^nW_{i-k-ms_1,j-l-ns_2}, \quad (7)$$

where

$$W_{ij}=\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\phi_k(d_1)\phi_l(d_2)\phi_m(D_1)\phi_n(D_2)\varepsilon_{i-k-ms_1,j-l-ns_2}, \quad (8)$$

with

$$\phi_k(d_1) = \begin{cases} \dfrac{\Gamma(k+d_1)}{\Gamma(k+1)\Gamma(d_1)} & if\, k \in \mathbb{Z}_+ \\ 0 & if\, k \notin \mathbb{Z}_+ \end{cases} ; \quad \phi_l(d_2) = \begin{cases} \dfrac{\Gamma(l+d_2)}{\Gamma(t+1)\Gamma(d_2)} & if\, l \in \mathbb{Z}_+ \\ 0 & if\, l \notin \mathbb{Z}_+ \end{cases} \tag{9}$$

and

$$\phi_m(D_1) = \begin{cases} \dfrac{\Gamma(m+D_1)}{\Gamma(m+1)\Gamma(D_1)} & if\, m \in \mathbb{Z}_+ \\ 0 & if\, m \notin \mathbb{Z}_+ \end{cases} ; \quad \phi_n(D_2) = \begin{cases} \dfrac{\Gamma(n+D_2)}{\Gamma(n+1)\Gamma(D_2)} & if\, n \in \mathbb{Z}_+ \\ 0 & if\, n \notin \mathbb{Z}_+ \end{cases} \tag{10}$$

$\Gamma(.)$, is the Gamma function defined by $\Gamma(t) = \int_0^\infty x^{t-1-x}dx$ and $\{\varepsilon_{ij}\}_{i,j\in\mathbb{Z}_+}$ is a two-dimensional white noise process. Equations (7)-(8) have an unique solution if the polynomials $\Phi(z_1, z_2)$ and $\Psi(z_1, z_2)$ are such that all their roots lie outside the unit polydisk, i.e

i) $\mid \phi_{10} \mid < 1$, $\mid \phi_{01} \mid < 1$, $\mid \psi_{10} \mid < 1$ *and* $\mid \psi_{01} \mid < 1$
ii) $\left(1 + \phi_{10}^2 - \phi_{01}^2 - \phi_{10}^2\phi_{01}^2\right) - 4\phi_{10}\left(1 - \phi_{10}\phi_{01}\right) > 0$
iii) $\left(1 + \psi_{10}^2 - \psi_{01}^2 - \psi_{10}^2\psi_{01}^2\right) - 4\psi_{10}\left(1 - \psi_{10}\psi_{01}\right) > 0$

Proof. : The sketch of the proof is provided in Appendix. It derives from Basu and Reinsel (1993).

3. Some properties of the seasonal FISSAR model

We provide now the spectral density function of the process $\{W_{ij}\}$ and $\{X_{ij}\}$ and we establish the asymptotic mean of this function. We use this result to give the stationary conditions for the processes.

Proposition 2. *Let $\{W_{ij}\}$ be the process defined by (3) and $f_W(\lambda_1, \lambda_2)$ its spectral density. When $|d_i + D_i| < 0.5$ and $|d_i| < 0.5$ $(i = 1, 2)$, its spectral density is equal to:*

$$f_W(\lambda_1, \lambda_2) = \frac{\sigma^2}{4\pi^2}\left[2\sin\left(\frac{\lambda_1}{2}\right)\right]^{-2d_1}\left[2\sin\left(\frac{s_1\lambda_1}{2}\right)\right]^{-2D_1}\left[2\sin\left(\frac{\lambda_2}{2}\right)\right]^{-2d_2}\left[2\sin\left(\frac{s_2\lambda_2}{2}\right)\right]^{-2D_2} \tag{11}$$

with λ_1 and $\lambda_2 \in]0, \pi]$.

Proof. : The proof of this Proposition is provided in the Appendix.

Proposition 3. *Let $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ be the Seasonal FISSAR process defined in (4), the spectral density function $f_X(\lambda_1, \lambda_2)$ of this process is equal to*

$$f_X(\lambda_1, \lambda_2) = \left|\Phi\left({}^{-i\lambda_1}, {}^{-i\lambda_2}\right)\right|^{-2}\left|\Psi\left({}^{-is\lambda_1}, {}^{-is\lambda_2}\right)\right|^{-2} f_W(\lambda_1, \lambda_2) \tag{12}$$

where $f_W(\lambda_1, \lambda_2)$ is the spectral density function of the process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ given in (11) and $\Phi(.,.)$ and $\Psi(.,.)$ are respectively defined in (5) and (6) with λ_1 and $\lambda_2 \in]0, \pi]$.

Proof. : This result derived from the definition of the spectral density function.

Corollary 1. *The spectral density of the process* $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ *defined in* (2) *can be rewritten as*

$$\begin{aligned} f_X(\lambda_1,\lambda_2) \;=\; & \left(1-2\phi_{10}\cos(\lambda_1)+\phi_{10}^2\right)^{-1}\left(1-2\psi_{10}\cos(s_1\lambda_1)+\psi_{10}^2\right)^{-1} \\ & \left(1-2\phi_{01}\cos(\lambda_2)+\phi_{01}^2\right)^{-1}\left(1-2\psi_{01}\cos(s_2\lambda_2)+\psi_{01}^2\right)^{-1} f_W(\lambda_1,\lambda_2) \end{aligned} \tag{13}$$

where $f_W(\lambda_1,\lambda_2)$ *is given in* (11).

We analyse now the behaviour of the spectral density for the processes $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ and $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ near the seasonal frequencies.

Proposition 4. *The asymptotic expression of the spectral density of the process* $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ *near the seasonal frequencies is such that*

(i) For $\lambda_0 = 0$,

$$f_W(\lambda_1,\lambda_2) \sim C_1\,|\lambda_1-\lambda_0|^{-2(d_1+D_1)}\,|\lambda_2-\lambda_0|^{-2(d_2+D_2)}, \quad \textit{when } (\lambda_1,\lambda_2)\longrightarrow(0,0), \tag{14}$$

with

$$C_1 = \frac{\sigma_\varepsilon^2}{4\pi^2} s_1^{-2D_1} s_2^{-2D_2} \tag{15}$$

(ii) For $\lambda_i = \frac{2\pi i}{s_1}$, $\lambda_j = \frac{2\pi j}{s_2}$, $i = 1,\ldots,[s_1/2]$ *and* $j = 1,\ldots,[s_2/2]$, *where* $[x]$ *means the integer part of* x,

$$f_W(\lambda_1,\lambda_2) \sim C_2\,|\lambda_1-\lambda_i|^{-2D_1}\,|\lambda_2-\lambda_j|^{-2D_2}, \quad \textit{when } (\lambda_1,\lambda_2)\longrightarrow(\lambda_i,\lambda_j) \tag{16}$$

with

$$C_2 = \frac{\sigma_\varepsilon^2}{4\pi^2} s_1^{-2D_1} s_2^{-2D_2}\left[2\sin\left(\frac{\lambda_i}{2}\right)\right]^{-2d1}\left[2\sin\left(\frac{\lambda_j}{2}\right)\right]^{-2d2} \tag{17}$$

Proof. : The proof of this Proposition is provided in the Appendix.

Proposition 5. *The asymptotic expression of the spectral density of the process* $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ *near the seasonal frequencies is such that*

(i) For $\lambda_0 = 0$,

$$f_X(\lambda_1,\lambda_2) \sim C_3\,|\lambda_1-\lambda_0|^{-2(d_1+D_1)}\,|\lambda_2-\lambda_0|^{-2(d_2+D_2)}, \quad \textit{when } (\lambda_1,\lambda_2)\longrightarrow(0,0) \tag{18}$$

with

$$\begin{aligned} C_3 \;&=\; \frac{\sigma_\varepsilon^2}{4\pi^2} s_1^{-2D_1} s_2^{-2D_2}\left|\Phi\left(^{-i\lambda_0},^{-i\lambda_0}\right)\right|^{-2}\left|\Psi\left(^{-i\lambda_0},^{-i\lambda_0}\right)\right|^{-2} \\ &=\; \frac{\sigma_\varepsilon^2}{4\pi^2} s_1^{-2D_1} s_2^{-2D_2}(1-\phi_{10})^{-2}(1-\psi_{10})^{-2}(1-\phi_{01})^{-2}(1-\psi_{10})^{-2}. \end{aligned} \tag{19}$$

(ii) For $\lambda_i = \frac{2\pi i}{s_1}$, $\lambda_j = \frac{2\pi j}{s_2}$, $i = 1,\ldots,[s_1/2]$ *and* $j = 1,\ldots,[s_2/2]$, *where* $[x]$ *means the integer part of* x,

$$f_X(\lambda_1,\lambda_2) \sim C_4\,|\lambda_1-\lambda_i|^{-2D_1}\,|\lambda_2-\lambda_j|^{-2D_2}, \quad \textit{when } (\lambda_1,\lambda_2)\longrightarrow(\lambda_i,\lambda_j) \tag{20}$$

with

$$C_4 = \frac{\sigma_\varepsilon^2}{4\pi^2} s_1^{-2D_1} s_2^{-2D_2} \left[2\sin\left(\frac{\lambda_i}{2}\right)\right]^{-2d1} \left[2\sin\left(\frac{\lambda_j}{2}\right)\right]^{-2d2} \left|\Phi\left(^{-i\lambda_i}, ^{-i\lambda_j}\right)\right|^{-2} \left|\Psi\left(^{-is_1\lambda_0}, ^{-is_2\lambda_0}\right)\right|^{-2}, \tag{21}$$

the polynomials $\Phi(.,.)$ *and* $\Psi(.,.)$ *are introduced in* (5) *and* (6).

Proof. : The proof is given in the Appendix.

We investigate now the stationary conditions for the model (1) as well as its long memory behaviour. We give also two expressions for the autocovariance function of the Seasonal FISSAR process.

Proposition 6. *The two-dimensional process* $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ *defined in (3)*

(i) is stationary when $d_i + D_i < 0.5$, $D_i < 0.5$, $i = 1, 2$.
(ii) has a long memory behaviour when $0 < d_i + D_i < 0.5$, $0 < D_i < 0.5$, $i = 1, 2$.

Proof. : The proof is given in the Appendix.

Proposition 7. *Let* $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ *be a Seasonal FISSAR process defined in* (1). *The process* $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$

(i) is stationary when $d_i + D_i < 0.5$, $D_i < 0.5$, $i = 1, 2$ *and* $\Phi(z_1, z_2)\Psi(z_1^s, z_2^s) \neq 0$ *for* $|z_1| < 1$ *and* $|z_2| < 1$.
(ii) has long memory property when $0 < d_i + D_i < 0.5$, $0 < D_i < 0.5$, $i = 1, 2$ *and* $\Phi(z_1, z_2)\Psi(z_1^{s_1}, z_2^{s_2}) \neq 0$, *for* $|z_1| \leq 1$ *and* $|z_2| \leq 1$.

Proof. : The proof is given in the Appendix.

To investigate the autocovariance function of the process defined in (2), we show that its autocovariance function can be written as a product of the autocovariance function for two processes $\{Z_{ij}\}_{i,j\in\mathbb{Z}_+}$ and $\{Y_{ij}\}_{i,j\in\mathbb{Z}_+}$ defined in the following way.
Let respectively $\{\varepsilon_{ij}^*\}$, $\{\varepsilon_{ij}'\}$ be two orthogonal two-dimensional white noise processes with mean zero and respectively variance $\sigma_{\varepsilon^*}^2$ and $\sigma_{\varepsilon'}^2$, we define the processes $\{Z_{ij}\}_{i,j\in\mathbb{Z}_+}$ and $\{Y_{ij}\}_{i,j\in\mathbb{Z}_+}$:

$$(1 - B_1^{s_1})^{D_1}(1 - B_2^{s_2})^{D_2} Z_{ij} = \varepsilon_{ij}^* \tag{22}$$

$$(1 - B_1)^{d_1}(1 - B_2)^{d_2} Y_{ij} = \varepsilon_{ij}' \tag{23}$$

Shitan (2008) prove that the autocovariance function of the process $\{Y_{ij}\}_{i,j\in\mathbb{Z}_+}$ is such that:

$$\gamma_Y(h_1, h_2) = \sigma_{\varepsilon'}^2 \frac{(-1)^{h_1+h_2}\Gamma(1 - 2d_1)\Gamma(1 - 2d_2)}{\Gamma(h_1 - d_1 + 1)\Gamma(1 - h_1 - d_1)\Gamma(h_2 - d_2 + 1)\Gamma(1 - h_2 - d_2)} \tag{24}$$

We can derive the expression of the process $\{Z_{ij}\}_{i,j\in\mathbb{Z}_+}$ from (24) and obtain

$$\gamma_Z(s_1h_1+\xi_1, s_2h_2+\xi_2) = \sigma^2_{\varepsilon^*}\frac{(-1)^{h_1+h_2}\Gamma(1-2D_1)\Gamma(1-2D_2)}{\Gamma(h_1-D_1+1)\Gamma(1-h_1-D_1)\Gamma(h_2-D_2+1)\Gamma(1-h_2-D_2)} \text{ if } (\xi_1,\xi_2)=(0,0) \quad (25)$$

$$\gamma_Z(s_1h_1+\xi_1, s_2h_2+\xi_2) = \gamma_Z(s_1h_1+\xi_1, s_2h_2+\xi_2) = 0 \text{ if } (\xi_1,\xi_2)\in A_1\times A_2 \quad (26)$$

where $A_1=\{1,\dots,s_1-1\}$ and $A_2=\{1,\dots,s_2-1\}$.
We can now give the autocovariance function of $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ introduced in (2):

Proposition 8. *Let $\ell_1,\ell_2\in\mathbb{Z}_+$, $(\xi_1,\xi_2)\in A_1\times A_2$ where $A_1=\{1,\dots,s_1-1\}$ and $A_2=\{1,\dots,s_2-1\}$. The autocovariance function of the process $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ is given by:*

$$\gamma_X(h_1,h_2) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\sum_{p=0}^{+\infty}\sum_{q=0}^{+\infty}\sum_{r=0}^{+\infty}\sum_{t=0}^{+\infty}\phi_{10}^{k+p}\phi_{01}^{l+q}\psi_{10}^{m+r}\psi_{01}^{n+t} \times\gamma_W\left(h_1+k+s_1(m-r)-p, h_2+l+s_2(n-t)-q\right) \quad (27)$$

where

$$\gamma_W(h_1,h_2) = \sigma^2_\varepsilon\sum_{\nu_1=0}^{+\infty}\sum_{\nu_2=0}^{+\infty}\gamma_Z(s_1\nu_1,s_2\nu_2) \times\gamma_Y(h_1-s_1\nu_1,h_2-s_2\nu_2) \text{ if } (h_1,h_2)=(s_1\ell_1,s_2\ell_2) \quad (28)$$

$$\gamma_W(h_1,h_2) = 0, \text{ if } (h_1,h_2)=(s_1\ell_1+\xi_1,s_2\ell_2+\xi_2) \quad (29)$$

with $\gamma_Z(.,.)$ and $\gamma_Y(.,.)$ given respectively in (25)-(26) and (24).

Proof. : The proof is given in the Appendix.

Corollary 2. *The variance of the Seasonal FISSAR process has the following expression*

$$\gamma_X(0,0) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\sum_{p=0}^{+\infty}\sum_{q=0}^{+\infty}\sum_{r=0}^{+\infty}\sum_{t=0}^{+\infty}\phi_{10}^{k+p}\phi_{01}^{l+q}\psi_{10}^{m+r}\psi_{01}^{n+t} \times\gamma_W\left(k+s_1(m-r)-p, l+s_2(n-t)-q\right) \quad (30)$$

where $\gamma_W(.,.)$ is given by (28)-(29) where $h_1=h_2=0$.

For practical purpose, we propose a general formula of the autocovariance function of the stationary process $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ which does not depend on the two-dimensional seasonal fractionally integrated white noise ($\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$). For that, we introduce two new processes $\{U_{ij}\}_{i,j\in\mathbb{Z}_+}$ and $\{V_{ij}\}_{i,j\in\mathbb{Z}_+}$.
Let respectively $\{\widetilde{\varepsilon}_{ij}\}$, $\left\{\widetilde{\widetilde{\varepsilon}}_{ij}\right\}$ be two 2-dimensional white noise processes with mean zero and respectively variances $\sigma^2_{\widetilde{\varepsilon}_{ij}}$ and $\sigma^2_{\widetilde{\widetilde{\varepsilon}}_{ij}}$. We introduce respectively the processes $\{U_{ij}\}_{i,j\in\mathbb{Z}_+}$ and $\{V_{ij}\}_{i,j\in\mathbb{Z}_+}$:

$$\Psi\left(B_1^{s_1},B_2^{s_2}\right)\left(1-B_1^{s_1}\right)^{D_1}\left(1-B_2^{s_2}\right)^{D_2}U_{ij} = \widetilde{\varepsilon}_{ij} \quad (31)$$

$$\Phi\left(B_1,B_2\right)\left(1-B_1\right)^{d_1}\left(1-B_2\right)^{d_2}V_{ij} = \widetilde{\widetilde{\varepsilon}}_{ij} \quad (32)$$

where $\Psi\left(B_1^s, B_2^s\right)$ and $\Phi\left(B_1, B_2\right)$ are respectively defined in (5) and (6).

Note that the process $\{U_{ij}\}_{i,j\in\mathbb{Z}_+}$ generalizes the process $\{Z_{ij}\}_{i,j\in\mathbb{Z}_+}$ introduced in (22) through the operator $\Psi\left(B_1^s, B_2^s\right)$ and the process $\{V_{ij}\}_{i,j\in\mathbb{Z}_+}$ generalizes the process $\{Y_{ij}\}_{i,j\in\mathbb{Z}_+}$ introduced in (23) through the operator $\Phi\left(B_1, B_2\right)$.

Proposition 9. *The autocovariance function of the stationary process $\{U_{ij}\}_{i,j\in\mathbb{Z}_+}$ in spatial lags (h_1, h_2) is equal to:*

$$\begin{aligned}\gamma_U(h_1,h_2) &= \sigma^2_{\tilde{\varepsilon}}\sum_{\nu_1=0}^{+\infty}\sum_{\nu_2=0}^{+\infty}\gamma_{\widetilde{U}}(s_1\nu_1, s_2\nu_2)\\ &\quad\times\gamma_Z(h_1-s_1\nu_1, h_2-s_2\nu_2),\ \ \textit{if}(h_1,h_2)=(s_1\ell_1, s_2\ell_2) \qquad (33)\\ \gamma_U(h_1,h_2) &= 0,\ \ \textit{if}\,(h_1,h_2)=(s_1\ell_1+\xi_1, s_2\ell_2+\xi_2) \qquad (34)\end{aligned}$$

where $\widetilde{U}$ is equal to:

$$\Psi\left(B_1^{s_1}, B_2^{s_2}\right)\widetilde{U}_{ij} = \widetilde{\varepsilon^*}_{ij},$$

$$\gamma_{\widetilde{U}}(s_1\nu_1, s_2\nu_2) = \sigma^2_{\widetilde{\varepsilon^*}}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi^1_{\nu_1+m}\varphi^1_m\varphi^2_{\nu_2+n}\varphi^2_n,$$

and $\gamma_Z(.,.)$ is introduced in (25)-(26). The coefficients φ^1_k and φ^2_l are linked by the relationship

$$\Psi^{-1}\left(z_1^s, z_2^s\right) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi^1_k\varphi^2_l z_1^{s_1k}z_2^{s_2l}$$

.

Proof. : The proof is given in the Appendix.

Proposition 10. *The autocovariance function of the stationary processes $\{V_{ij}\}_{i,j\in\mathbb{Z}_+}$ in spatial lags (h_1, h_2) is equal to:*

$$\gamma_V(h_1,h_2) = \sigma^2_{\tilde{\varepsilon}}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\gamma_{\widetilde{V}}(k,l)\gamma_Y(h_1-k, h_2-l) \qquad (35)$$

where $\widetilde{V}$ is given by:

$$\Phi\left(B_1, B_2\right)\widetilde{V}_{ij} = \widetilde{\varepsilon'}_{ij},$$

$$\gamma_{\widetilde{V}}(k,l) = \sigma^2_{\widetilde{\varepsilon'}}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi^1_{k+m}\varphi^1_m\varphi^2_{l+n}\varphi^2_n,$$

$\gamma_Y(.,.)$ being defined by (24) and the coefficients φ^1_k and φ^2_l are linked by the relationship

$$\Phi^{-1}\left(z_1, z_2\right) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi^1_k\varphi^2_l z_1^k z_2^l$$

Proof. : The proof of this Proposition is given in the Appendix.

Now we provide the autocovariance function of the Seasonal FISSAR process defined in (4).

Proposition 11. *Let* $\ell_1, \ell_2 \in \mathbb{Z}_+$ *,* $\xi \in A$ *where* $A = \{1, \ldots, s-1\}$.
The Seasonal FISSAR stationary process $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ *has autocovariance function at spatial lags* (h_1, h_2) *given by*

$$\gamma_X(h_1,h_2) = \sigma_\varepsilon^2 \sum_{\nu_1=0}^{+\infty}\sum_{\nu_2=0}^{+\infty} \gamma_U(s_1\nu_1, s_2\nu_2) \times\gamma_V(h_1 - s_1\nu_1, h_2 - s_2\nu_2), \text{ if } (h_1,h_2) = (s_1\ell_1, s_2\ell_2) \quad (36)$$

$$\gamma_X(h_1,h_2) = 0, \text{ if } (h_1,h_2) = (s_1\ell_1+\xi_1, s_2\ell_2+\xi_2). \quad (37)$$

where the autocovariance functions $\gamma_U(.,.)$ *and* $\gamma_V(.,.)$ *are defined respectively in* (33)-(34) *and* (35).

Proof. : The sketch of the proof is provided in the Appendix.

Corollary 3. *The variance for this second representation of the Seasonal FISSAR process is given by,*

$$\gamma_X(0,0) = \sigma_\varepsilon^2 \sum_{\nu_1=0}^{+\infty}\sum_{\nu_2=0}^{+\infty} \gamma_U(s_1\nu_1, s_2\nu_2)\gamma_V(s_1\nu_1, s_2\nu_2) \quad (38)$$

where the autocovariance functions $\gamma_U(.,.)$ *and* $\gamma_V(.,.)$ *are defined respectively in* (33)-(34) *and* (35) *with* $h_1 = h_2 = 0$.

4. Illustrations

A realisation of the two-dimensional seasonal fractionally integrated white noise processes $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ with $d_1 = 0.1, d_2 = 0.1, D_1 = 0.15, D_2 = 0.2, s_1 = s_2 = 4$ is shown in Figure 1. In this study, we generated 100×100 grid and we use only the values in south east corner in the matrix (they correspond to the interior values of grid size 30×30).
The spatial white noise process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ can be considered as a special case of the Seasonal FISSAR model. However, it is rare to see applications in a phenomenon that is only modelled by white noise.
We simulated the Seasonal FISSAR process in two stages. First we generate the two dimensional white noise $\{\varepsilon_{ij}\}_{i,j\in\mathbb{Z}_+}$ and second using (3) we obtained $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$. Then using the relationship (2), we get $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$. We use also the 30×30 values in south east corner by simulating 100×100 values in a regular grid with $d_1 = 0.1, d_2 = 0.1, D_1 = 0.1, D_2 = 0.2, \phi_{10} = 0.1, \phi_{01} = 0.15, \psi_{10} = 0.1, \phi_{0.2}$ and $s_1 = s_2 = 4$.
In practice, the Seasonal FISSAR model has many possible applications of real data sets from different fields when the observations are collected during different seasons at different locations: temperature data, agricultural data, systemic risk etc. An possible application may concern variability of the rice production over Senegal river valley. Indeed yields vary widely from season to season and depending on the growing areas in the valley. Thus, our model could be applied to these data for better management of forecasting yields, which would be a considerable contribution to the management of rice production, an influential factor on economic issues of the country and the West African sub region.

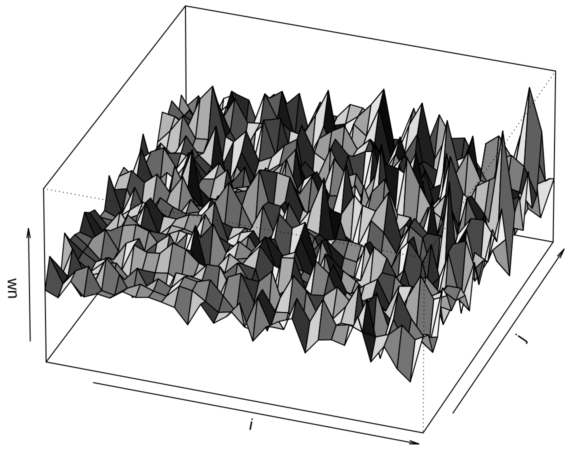

Fig. 1. 2D seasonal fractionally integrated white noise process, $d_1 = 0.1, d_2 = 0.1, D_1 = 0.15, D_2 = 0.2, s_1 = s_2 = 4$ and size 30×30.

In practice then, many observations are reporting by longitude and altitude and this new modelling is defined in two dimensional regular lattices. In this case we re-coded the position of the stations by assigning an integer value from number for both longitude and altitude, reflecting the relative position on the lattice into which the study region has been mapped.

5. Conclusion

The spatial modelling has a lot of applications in different fields. To take into account at the same time existence of short memory behaviour and long memory behaviour in time and space permits a greater flexibility for the use of these modellings. It is the objective of this paper which introduces and investigates the statistical properties of a new class of model called Fractionally Integrated Separable Spatial Autoregressive processes with Seasonality. The stationary conditions, an explicit expression form of the autocovariance function and spectral density function have also been given. On another hand, a practical formula of the autocovariance function as a product of covariance for the Seasonal FISSAR process is given. Extension of the results to the spatio-temporal data or d-dimensional ($d > 2$) fields is immediate but not provided in this paper. For the spatio-temporal representation, time can be represented by the direction i and the spatial components by the direction j taken in $\mathbb{Z}^d$, $d \geq 2$. We provide some representations of these models. It remains to provide a way to identify and estimate these models from data sets: this will be the purpose of a companion paper.

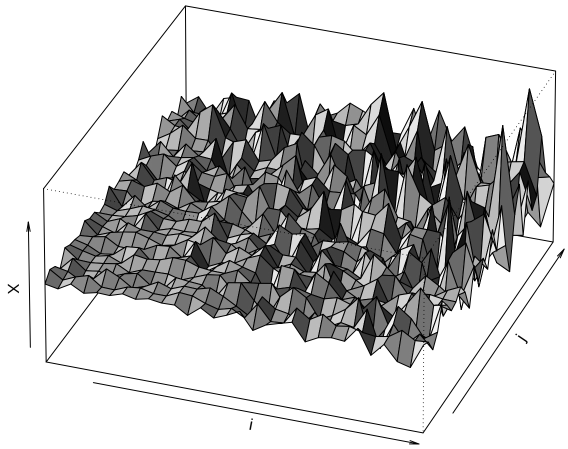

Fig. 2. Seasonal FISSAR process: $d_1 = 0.1, d_2 = 0.1, D_1 = 0.1, D_2 = 0.2, \phi_{10} = 0.1, \phi_{01} = 0.15, \psi_{10} = 0.1, \phi_{0.2}, s_1 = s_2 = 4$ and size $N \times N = 30 \times 30$.

6. Appendix

In this section we establish the main results and give the necessary technical proofs for the propositions.

Proof. of the Proposition 1.
According to equation (2), we have

$$X_{ij} = (1 - \phi_{10}B_1)^{-1} (1 - \psi_{10}B_1^{s_1})^{-1} (1 - \phi_{01}B_2)^{-1} (1 - \psi_{01}B_2^{s_2})^{-1} W_{ij}$$

Thus,

$$\begin{aligned} X_{ij} &= \left(\sum_{k=0}^{+\infty} \phi_{10}^k B_1^k\right) \left(\sum_{m=0}^{+\infty} \psi_{10}^m B_1^{ms_1}\right) \left(\sum_{l=0}^{+\infty} \phi_{01}^l B_2^l\right) \left(\sum_{n=0}^{+\infty} \psi_{01}^n B_2^{ns_2}\right) W_{ij} \\ &= \left(\sum_{k=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{n=0}^{+\infty} \phi_{10}^k B_1^k \psi_{10}^m B_1^{ms_1} \phi_{01}^l B_2^l \psi_{01}^n B_2^{ns_2}\right) W_{ij}. \end{aligned}$$

If $\Phi(z_1, z_2)$ and $\Psi(z_1, z_2)$ have their roots outside the unit polydisk then we have the convergent representation (7), see Proposition 1 in Basu and Reisel (1993).

Proof. of the Proposition 2.
We consider (3) and denote $f_\varepsilon(\lambda_1, \lambda_2)$ the spectral density of the process $\{\varepsilon_{ij}\}$. Let

$$\Psi(z_1, z_2) = (1 - z_1)^{-d_1} (1 - z_1^{s_1})^{-D_1} (1 - z_2)^{-d_2} (1 - z_2^{s_2})^{-D_2},$$

Then

$$\begin{aligned} f_W(\lambda_1,\lambda_2) &= \Psi\left({}^{i\lambda_1},{}^{i\lambda_2}\right)\Psi\left({}^{-i\lambda_1},{}^{-i\lambda_2}\right)f_\varepsilon(\lambda_1,\lambda_2) \\ &= \left(1-{}^{i\lambda_1}\right)^{-d_1}\left(1-{}^{is_1\lambda_1}\right)^{-D_1}\left(1-{}^{i\lambda_2}\right)^{-d_2}\left(1-{}^{is_2\lambda_2}\right)^{-D_2} \\ &\quad\times\left(1-{}^{-i\lambda_1}\right)^{-d_1}\left(1-{}^{-is_1\lambda_1}\right)^{-D_1}\left(1-{}^{-i\lambda_2}\right)^{-d_2}\left(1-{}^{-is_2\lambda_2}\right)^{-D_2}f_\varepsilon(\lambda_1,\lambda_2) \\ &= \left[\left(1-{}^{i\lambda_1}\right)\left(1-{}^{-i\lambda_1}\right)\right]^{-d_1}\left[\left(1-{}^{is_1\lambda_1}\right)\left(1-e^{-is_1\lambda_1}\right)\right]^{-D_1} \\ &\quad\times\left[\left(1-{}^{i\lambda_2}\right)\left(1-{}^{-i\lambda_2}\right)\right]^{-d_2}\left[\left(1-{}^{is_2\lambda_2}\right)\left(1-{}^{-is_2\lambda_2}\right)\right]^{-D_2}f_\varepsilon(\lambda_1,\lambda_2) \end{aligned}$$

Thus

$$f_W(\lambda_1,\lambda_2)=\left|1-{}^{-i\lambda_1}\right|^{-2d_1}\left|1-{}^{-is_1\lambda_1}\right|^{-2D_1}\left|1-{}^{-i\lambda_2}\right|^{-2d_2}\left|1-{}^{-is_2\lambda_2}\right|^{-2D_2}f_\varepsilon(\lambda_1,\lambda_2)$$

as soon as

$$\left(1-{}^{i\lambda_1}\right)\left(1-{}^{-i\lambda_1}\right)=\left|1-{}^{-i\lambda_1}\right|^2=\left[2\sin\left(\frac{\lambda_1}{2}\right)\right]^2,$$

we obtain (11) since $f_\varepsilon(\lambda_1,\lambda_2)=\dfrac{\sigma^2}{4\pi^2}$.

Proof. of the Proposition 4.(i) We consider the spectral density function of the process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ defined in (11) and we use the following approximations:

$$\lim_{\lambda\longrightarrow 0}\frac{\sin(s\lambda)}{s\lambda}=1 \text{ and } \sin(s\lambda)\simeq s\lambda,$$

then

$$\begin{aligned} f_W(\lambda_1,\lambda_2) &= \frac{\sigma_\varepsilon^2}{4\pi^2}|\lambda_1|^{-2d_1}s_1^{-2D_1}|\lambda_1|^{-2D_1}|\lambda_2|^{-2d_2}s_2^{-2D_2}|\lambda_2|^{-2D_2} \\ &= \frac{\sigma_\varepsilon^2}{4\pi^2}|\lambda_1|^{-2(d_1+D_1)}|\lambda_2|^{-2(d_2+D_2)}s_1^{-2D_1}s_2^{-2D_2} \end{aligned}$$

when $(\lambda_1,\lambda_2)\longrightarrow(0,0)$. As soon as $\lambda_0=0$ we obtain (14).

(ii) Let $\lambda_i=\frac{2\pi i}{s_1}$ and $\lambda_j=\frac{2\pi j}{s_2}$ for all $i=1,\dots,[s_1/2]$ and $j=1,\dots,[s_2/2]$, where $[x]$ means the integer part of x, then

$$\begin{aligned} f_W(\lambda_1+\lambda_i,\lambda_2+\lambda_j)=\frac{\sigma^2}{4\pi^2}&\left[2\sin\left(\frac{\lambda_1}{2}+\frac{\lambda_i}{2}\right)\right]^{-2d_1}\left[2\sin\left(\frac{s_1\lambda_1}{2}+\frac{s_1\lambda_i}{2}\right)\right]^{-2D_1} \\ &\left[2\sin\left(\frac{\lambda_2}{2}+\frac{\lambda_j}{2}\right)\right]^{-2d_2}\left[2\sin\left(\frac{s_2\lambda_2}{2}+\frac{s_2\lambda_j}{2}\right)\right]^{-2D_2} \end{aligned}$$

If $\lambda\longrightarrow 0$ then

$$\left[2\sin\left(\frac{s\lambda}{2}+\frac{s\lambda_j}{2}\right)\right]^{-2D}\simeq s^{-2D}|\lambda|^{-2D}$$

Therefore,

$$f_W(\lambda_1+\lambda_i,\lambda_2+\lambda_j) \simeq \frac{\sigma_\varepsilon^2}{4\pi^2} s_1^{-2D_1}|\lambda_1|^{-2D1} s_2^{-2D_2}|\lambda_2|^{-2D_2}\left[2\sin\left(\frac{\lambda_i}{2}\right)\right]^{-2d1}\left[2\sin\left(\frac{\lambda_j}{2}\right)\right]^{-2d2} \tag{39}$$

Replacing λ_1 by $\lambda_1-\lambda_i$ and λ_2 by $\lambda_2-\lambda_j$ in (39), we obtain (16).

Proof. of the Proposition 5.
(i) For this proof we need to use the corollary (1).
Suppose that the process $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ defined in (1) is causal and invertible. Using the expressions (13),and $\cos(s\lambda)\simeq 1$, $\lambda\longrightarrow 0$, then

$$\begin{aligned} f_W(\lambda_1,\lambda_2) &= \frac{\sigma_\varepsilon^2}{4\pi^2}|\lambda_1|^{-2d_1} s_1^{-2D_1}|\lambda_1|^{-2D_1}|\lambda_2|^{-2d_2} s_2^{-2D_2}|\lambda_2|^{-2D_2} \\ &\quad (1-\phi_{10})^{-2}(1-\psi_{10})^{-2}(1-\phi_{01})^{-2}(1-\psi_{10})^{-2} \\ &= \frac{\sigma_\varepsilon^2}{4\pi^2}|\lambda_1|^{-2(d_1+D_1)}|\lambda_2|^{-2(d_2+D_2)} s_1^{-2D_1}s_2^{-2D_2} \\ &\quad (1-\phi_{10})^{-2}(1-\psi_{10})^{-2}(1-\phi_{01})^{-2}(1-\psi_{10})^{-2} \end{aligned}$$

when $(\lambda_1,\lambda_2)\longrightarrow(0,0)$. For $\lambda_0=0$ we obtain (18).

(ii) Let $\lambda_i=\frac{2\pi i}{s_1}$ and $\lambda_j=\frac{2\pi j}{s_2}$ for all $i=1,\dots,[s_1/2]$ and $j=1,\dots,[s_2/2]$, where $[x]$ means the integer part of x.

$$\begin{aligned} f_X(\lambda_1+\lambda_i,\lambda_2+\lambda_j) &= \left|\Phi\left(^{-i(\lambda_1+\lambda_i)},^{-i(\lambda_2+\lambda_j)}\right)\right|^{-2}\left|\Psi\left(^{-is_1(\lambda_1+\lambda_i)},^{-is_2(\lambda_2+\lambda_j)}\right)\right|^{-2} \\ &\quad f_W(\lambda_1+\lambda_j,\lambda_2+\lambda_j) \\ &= \frac{\sigma^2}{4\pi^2}\left[2\sin\left(\frac{\lambda_1}{2}+\frac{\lambda_i}{2}\right)\right]^{-2d_1}\left[2\sin\left(\frac{s_1\lambda_1}{2}+\frac{s_1\lambda_j}{2}\right)\right]^{-2D_1} \\ &\quad \left[2\sin\left(\frac{\lambda_2}{2}+\frac{\lambda_j}{2}\right)\right]^{-2d_2}\left[2\sin\left(\frac{s_2\lambda_2}{2}+\frac{s_2\lambda_j}{2}\right)\right]^{-2D_2} \\ &\quad \left|\Phi\left(^{-i\lambda_i},^{-i\lambda_j}\right)\right|^{-2}\left|\Psi\left(^{-is_1\lambda_0},^{-is_2\lambda_0}\right)\right|^{-2} \end{aligned}$$

If $\lambda\longrightarrow 0$ then

$$\left[2\sin\left(\frac{s\lambda}{2}+\frac{s\lambda_j}{2}\right)\right]^{-2D}\simeq s^{-2D}|\lambda|^{-2D}$$

Therefore,

$$\begin{aligned} f_X(\lambda_1+\lambda_i,\lambda_2+\lambda_j) &\simeq \frac{\sigma_\varepsilon^2}{4\pi^2} s_1^{-2D_1}|\lambda_1|^{-2D1} s_2^{-2D_2}|\lambda_2|^{-2D_2}\left[2\sin\left(\frac{\lambda_i}{2}\right)\right]^{-2d1}\left[2\sin\left(\frac{\lambda_j}{2}\right)\right]^{-2d2} \\ &\quad \left|\Phi\left(^{-i\lambda_i},^{-i\lambda_j}\right)\right|^{-2}\left|\Psi\left(^{-is_1\lambda_0},^{-is_2\lambda_0}\right)\right|^{-2} \end{aligned} \tag{40}$$

Replacing λ_1 by $\lambda_1-\lambda_i$ and λ_2 by $\lambda_2-\lambda_j$ in (40), we obtain (20).

Proof. of the Proposition 6.
(i) Let $f_W(.,.)$ the spectral density function of the process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ given in (11). Then $f_W(\lambda_1,\lambda_2) = f_W(-\lambda_1,-\lambda_2)$ and $f_W(\lambda_1,\lambda_2) \geq 0$. Therefore the processus is stationary if

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_W(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 = 4\int_{0}^{\pi}\int_{0}^{\pi} f_W(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 < \infty \tag{41}$$

From (14) and (16) we have

$$C_1\int_0^{\pi} |\lambda_1|^{-2(d_1+D_1)}\, d\lambda_1 \int_0^{\pi} |\lambda_2|^{-2(d_2+D_2)}\, d\lambda_2 < \infty$$

and

$$C_2\int_0^{\pi} |\lambda_1-\lambda_j|^{-2D_1}\, d\lambda_1 \int_0^{\pi} |\lambda_2-\lambda_j|^{-2D_2}\, d\lambda_2 < \infty$$

when $d_i + D_i < 0.5$ and $D_i < 0.5$, $i = 1, 2$. Thus (41) is verified, and the process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ is stationary.

(ii) From the asymptotic expression of the spectral density function of the process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ and using Proposition 2 we derive that the process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ has long memory property if $0 < d_i + D_i < 0.5$ and $0 < D_i < 0.5$, $i = 1, 2$.

Proof. of the Proposition 7.
(i) The process $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ can be rewritten as

$$X_{ij} = \Phi\left(B_1,B_2\right)^{-1}\Psi\left(B_1^{s_1},B_2^{s_2}\right)^{-1}\left(1-B_1\right)^{-d_1}\left(1-B_1^{s_1}\right)^{-D_1}\left(1-B_2\right)^{-d_2}\left(1-B_2^{s_2}\right)^{-D_2}\varepsilon_{ij}$$

Let

$$\pi(z_1,z_2) = \Phi\left(z_1,z_2\right)^{-1}\Psi\left(z_1^{s_1},z_2^{s_2}\right)^{-1}\left(1-z_1\right)^{-d_1}\left(1-z_1^{s_1}\right)^{-D_1}\left(1-z_2\right)^{-d_2}\left(1-z_2^{s_2}\right)^{-D_2}\varepsilon_{ij}$$

Then

$$X_{ij} = \pi(B_1,B_2)\varepsilon_{ij}$$

If $d_i + D_i < 0.5$ and $D_i < 0.5$, $i = 1, 2$ the item (i) of Proposition 6 assures that the power series expansion of $(1-z_1)^{-d_1}(1-z_1^{s_1})^{-D_1}(1-z_2)^{-d_2}(1-z_2^{s_2})^{-D_2}$ converges for $|z_1| \leq 1$ and $|z_2| \leq 1$. In another hand, the polynomial $(\Phi(z_1,z_2)\Psi(z_1^s,z_2^s))^{-1}$ converges for $|z_1| \leq 1$ and $|z_2| \leq 1$ when the roots of $\Phi(z_1,z_2)\Psi(z_1^{s_1},z_2^{s_2}) = 0$ are outside the unit disk. Therefore, the power series $\pi(z_1,z_2)$ converges for all $|z_1| \leq 1$ and $|z_2| \leq 1$ and the process $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ is stationary.

(ii) Let $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a Seasonal FISSAR process in (4) whose all roots of $\Phi(z_1,z_2)\Psi(z_1^{s_1},z_2^{s_2}) = 0$ are outside the unit polydisk. From the asymptotic expression of the spectral density function of $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ and the Proposition 3 the Seasonal FISSAR process has long memory property when $0 < d_i + D_i < 0.5$ and $0 < D_i < 0.5$, $i = 1, 2$ if all the roots of $\Phi(z_1,z_2)\Psi(z_1^s,z_2^s) = 0$ are outside the unit polydisk.

Proof. of the Proposition 8.
First, we prove the expression of the autocovariance function for the process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ as a product of the autocovariance function of $\{Z_{ij}\}_{i,j\in\mathbb{Z}_+}$ and $\{Y_{ij}\}_{i,j\in\mathbb{Z}_+}$.
Let $\{Z_{ij}\}_{i,j\in\mathbb{Z}_+}$ the process defined in (22). Then

$$Z_{ij} = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi_k(D_1)B_1^{s_1k}\varphi_l(D_2)B_2^{s_2l}\left(\varepsilon^*_{ij}\right) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi_k(D_1)\varphi_l(D_2)\varepsilon^*_{i-s_1k,j-s_2l} \tag{42}$$

where the quantity $\varphi_k(D_1)$ and $\varphi_l(D_2)$ are

$$\phi_k(D_1) = \frac{\Gamma(k+D_1)}{\Gamma(k+1)\Gamma(D_1)}\ ;\ \phi_l(D_2) = \frac{\Gamma(l+D_2)}{\Gamma(l+1)\Gamma(D_2)}. \tag{43}$$

For an easier representation we note in the following $\varphi_k(D_1) = \varphi^1_k$ and $\varphi_l(D_2) = \varphi^2_l$.
Therefore

$$\begin{aligned}\gamma_Z(h_1,h_2) &= \operatorname{Cov}\left(Z_{i+h_1,j+h_2}, Z_{ij}\right)\\ \gamma_Z(h_1,h_2) &= \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi^1_k\varphi^2_l\varphi^1_m\varphi^2_n\gamma_{\varepsilon^*}(h_1-s_1k+s_1m, h_2-s_2l+s_2n) \end{aligned}\tag{44}$$

When $h_1 - s_1k + s_1m = 0$ and $h_2 - s_2 + s_2n = 0$, we have $k = \frac{h_1}{s_1} + m$ and $l = \frac{h_2}{s_2} + n$, thus (44) can be rewritten as

$$\gamma_Z(h_1,h_2) = \sigma^2_{\varepsilon^*}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi^1_{\frac{h_1}{s_1}+m}\varphi^2_{\frac{h_2}{s_2}+n}\varphi^1_m\varphi^2_n. \tag{45}$$

Taking $(h_1,h_2) = (s_1\ell_1, s_2\ell_2)$ for $\ell_1,\ell_2\in\mathbb{Z}_+$, then

$$\gamma_Z(s_1\ell_1, s_2\ell_2) = \sigma^2_{\varepsilon^*}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi^1_{\ell_1+m}\varphi^2_{\ell_2+n}\varphi^1_m\varphi^2_n,$$

if $(h_1,h_2) = (s_1\ell_1+\xi_1, s_2\ell_2+\xi_2)$ for $\ell_1,\ell_2\in\mathbb{Z}_+$, $(\xi_1,\xi_2)\in A_1\times A_2$, where $A_1 = \{1,\dots,s_1-1\}$, $A_2 = \{1,\dots,s_2-1\}$ then $\gamma_Z(h_1,h_2) = 0$.
Thus the autocovariance function of the stationary process $\{Z_{ij}\}_{i,j\in\mathbb{Z}_+}$ is given by

$$\gamma_Z(h_1,h_2) = \begin{cases}\sigma^2_{\varepsilon^*}\displaystyle\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi^1_{\ell_1+m}\varphi^1_m\varphi^2_{\ell_2+n}\varphi^2_n & \text{if } (h_1,h_2) = (s_1\ell_1, s_2\ell_2)\\ 0 & \text{if } (h_1,h_2) = (s_1\ell_1+\xi_1, s_1\ell_2+\xi_2).\end{cases} \tag{46}$$

Now the process $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ can be rewritten by

$$W_{ij} = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi^1_k\varphi^2_lY_{i-s_1k,j-s_2l}$$

Then its autocovarianec function is given by

$$\begin{aligned}\gamma_W(h_1,h_2) &= \mathrm{Cov}\left(W_{i+h_1,j+h_2},W_{ij}\right)\\ &= \mathrm{Cov}\left(\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi_k^1\varphi_l^2Y_{i+h_1-s_1k,j+h_2-s_2l},\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_m^1\varphi_n^2Y_{i-ms_1,j-ns_2}\right)\\ &= \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_k^1\varphi_l^2\varphi_m^1\varphi_n^2\mathrm{Cov}\left(Y_{i+h_1-s_1k,j+h_2-s_2l},Y_{i-ms_1,j-ns_2}\right)\\ &= \sigma^2_{\varepsilon'}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_k^1\varphi_l^2\varphi_m^1\varphi_n^2\gamma_Y(h_1-s_1k+s_1m,h_2-s_2l+s_2n).\end{aligned}$$

Thus

$$\gamma_W(h_1,h_2)=\sigma^2_{\varepsilon'}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_k^1\varphi_l^2\varphi_m^1\varphi_n^2\gamma_Y\left(h_1-s_1(k-m),h_2-s_2(l-n)\right). \quad (47)$$

Taking $\nu_1=k-m$ and $\nu_2=l-n$ in (47), we get

$$\gamma_W(h_1,h_2)=\sigma^2_{\varepsilon'}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_{\nu_1+m}^1\varphi_{\nu_2+n}^2\varphi_m^1\varphi_n^2\gamma_Y\left(h_1-s_1\nu_1),h_2-s_2\nu_2\right). \quad (48)$$

Using (46) and denoting $\sigma^2_\varepsilon=\sigma^2_{\varepsilon'}/\sigma^2_{\varepsilon^*}$ the variance of the two-dimensional white noise process $\{\varepsilon_{ij}\}_{i,j\in\mathbb{Z}_+}$ we obtain (28) and (29).
We give now the proof of the of the expression of the autocovariance function for the Seasonal FISSAR model defined in (2). Since $\mathbb{E}(W_{ij})=0$ we have $\mathbb{E}(X_{ij})=0$ and

$$\gamma_X(h_1,h_2) = \mathbb{E}\left(X_{i+h_1,j+h_2}X_{ij}\right).$$

Thus

$$\gamma_X(h_1,h_2)=\mathbb{E}\left[\sum_{p=0}^{+\infty}\sum_{q=0}^{+\infty}\sum_{r=0}^{+\infty}\sum_{t=0}^{+\infty}\phi_{10}^p\phi_{01}^q\psi_{10}^r\psi_{01}^tW_{i+h_1-p-rs_1,j+h_2-q-ts_2}\right.$$
$$\left.\times\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\phi_{10}^k\phi_{01}^l\psi_{10}^m\psi_{01}^nW_{i-k-ms_1,j-l-ns_2}\right]$$

and

$$\gamma_X(h_1,h_2)=\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\sum_{p=0}^{+\infty}\sum_{q=0}^{+\infty}\sum_{r=0}^{+\infty}\sum_{t=0}^{+\infty}\phi_{10}^{k+p}\phi_{01}^{l+q}\psi_{10}^{m+r}\psi_{01}^{n+t}$$
$$\times\,\mathbb{E}\left(W_{i+h_1-p-rs_1,j+h_2-q-ts_2}W_{i-k-ms_1,j-l-ns_2}\right).$$

Now,

$$\begin{aligned}\mathbb{E}\left(W_{i+h_1-p-rs,j+h_2-q-ts}W_{i-k-ms,j-l-ns}\right) &= \gamma_W\left(h_1+k+ms_1-p-rs_1,h_2+l+ns_2-q-ts_2\right)\\ &= \gamma_W\left(h_1+k+s_1(m-r)-p,h_2+l+s_2(n-t)-q\right),\end{aligned}$$

then we obtain (27).

Proof. of the Proposition 9.
Let $\widetilde{U}$ a causal and stationary process,

$$\widetilde{U}_{ij} = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} \varphi_k^1 \varphi_l^2 \widetilde{\varepsilon^*}_{i-s_1k,j-s_2l}$$

where the coefficients φ_k^1 and φ_l^2 are such that,

$$\Psi^{-1}\left(z_1^{s_1}, z_2^{s_1}\right) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} \varphi_k^1 \varphi_l^2 z_1^{s_1k} z_2^{s_2l}.$$

Therefore

$$\gamma_{\widetilde{U}}(h_1,h_2) = \mathrm{Cov}\left(\widetilde{U}_{i+h_1,j+h_2}, \widetilde{U}_{ij}\right)$$

$$\gamma_{\widetilde{U}}(h_1,h_2) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \varphi_k^1\varphi_l^2\varphi_m^1\varphi_n^2 \gamma_{\widetilde{\varepsilon^*}}(h_1 - s_1k + s_1m, h_2 - s_2l + s_2n). \tag{49}$$

When $h_1 - s_1k + s_2m = 0$ and $h_2 - s_2l + s_2n = 0$ in (49) we have $k = \frac{h_1}{s_1} + m$ and $l = \frac{h_2}{s_2} + n$ then (49) can be rewritten as

$$\gamma_{\widetilde{U}}(h_1,h_2) = \sigma^2_{\widetilde{\varepsilon^*}} \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \varphi^1_{\frac{h_1}{s_1}+m} \varphi^2_{\frac{h_2}{s_2}+n} \varphi_m^1 \varphi_n^2. \tag{50}$$

Taking $(h_1,h_2) = (s_1\ell_1, s_2\ell_2)$ in (50) for $\ell_1, \ell_2 \in \mathbb{Z}_+$ then

$$\gamma_{\widetilde{U}}(s_1\ell_1, s_2\ell_2) = \sigma^2_{\widetilde{\varepsilon^*}} \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \varphi^1_{\ell_1+m}\varphi^2_{\ell_2+n}\varphi^1_m\varphi^2_n.$$

If $(h_1,h_2) = (s_1\ell_1+\xi_1, s_2\ell_2+\xi_2)$ for $\ell_1,\ell_2 \in \mathbb{Z}_+$, $(\xi_1,\xi_2) \in A_1 \times A_2$, where $A_1 = \{1,\dots,s_1-1\}$, $A_2 = \{1,\dots,s_2-1\}$ then $\gamma_Z(h_1,h_2) = 0$. Therefore the autocovariance function of the process $\{\widetilde{U}_{ij}\}_{i,j\in\mathbb{Z}_+}$ is equal to

$$\gamma_{\widetilde{U}}(h_1,h_2) = \begin{cases} \sigma^2_{\widetilde{\varepsilon^*}} \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \varphi^1_{\ell_1+m}\varphi^1_m\varphi^2_{\ell_2+n}\varphi^2_n & \text{if } (h_1,h_2) = (s_1\ell_1, s_2\ell_2) \\ 0 & \text{if } (h_1,h_2) = (s_1\ell_1+\xi_1, s_2\ell_2+\xi_2). \end{cases} \tag{51}$$

Now the process $\{U_{ij}\}_{i,j\in\mathbb{Z}_+}$ can be rewritten by

$$U_{ij} = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} \varphi_k^1\varphi_l^2 Z_{i-s_1k,j-s_2l},$$

where the process $\{Z_{ij}\}_{i,j\in\mathbb{Z}_+}$ is given by (22). Then its autocovariance function is equal to

$$\begin{aligned}
\gamma_U(h_1,h_2) &= \text{Cov}\left(\widetilde{U}_{i+h_1,j+h_2},\widetilde{U}_{ij}\right)\\
&= \text{Cov}\left(\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi_k^1\varphi_l^2 Y_{i+h_1-s_1k,j+h_2-s_2l},\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_m^1\varphi_n^2 Z_{i-ms_1,j-ns_2}\right)\\
&= \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_k^1\varphi_l^2\varphi_m^1\varphi_n^2\text{Cov}\left(Z_{i+h_1-s_1k,j+h_2-s_2l},Z_{i-ms_1,j-ns_2}\right)\\
&= \sigma^2_{\varepsilon^*}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_k^1\varphi_l^2\varphi_m^1\varphi_n^2\gamma_Z(h_1-s_1k+s_1m,h_2-s_2l+s_2n)\\
&= \sigma^2_{\varepsilon*}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_k^1\varphi_l^2\varphi_m^1\varphi_n^2\gamma_Z\left(h_1-s_1(k-m),h_2-s_2(l-n)\right)
\end{aligned}$$

Taking $\nu_1=k-m$ and $\nu_2=l-n$, we get

$$\gamma_U(h_1,h_2)=\sigma^2_{\varepsilon^*}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi^1_{\nu_1+m}\varphi^2_{\nu_2+n}\varphi_m^1\varphi_n^2\gamma_Z\left(h_1-s_1\nu_1),h_2-s_2\nu_2\right). \tag{52}$$

Using (51) and denoting $\sigma^2_{\widetilde{\varepsilon}}=\sigma^2_{\varepsilon^*}/\sigma^2_{\widetilde{\varepsilon^*}}$ the variance of the two-dimensional white noise process $\{\varepsilon^*_{ij}\}_{i,j\in\mathbb{Z}_+}$ we obtain the results (33) and (34).

Proof. of the Proposition 10.
Let $\widetilde{V}$ a causal and stationary process,

$$\widetilde{V}_{ij}=\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi_k^1\varphi_l^2\widetilde{\varepsilon'}_{i-k,j-l}$$

where the coefficients φ_k^1 and φ_l^2 are given in

$$\Phi^{-1}(z_1,z_2)=\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi_k^1\varphi_l^2 z_1^k z_2^l,$$

then

$$\begin{aligned}
\gamma_{\widetilde{V}}(h_1,h_2) &= \text{Cov}\left(\widetilde{V}_{i+h_1,j+h_2},\widetilde{V}_{ij}\right)\\
\gamma_{\widetilde{V}}(h_1,h_2) &= \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_k^1\varphi_l^2\varphi_m^1\varphi_n^2\gamma_{\widetilde{\varepsilon'}}(h_1-k+m,h_2-l-n). \qquad (53)
\end{aligned}$$

When $h_1-k+m=0$ and $h_2-l+n=0$, we have $k=h_1+m$ and $l=h_2+n$.
Now (53) can be rewritten as

$$\gamma_{\widetilde{V}}(h_1,h_2)=\sigma^2_{\widetilde{\varepsilon'}}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi^1_{h_1+m}\varphi^1_m\varphi^2_{h_2+n}\varphi^2_n, \tag{54}$$

and the process $\{V\}_{ij\in\mathbb{Z}_+}$ is equal to

$$V_{ij} = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi_k^1\varphi_l^2 Y_{i-k,j-l}$$

where $\{Y_{ij}\}_{i,j\in\mathbb{Z}_+}$ is given by (23). Then its autocovariance function is given by

$$\begin{aligned}\gamma_V(h_1,h_2) &= \mathrm{Cov}\left(\widetilde{V}_{i+h_1,j+h_2},\widetilde{V}_{ij}\right)\\ &= \mathrm{Cov}\left(\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\varphi_k^1\varphi_l^2 Y_{i+h_1-k,j+h_2-l},\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_m^1\varphi_n^2 Y_{i-m,j-n}\right)\\ &= \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_k^1\varphi_l^2\varphi_m^1\varphi_n^2\mathrm{Cov}\left(Y_{i+h_1-k,j+h_2-l},Y_{i-m,j-n}\right)\end{aligned}$$

and

$$\gamma_V(h_1,h_2) = \sigma_{\varepsilon^*}^2\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\varphi_k^1\varphi_l^2\varphi_m^1\varphi_n^2\gamma_Y(h_1-k+m,h_2-l+n). \tag{55}$$

Applying (54) into (55) , with $\sigma^2_{\widetilde{\widetilde{\varepsilon}}} = \sigma^2_{\varepsilon'}/\sigma^2_{\widetilde{\varepsilon'}}$ the variance of the two-dimensional white noise process $\{\widetilde{\widetilde{\varepsilon}}_{ij}\}_{i,j\in\mathbb{Z}_+}$, we obtain (35).

Proof. of the Proposition 11.
We obtain the autocovariance function of the Seasonal FISSAR stationary process by repeating the same method as in the proof of the Propostion (9) where the processes $\{U_{ij}\}_{i,j\in\mathbb{Z}_+}$ and $\{V_{ij}\}_{i,j\in\mathbb{Z}_+}$ are respectively defined by (31) and (32) and taking the variance of the two-dimensional white noise process $\{\varepsilon_{ij}\}_{i,j\in\mathbb{Z}_+}$ equal to $\sigma^2_{\varepsilon} = \sigma^2_{\widetilde{\varepsilon}}/\sigma^2_{\widetilde{\widetilde{\varepsilon}}}$.

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