



A Review on asymptotic normality of sums of associated random variables

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Abstract. Association between random variables is a generalization of independence of these random variables. This concept is more and more commonly used in current trends in any research fields in Statistics. In this paper, we proceed to a simple, clear and rigorous introduction to it. We will present the fundamental asymptotic normality theorem on stationary and associated sequences of random variables. A coherent and modern frame is used. This review will be profitable to new researchers in the topic.

Résumé. Le concept de variables aléatoires associées est une généralisation de l'indépendance entre variables aléatoires. Il devient de plus en plus important dans les probabilités et les applications statistiques de tous les jours. Dans ce papier, nous introduisons à ce concept et présentons le théorème fondamental de la normalité asymptotique de sommes partielles d'une suite stationnaire de variables aléatoires associées. Nous utilisons un cadre moderne et cohérent qui sera profitable aux chercheurs débutants dans ce domaine.

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1. A brief reminder of association

We begin to introduce to the associated random variables concept which goes back to [Lehmann \(1966\)](#) in the bivariate case. The concept of association for random variables generalizes that of independence and seems to model a great variety of stochastic models.

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This property also arises in Physics, and is quoted under the name of FKG property (Fortuin *et al.*, 1971), in percolation theory and even in Finance (see Jiazhu, 2002).

The definite definition is given by Esary *et al.* (1967) as follows.

Definition 1. The rv's X_1, \dots, X_n are associated, or equivalently the finite sequence X_1, \dots, X_n is associated, if for any couple of real and coordinatewise nondecreasing functions h and g defined on \mathbb{R}^n , we have

$$\text{Cov}(h(X_1, \dots, X_n), g(X_1, \dots, X_n)) \geq 0$$

An infinite sequence of rv's is associated whenever all its finite subsequences are associated.

We have a few number of interesting properties to be found in (Prakasa Rao, 2012) :

(P1) A sequence of independent rv's is associated.

(P2) Partial sums of associated rv's are associated.

(P3) Order statistics of independent rv's are associated.

(P4) Non-decreasing functions and non-increasing functions of associated variables are associated.

(P5) Let the sequence Z_1, Z_2, \dots, Z_n be associated and let $(a_i)_{1 \leq i \leq n}$ be positive numbers and $(b_i)_{1 \leq i \leq n}$ real numbers. Then the rv's $a_i(Z_i - b_i)$ are associated.

As immediate other examples of associated sequences, we may cite Gaussian random vectors with nonnegatively correlated components (see Pitt, 1982) and homogenous Markov chains (Daley, 1968).

Demimartingales are set from associated centered variables exactly as martingales are derived from partial sums of centered independent random variables. We have

Definition 2. A sequence of rv's $\{S_n, n \geq 1\} \subset L^1(\Omega, \mathcal{A}, \mathbb{P})$ is a demimartingale if for any $j \geq 1$, for any coordinatewise nondecreasing function g defined on \mathbb{R}^j , we have

$$\mathbb{E}((S_{j+1} - S_j) g(S_1, \dots, S_j)) \geq 0. \tag{1}$$

Two particular cases should be highlighted. First any martingale is a demimartingale. Secondly, partial sums $S_0 = 0, S_n = X_1 + \dots + X_n, n \geq 1$, of associated and centered random variables X_1, X_2, \dots form a demimartingale for, in this case, (1) becomes :

$$\mathbb{E}\{(S_{j+1} - S_j) g(S_1, \dots, S_j)\} = \mathbb{E}\{X_{j+1} g(S_1, \dots, S_j)\} = \text{Cov}\{X_{j+1}, g(S_1, \dots, S_j)\},$$

since $\mathbb{E}X_{j+1} = 0$. Since $(x_1, \dots, x_{j+1}) \mapsto x_{j+1}$ et $(x_1, \dots, x_{j+1}) \mapsto g(x_1, \dots, x_j)$ are coordinate-wise nondecreasing functions and since the X_1, X_2, \dots are associated, we get

$$\mathbb{E}\{(S_{j+1} - S_j) g(S_1, \dots, S_j)\} = \text{Cov}\{X_{j+1} g(S_1, \dots, S_j)\} \geq 0.$$

2. Key results for associated sequences

Lemma 1. *Let (X, Y) be a bivariate random vector such that $\mathbb{E}(X^2) < \infty$ and $\mathbb{E}(Y^2) < \infty$. If (X_1, Y_1) and (X_2, Y_2) are two independent copies of (X, Y) , then We have*

$$2Cov(X, Y) = \mathbb{E}(X_1 - X_2)(Y_1 - Y_2).$$

We also have

$$Cov(X, Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H(x, y) dx dy,$$

where,

$$H(x, y) = \mathbb{P}(X > x, Y > y) - \mathbb{P}(X > x)\mathbb{P}(Y > y).$$

Before the proof of the lemma, we observe that :

$$H(x, y) = \mathbb{P}(X > x, Y > y) - \mathbb{P}(X > x)\mathbb{P}(Y > y) = \mathbb{P}(X \leq x, Y \leq y) - \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y). \quad (2)$$

Indeed we have

$$\begin{aligned} \mathbb{P}(X > x, Y > y) - \mathbb{P}(X > x)\mathbb{P}(Y > y) &= \mathbb{E}(\mathbb{I}_{(X>x)}\mathbb{I}_{(Y>y)}) - \mathbb{E}(\mathbb{I}_{(X>x)})\mathbb{E}(\mathbb{I}_{(Y>y)}) \\ &= Cov(\mathbb{I}_{(X>x)}, \mathbb{I}_{(Y>y)}) \\ &= Cov(1 - \mathbb{I}_{(X \leq x)}, 1 - \mathbb{I}_{(Y \leq y)}) \\ &= Cov(\mathbb{I}_{(X \leq x)}, \mathbb{I}_{(Y \leq y)}) \\ &= \mathbb{P}(X \leq x, Y \leq y) - \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y). \end{aligned}$$

Proof. We have

$$\begin{aligned} \mathbb{E}(X_1 - X_2)(Y_1 - Y_2) &= \mathbb{E}(X_1 Y_1) - \mathbb{E}(X_1)\mathbb{E}(Y_2) - \mathbb{E}(X_2)\mathbb{E}(Y_1) + \mathbb{E}(X_2 Y_2) \\ &= 2\mathbb{E}(X_1 Y_1) - 2\mathbb{E}(X_1)\mathbb{E}(Y_1) \\ &= 2Cov(X_1, Y_1). \end{aligned}$$

Next, for $a \in \mathbb{R}$, by Fubini's Theorem for nonnegative random variables,

$$\begin{aligned} \int_a^\infty \int_a^\infty \mathbb{P}(X > x, Y > y) dx dy &= \mathbb{E} \int_a^\infty \int_a^\infty \mathbb{I}_{(X>x)}\mathbb{I}_{(Y>y)} dx dy \\ &= \mathbb{E} \left(\int_a^X dx \int_a^Y dy \right) \\ &= \mathbb{E}[(X - a)(Y - a)]. \end{aligned}$$

We have

$$\begin{aligned} 2Cov(X_1, Y_1) &= \mathbb{E}(X_1 - X_2)(Y_1 - Y_2) \\ &= \mathbb{E}(\{(X_1 - a) - (X_2 - a)\} \{(Y_1 - a) - (Y_2 - a)\}) \\ &= \mathbb{E}(X_1 - a)(Y_1 - a) - \mathbb{E}(X_1 - a)(Y_2 - a) \\ &\quad - \mathbb{E}(X_2 - a)(Y_1 - a) + \mathbb{E}(X_2 - a)(Y_2 - a) \\ &= \int_a^\infty \int_a^\infty \mathbb{P}(X_1 > x, Y_1 > y) dx dy - \int_a^\infty \int_a^\infty \mathbb{P}(X_1 > x, Y_2 > y) dx dy \end{aligned}$$

$$- \int_a^\infty \int_a^\infty \mathbb{P}(X_2 > x, Y_1 > y) dx dy + \int_a^\infty \int_a^\infty \mathbb{P}(X_2 > x, Y_2 > y) dx dy.$$

By the independence of $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$, $\mathbb{P}(X_1 > x, Y_2 > y) = \mathbb{P}(X_1 > x) \times \mathbb{P}(Y_1 > y)$ and $\mathbb{P}(X_2 > x, Y_1 > y) = \mathbb{P}(X_1 > x) \times \mathbb{P}(Y_1 > y)$,

$$2Cov(X, Y) = 2 \left(\int_a^\infty \int_a^\infty \{ \mathbb{P}(X_1 > x, Y_1 > y) - \mathbb{P}(X_1 > x) \times \mathbb{P}(Y_1 > y) \} dx dy \right).$$

We get the final result by letting $a \rightarrow -\infty$. □

Lemma 2. *Suppose that X, Y are two random variables with finite variance and, f and g are C^1 complex valued functions on \mathbb{R}^1 with bounded derivatives f' and g' . Then*

$$|Cov(f(X), h(Y))| \leq \|f'\|_\infty \|g'\|_\infty Cov(X, Y)$$

Proof. By Lemma 1, we have

$$\begin{aligned} 2Cov(f(X), g(Y)) &= \mathbb{E}(f(X_1) - f(X_2))(g(Y_1) - g(Y_2)) \\ &= \mathbb{E} \left(\int_{X_1}^{X_2} f'(x) dx \int_{Y_1}^{Y_2} g'(x) dx \right). \end{aligned}$$

But

$$\begin{aligned} \int_{X_1}^{X_2} f'(x) dx &= \int_{X_1}^{+\infty} f'(x) dx - \int_{X_2}^{+\infty} f'(x) dx \\ &= \int_{\mathbb{R}} f'(x) \{ 1_{(X_1 \leq x)} - 1_{(X_2 \leq x)} \} dx \end{aligned}$$

Applying this to $\int_{Y_1}^{Y_2} g'(x) dx$ and combining all that, leads to

$$2Cov(f(X), g(Y)) = \mathbb{E} \int_{\mathbb{R}^2} f'(x) g'(y) \{ 1_{(X_1 \leq x)} - 1_{(X_2 \leq x)} \} \{ 1_{(Y_1 \leq y)} - 1_{(Y_2 \leq y)} \} dx dy. \quad (3)$$

It is easy to see that

$$\begin{aligned} &\mathbb{E} \{ 1_{(X_1 \leq x)} - 1_{(X_2 \leq x)} \} \{ 1_{(Y_1 \leq y)} - 1_{(Y_2 \leq y)} \} \\ &= 2(\mathbb{P}(X \leq x, Y \leq y) - \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)) \end{aligned}$$

and by (2), this is equal to $2H(x, y)$. By applying Fubini's theorem in (3), we get

$$2Cov(f(X), g(Y)) = 2 \int_{\mathbb{R}^2} f'(x) g'(y) H(x, y) dx dy.$$

This gives, since $H(x, y) \geq 0$ for associated rv 's,

$$|Cov(f(X), g(Y))| \leq \|f'\|_\infty \|g'\|_\infty \int_{\mathbb{R}^2} H(x, y) dx dy.$$

And we complete the proof by applying Lemma 1. □

Remark : We used the proof of Yu (1993) here.

Theorem 1. Let X_1, X_2, \dots, X_n be associated, then we have for all $t = (t_1, \dots, t_n) \in \mathbb{R}^k$,

$$\left| \psi_{(X_1, X_2, \dots, X_n)}(t) - \prod_{i=1}^n \psi_{X_i}(t_i) \right| \leq \frac{1}{2} \sum_{1 \leq i \neq j \leq n} |t_i t_j| |Cov(X_i, X_j)|. \quad (4)$$

Proof : First, we prove this for $n = 2$. Use the Newman inequality in Lemma 2. Let X and Y be two associated random variables. For $(s, t) \in \mathbb{R}^2$, put $U = f(X) =: e^{isX}$ and $V = g(Y) =: e^{itY}$. We have

$$Cov(U, V) = E(e^{(isX+itY)}) - E(e^{isX})E(e^{itY}) = \psi_{(X,Y)}(s, t) - \psi_X(s)\psi_Y(t).$$

But Lemma 2 implies

$$\begin{aligned} |Cov(U, V)| &= |Cov(f(X), g(Y))| \leq |st| \|f'\|_\infty \|g'\|_\infty |Cov(X, Y)| = |st| |Cov(X, Y)| \\ &= \frac{1}{2} |st| |(Cov(X, Y) + cov(Y, X))|. \end{aligned}$$

And (4) is valid for $n = 2$. Now we proceed by induction and suppose that (4) is true up to n . Consider associated random variables X_1, X_2, \dots, X_{n+1} and let $t = (t_1, \dots, t_{n+1}) \in \mathbb{R}^{n+1}$. If all the t_i are nonnegative, we have $U = t_1 X_1 + \dots + t_n X_n$ and $V = X_{n+1}$ are associated. We have

$$\psi_{(X_1, X_2, \dots, X_{n+1})}(t) = \psi_{(U, V)}(1, t_{n+1}) \text{ and } \psi_U(1) = \psi_{(X_1, X_2, \dots, X_n)}(t_1, \dots, t_n).$$

By the induction hypothesis, we have

$$\begin{aligned} &\left| \psi_{(X_1, X_2, \dots, X_{n+1})}(t) - \psi_{(X_1, X_2, \dots, X_n)}(t_1, \dots, t_n) \psi_{X_{n+1}}(t_{n+1}) \right| \\ &\leq |t_{n+1}| |cov(X_{n+1}, t_1 X_1 + \dots + t_n X_n)| \\ &\leq \frac{1}{2} \sum_{j=1}^n |t_j t_{n+1}| |cov(X_{n+1}, X_j)|. \end{aligned} \quad (5)$$

Next

$$\begin{aligned} &\left| \psi_{(X_1, X_2, \dots, X_{n+1})}(t) - \prod_{i=1}^{n+1} \psi_{X_i}(t_i) \right| \\ &\leq \left| \psi_{(X_1, X_2, \dots, X_{n+1})}(t) - \psi_{(X_1, X_2, \dots, X_n)}(t_1, \dots, t_n) \psi_{X_{n+1}}(t_{n+1}) \right| \\ &\quad + \left| \psi_{(X_1, X_2, \dots, X_n)}(t_1, \dots, t_n) \psi_{X_{n+1}}(t_{n+1}) - \prod_{i=1}^{n+1} \psi_{X_i}(t_i) \right|. \end{aligned}$$

The first term in the right side member is bounded as in (5). The second term is bounded, due to the induction hypothesis, by

$$\left| \psi_{X_{n+1}}(t_{n+1}) \right| \left| \psi_{(X_1, X_2, \dots, X_n)}(t_1, \dots, t_n) - \prod_{i=1}^n \psi_{X_i}(t_i) \right|$$

$$\begin{aligned}
 &= \left| \psi_{(X_1, X_2, \dots, X_n)}(t_1, \dots, t_n) - \prod_{i=1}^n \psi_{X_i}(t_i) \right| \\
 &\leq \frac{1}{2} \sum_{1 \leq i \neq j \leq n} |t_i t_j| |\text{cov}(X_i, X_j)|.
 \end{aligned} \tag{6}$$

By putting (5) and (6) together, we get that (4) is valid. By re-arranging the t_i , we observe that we have proved (4) for $n = 3$, if at least n of the t_i are nonnegative. Also, if at least n of them are nonpositive, we consider the sequence $-X_1, \dots, -X_{n+1}$ that is also associated and get the same conclusion. This means that (4) is true. It remains the case where exactly p of the t_i 's are nonnegative with $2 \leq p \leq n - 2$. By re-arranging the t_i if necessary, we may consider that $t_i \geq 0$ for $1 \leq i \leq p$ and $t_i < 0$ for $i > p$. Now, put $U = t_1 X_1 + \dots + t_p X_p$ and $V = t_{p+1} X_{p+1} + \dots + t_{n+1} X_{n+1}$. Since U et $-V$ are associated and since

$$\psi_{(X_1, X_2, \dots, X_{n+1})}(t) = \psi_{(U, -V)}(1, -1),$$

we have by the induction hypothesis

$$\left| \psi_{(X_1, X_2, \dots, X_{n+1})}(t) - \psi_U(1)\psi_{-V}(-1) \right| \leq \frac{1}{2} |\text{Cov}(U, -V)| \leq \frac{1}{2} \sum_{i=1}^p \sum_{j=p+1}^{n+1} |t_i t_j| |\text{cov}(X_i, X_j)| \tag{7}$$

Now use

$$\begin{aligned}
 &\left| \psi_{(X_1, X_2, \dots, X_{n+1})}(t) - \prod_{i=1}^{n+1} \psi_{X_i}(t_i) \right| \leq \left| \psi_{(X_1, X_2, \dots, X_{n+1})}(t) - \psi_U(1)\psi_{-V}(-1) \right| \\
 &\quad + \left| \psi_U(1)\psi_{-V}(-1) - \psi_U(1) \prod_{i=p+1}^{n+1} \psi_{X_i}(t_i) \right| \\
 &\leq \left| \psi_{(X_1, X_2, \dots, X_{n+1})}(t) - \psi_U(1)\psi_{-V}(-1) \right| + \left| \psi_U(1)\psi_{-V}(-1) - \prod_{i=1}^p \psi_{X_i}(t_i)\psi_{-V}(-1)(t_i) \right| \\
 &\quad + \left| \prod_{i=1}^p \psi_{X_i}(t_i)\psi_{-V}(-1) - \prod_{i=1}^{n+1} \psi_{X_i}(t_i) \right|
 \end{aligned} \tag{8}$$

The first term already handled in (8). The second term is bounded as follows

$$\begin{aligned}
 &\left| \psi_U(1)\psi_{-V}(-1) - \prod_{i=1}^p \psi_{X_i}(t_i)\psi_{-V}(-1)(t_i) \right| = |\psi_{-V}(-1)(t_i)| \times \left| \psi_U(1) - \prod_{i=1}^p \psi_{X_i}(t_i) \right| \\
 &\leq \left| \psi_U(1) - \prod_{i=1}^p \psi_{X_i}(t_i) \right| = \left| \psi_{(X_1, X_2, \dots, X_p)}(t_1, \dots, t_p) - \prod_{i=1}^p \psi_{X_i}(t_i) \right| \\
 &\leq \frac{1}{2} \sum_{1 \leq i \neq j \leq p} |t_i t_j| |\text{cov}(X_i, X_j)|.
 \end{aligned} \tag{9}$$

where we used the induction hypothesis in the last formula. The last term is

$$\begin{aligned} \left| \prod_{i=1}^p \psi_{X_i}(t_i) \psi_{-V}(-1) - \prod_{i=1}^{n+1} \psi_{X_i}(t_i) \right| &= \left| \prod_{i=1}^p \psi_{X_i}(t_i) \psi_{-V}(-1) - \prod_{i=1}^{n+1} \psi_{X_i}(t_i) \right| \\ &\leq \left| \prod_{i=1}^p \psi_{X_i}(t_i) \right| \times \left| \psi_{-V}(-1) - \prod_{i=p+1}^{n+1} \psi_{X_i}(t_i) \right| \\ &\leq \left| \psi_{(X_{p+1}, \dots, X_{n+1})}(t_{p+1}, \dots, t_{n+1}) - \prod_{i=p+1}^{n+1} \psi_{X_i}(t_i) \right| \\ &\leq \frac{1}{2} \sum_{p+1 \leq i \neq j \leq n+1} |t_i t_j| |\text{cov}(X_i, X_j)|, \end{aligned} \tag{10}$$

where we used again the induction hypothesis. We complete the proof by putting (7), (8), (10) and (9) together, we arrive at the result (4). \square

3. Central limit theorem for a stricly stationary and associated sequence

In this section, we provide all the details of the sharpest result in this topic by Newman and Wright (1981). This came as a concluding paper for a series of papers by Newman.

We present here all the materials used in the proof of Newman and Wright in a detailed writing that makes it better understandable by a broad public.

First, we have this simple lemma.

Lemma 3. *Let X and Y be finite variance random variables such that*

$$E(X, Y 1_{(Y \leq 0)}) \geq 0. \tag{11}$$

Then, we have

$$\mathbb{E}[(\max(X, X + Y))^2] \leq E(X + Y)^2. \tag{12}$$

If X and Y are associated and X is mean zero, then (11) holds and (12) is true.

Proof. We have

$$\begin{aligned} \max(X, X + Y)^2 &= \{X 1_{(Y \leq 0)} + (X + Y) 1_{(Y > 0)}\}^2 \\ &= X^2 1_{(Y \leq 0)} + (X + Y)^2 1_{(Y > 0)} = X^2 1_{(Y \leq 0)} + (X^2 + Y^2 + 2XY) 1_{(Y > 0)} \\ &= X^2 + Y^2 - Y^2 1_{(Y \leq 0)} + 2(XY) 1_{(Y > 0)} \\ &= X^2 + Y^2 + 2XY - 2XY 1_{(Y \leq 0)} - Y^2 1_{(Y \leq 0)} \\ &= (X + Y)^2 - 2XY 1_{(Y \leq 0)} - Y^2 1_{(Y \leq 0)} \end{aligned}$$

We get the desired result whenever

$$E(XY1_{(Y \leq 0)}) = Cov(X, Y1_{(Y \leq 0)}) \geq 0$$

Now if X and Y are associated, we have

$$XY1_{(Y \leq 0)} = (-X)(-Y)1_{(-Y \geq 0)}.$$

Since $(-X)$ and $(-Y)$ are associated too and $1_{(-Y \geq 0)}$ is a nondecreasing function of $(-Y)$, and reminding that X is mean zero, we get that

$$E(XY1_{(Y \leq 0)}) = E((-X)(-Y)1_{(-Y \geq 0)}) = Cov((-X), (-Y)1_{(-Y \geq 0)}) \geq 0.$$

□

Theorem 2 (Maximal inequality of Newman and Wright). *Let X_1, X_2, \dots, X_n be associated, mean zero, finite variance, random variables and $M_n = \max(S_1, S_2, \dots, S_n)$ where $S_n = X_1 + X_2 + \dots + X_n$, we have*

$$\mathbb{E}(M_n^2) \leq V(S_n). \tag{13}$$

Proof. Let us prove (13) by induction. It is obviously true for $n = 1$ and for $n = 2$ by Lemma 3. Let us suppose that it is true for $j, 2 \leq j < n$. By putting $L_j = X_2 + \dots + X_j$, $j \geq 2$, we have

$$M_n = \max(X_1, X_1 + L_2, \dots, X_1 + L_n) = X_1 + \max(0, L_2, \dots, L_n).$$

Also

$$\max(X_1, X_1 + \max(L_2, \dots, L_n)) = X_1 + \max(0, \max(L_2, \dots, L_n))$$

We obviously have

$$\max(0, \max(L_2, \dots, L_n)) = \max(0, L_2, \dots, L_n).$$

Then

$$\mathbb{E}M_n^2 = E \max(X_1, X_1 + \max(L_2, \dots, L_n))^2$$

Since X_1 and $\max(L_2, \dots, L_n)$ are associated and X_1 is mean zero, use Lemma 3 to get

$$EM_n^2 = E \max(X_1, X_1 + \max(L_2, \dots, L_n))^2 \leq EX_1^2 + E \max(L_2, \dots, L_n)^2.$$

And then, apply (13) on $\mathbb{E} \max(L_2, \dots, L_n)^2$ for $(n - 1)$ mean zero associated rv's to have

$$\mathbb{E} \max(L_2, \dots, L_n)^2 \leq EX_2^2 + \dots + X_n^2.$$

We conclude that

$$EM_n^2 \leq \mathbb{E}X_1^2 + \mathbb{E}X_2^2 + \dots + \mathbb{E}X_n^2.$$

□

Lemma 4. *Let X_1, X_2, \dots, X_n be a second-order stationary sequence with $\sigma^2 = V(X_1) + 2 \sum_{j=2}^{\infty} |Cov(X_1, X_j)| < \infty$, then*

$$V\left(\frac{S_n}{\sqrt{n}}\right) \rightarrow \sigma^2 = V(X_1) + 2 \sum_{j=2}^{\infty} Cov(X_1, X_j).$$

Proof. We have

$$\alpha_n = V\left(\frac{S_n}{\sqrt{n}}\right) = \frac{1}{n} \left\{ \sum_{j=1}^n V(X_j) + \sum_{1 \leq i \neq j \leq n} Cov(X_i, X_j) \right\}.$$

By stationarity, we have

$$\begin{aligned} V\left(\frac{S_n}{\sqrt{n}}\right) &= V(X_1) + \frac{2}{n} \sum_{1 \leq i < j \leq n} Cov(X_i, X_j) \\ &= V(X_1) + \frac{2}{n} \sum_{j=2}^n (n-j+1)Cov(X_1, X_j). \end{aligned}$$

Let $\epsilon > 0$. Since $\sum_{j=2}^{\infty} Cov(X_1, X_j) < +\infty$, there exists $K > 0$ such that for any $k \geq K$,

$$\sum_{j \geq k+1} Cov(X_1, X_j) < \epsilon.$$

We fix that $k \geq K$ and write,

$$\alpha_n = V(X_1) + 2 \left[\sum_{j=2}^k \left(1 - \frac{j-1}{n}\right) Cov(X_1, X_j) + \sum_{j=k+1}^n \left(1 - \frac{j-1}{n}\right) Cov(X_1, X_j) \right]$$

and observe that

$$\left| \alpha_n - V(X_1) - 2 \sum_{j=2}^k \left(1 - \frac{j-1}{n}\right) Cov(X_1, X_j) \right| \leq 2\epsilon.$$

Thus, we get

$$\begin{aligned} \liminf V(X_1) + 2 \sum_{j=2}^k \left(1 - \frac{j-1}{n}\right) Cov(X_1, X_j) - 2\epsilon &\leq \liminf \alpha_n \\ &\leq \limsup \alpha_n \leq \limsup V(X_1) + 2 \sum_{j=2}^k \left(1 - \frac{j-1}{n}\right) Cov(X_1, X_j) + 2\epsilon. \end{aligned}$$

Therefore, for any $k \geq K$,

$$\begin{aligned} V(X_1) + 2 \sum_{j=2}^k Cov(X_1, X_j) - 2\epsilon &\leq \liminf \alpha_n \leq \limsup \alpha_n \\ &\leq V(X_1) + 2 \sum_{j=2}^k Cov(X_1, X_j) + 2\epsilon. \end{aligned}$$

We finish the proof by letting $k \rightarrow \infty$ and next by letting $\epsilon \rightarrow 0$. □

Theorem 3. Let X_1, X_2, \dots, X_m be a strictly stationary, mean zero, associated random variables such that

$$\sigma^2 = V(X_1) + 2 \sum_{j=2}^{+\infty} Cov(X_1, X_j) < \infty,$$

then

$$\frac{S_n}{\sqrt{n}} = \frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}} \rightarrow N(0, \sigma^2) \text{ as } n \rightarrow \infty$$

Proof. Let us fix $\ell > 1$ an integer and let us set $m = \lceil \frac{n}{\ell} \rceil$, that is $m\ell \leq n \leq m\ell + \ell$. Let us define $\Psi_n(r) = \mathbb{E}(e^{irS_n/\sqrt{n}})$, $r \in \mathbb{R}$. First, we have for $r \in \mathbb{R}$,

$$\begin{aligned} |\Psi_n(r) - \Psi_{m\ell}(r)| &= |\mathbb{E}(e^{irS_n/\sqrt{n}}) - \mathbb{E}(e^{irS_{m\ell}/\sqrt{m\ell}})| \\ &= \left| \mathbb{E} \left[e^{irS_{m\ell}/\sqrt{m\ell}} \left(e^{ir[(S_n/\sqrt{n}) - (S_{m\ell}/\sqrt{m\ell})]} - 1 \right) \right] \right| \\ &\leq \mathbb{E} \left| e^{ir \left(\frac{S_n}{\sqrt{n}} - \frac{S_{m\ell}}{\sqrt{m\ell}} \right)} - 1 \right|. \end{aligned} \tag{14}$$

But for any $x \in \mathbb{R}$,

$$|e^{ix} - 1| = |(\cos x - 1) + i \sin x| = |2 \sin \frac{x}{2}| \leq |x|.$$

Thus the second member of (14) is, by the Cauchy-Schwarz's inequality, bounded by

$$|r| \mathbb{E} \left| \frac{S_n}{\sqrt{n}} - \frac{S_{m\ell}}{\sqrt{m\ell}} \right| \leq |r| V \left(\frac{S_n}{\sqrt{n}} - \frac{S_{m\ell}}{\sqrt{m\ell}} \right)^{\frac{1}{2}}.$$

Let us compute the quantity between brackets for fixed ℓ and $n \rightarrow \infty$ ($m \rightarrow \infty$), we get

$$\begin{aligned} \frac{S_n}{\sqrt{n}} - \frac{S_{m\ell}}{\sqrt{m\ell}} &= \frac{S_n}{\sqrt{n}} - \frac{S_{m\ell}}{\sqrt{n}} + \frac{S_{m\ell}}{\sqrt{n}} - \frac{S_{m\ell}}{\sqrt{m\ell}} \\ &= \frac{S_n - S_{m\ell}}{\sqrt{n}} - \frac{\sqrt{n} - \sqrt{m\ell}}{\sqrt{nm\ell}} S_{m\ell} \end{aligned}$$

and

$$\begin{aligned} \delta_{m,\ell} = V \left(\frac{S_n}{\sqrt{n}} - \frac{S_{m\ell}}{\sqrt{m\ell}} \right) &= V \left(\frac{S_n - S_{m\ell}}{\sqrt{n}} \right) + \left(\frac{\sqrt{n} - \sqrt{m\ell}}{\sqrt{n}} \right)^2 V \left(\frac{S_{m\ell}}{\sqrt{m\ell}} \right) \\ &\quad - 2 \frac{\sqrt{n} - \sqrt{m\ell}}{\sqrt{nm\ell}} Cov(S_n - S_{m\ell}, S_{m\ell}). \end{aligned}$$

$Cov(S_n - S_{m\ell}, S_{m\ell}) \geq 0$ by association. Thus

$$\delta_{m,\ell} \leq V \left(\frac{S_n - S_{m\ell}}{\sqrt{n}} \right) + \left(\frac{\sqrt{n} - \sqrt{m\ell}}{\sqrt{n}} \right)^2 V \left(\frac{S_{m\ell}}{\sqrt{m\ell}} \right).$$

Since $0 \leq n - m\ell \leq \ell$, and $Cov(X_1, X_j) \geq 0$ by association, we have

$$\begin{aligned} V(S_{n-m\ell}) &= \sum_{i=1}^{n-m\ell} V(X_i) + \sum_{1 \leq i \neq j \leq n-m\ell} Cov(X_i, X_j) \\ &\leq \sum_{i=1}^{\ell} V(X_i) + \sum_{1 \leq i \neq j \leq \ell} Cov(X_i, X_j) = A(\ell). \end{aligned}$$

Further, $m\ell \leq n \leq (m+1)\ell$ implies

$$0 \leq \frac{\sqrt{n} - \sqrt{m\ell}}{\sqrt{n}} \leq \left(1 - \sqrt{\frac{m\ell}{n}}\right) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Then when $m \rightarrow \infty$ ($n \rightarrow \infty$)

$$V\left(\frac{S_{m\ell}}{\sqrt{m\ell}}\right) \rightarrow V(X_1) + 2 \sum_{j=2}^{\infty} Cov(X_1, X_j) < \infty$$

and

$$\delta_{m,\ell} \leq \frac{A(\ell)}{n} + \left(1 - \sqrt{\frac{m\ell}{n}}\right)^2 V\left(\frac{S_{m\ell}}{\sqrt{m\ell}}\right) \rightarrow 0.$$

For fixed ℓ , $n \rightarrow \infty$, we get

$$|\Psi_n(r) - \Psi_{m\ell}(r)| \rightarrow 0.$$

Now, let us set $Y_j = (S_{j\ell} - S_{\ell(j-1)})/\sqrt{\ell}$, for a fixed ℓ . By strict stationarity, the Y_j 's are associated and identically distributed. Let Ψ_ℓ be the common characteristic function of Y_1, \dots, Y_m . Furthermore

$$\frac{S_{m\ell}}{\sqrt{m\ell}} = \frac{1}{\sqrt{m}\sqrt{\ell}} \sum_{j=1}^m (S_{j\ell} - S_{\ell(j-1)}) = \frac{1}{\sqrt{m}} \sum_{j=1}^m Y_j.$$

According to the Newman's Theorem (see Theorem 1)

$$\left| \Psi_{m\ell}(r) - \left(\Psi_\ell\left(\frac{r}{\sqrt{m}}\right) \right)^m \right| \leq \frac{r^2}{2m} \sum_{1 \leq j \neq k \leq m} Cov(Y_j, Y_k),$$

and we know that

$$V\left(\sum_{j=1}^m Y_j\right) = \sum_{j=1}^m V(Y_j) + \sum_{1 \leq j \neq k \leq m} Cov(Y_j, Y_k).$$

Thus, by using the stationarity again, we get

$$\frac{1}{m} \sum_{1 \leq j \neq k \leq m} Cov(Y_j, Y_k) = \frac{1}{m} V\left(\sum_{j=1}^m Y_j\right) - \frac{1}{m} \sum_{j=1}^m V(Y_j)$$

$$= V \left(\frac{1}{\sqrt{m}} \sum_{j=1}^m Y_j \right) - \frac{1}{m} \sum_{j=1}^m V(Y_j)$$

$$V \left(\frac{S_{m\ell}}{\sqrt{m\ell}} \right) - V \left(\frac{S_\ell}{\sqrt{\ell}} \right) = \sigma_{m\ell}^2 - \sigma_\ell^2,$$

where for any $p \geq 2$,

$$\sigma_p^2 = \frac{1}{p} \sum_{i=1}^p V(Y_i) + \frac{1}{p} \sum_{1 \leq i \neq j \leq p} Cov(Y_i, Y_j)$$

Now, when $m \rightarrow \infty$, $\sigma_{m\ell}^2 \rightarrow \sigma^2$ and

$$\left(\Psi_\ell \left(\frac{r}{\sqrt{m}} \right) \right)^m \rightarrow e^{-\sigma_\ell^2 r^2 / 2},$$

where σ_ℓ^2 is the common variance of Y_j 's,

$$\sigma_\ell^2 = \sum_{i=1}^{\ell} V(X_i) + \frac{1}{\ell} \sum_{1 \leq i \neq j \leq m} Cov(X_i, X_j).$$

Then it comes out that

$$\overline{\lim} \left| \Psi_{m\ell}(r) - e^{-\sigma_\ell^2 r^2 / 2} \right| \leq \frac{r^2}{2} (\sigma^2 - \sigma_\ell^2).$$

We complete the proof by letting $\ell \rightarrow \infty$. Thus $\sigma_\ell^2 - \sigma^2 \rightarrow 0$ and we get

$$\lim_{n \rightarrow \infty} \left| \Psi_n(r) - e^{-\sigma^2 r^2 / 2} \right| = 0.$$

□

Remark. We finish this exposition by these important facts. A number of CLT's and invariance principles are available in the literature for strictly stationary sequences of associated random variables and not stationary ones. The most general CLT seems to be the one provided by [Cox and Grimmett \(1983\)](#) for arbitrary associated rv's fulfilling a number of moment conditions. Dabrowski and co-authors (see [Burton et al., 1986](#) and [Dabrowski, 1958](#)) considered weakly associated random variables to establish principle invariances in the lines of [Newman and Wright \(1981\)](#), as well as Berry-Essen-type results and functional LIL's. But almost all these results use adaptations of the original method of Newman we have described here.

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